Different Kinds of Proof

Henry Towsner

This presentation looks best in full-screen mode.

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In this case the "toy" part is taken a bit literally.

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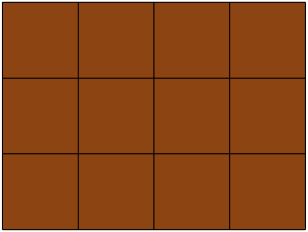
Chomp is a game played on a bar of chocolate

Chomp is a game played on a bar of chocolate that has been divided into a grid



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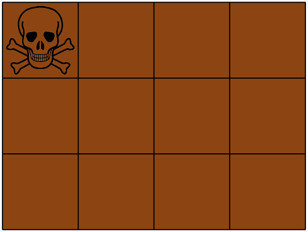

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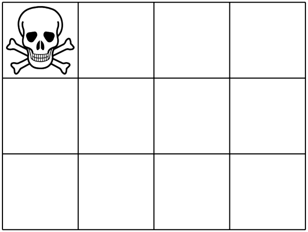

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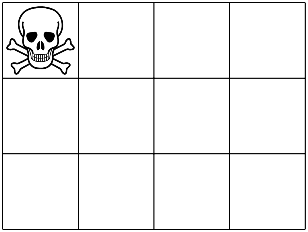

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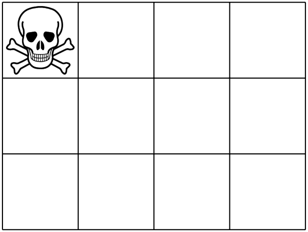

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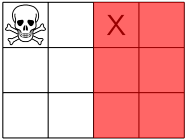

Like this

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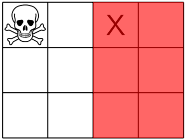

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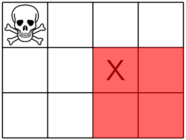

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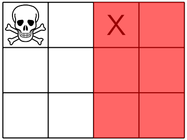

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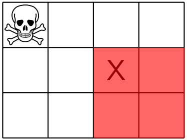

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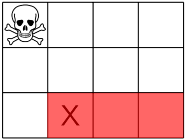

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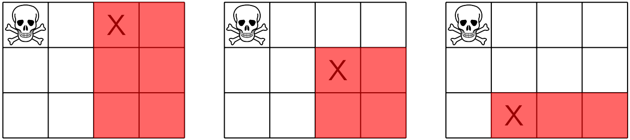

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The game is played between two players, who we'll call Alice and Bob. Alice and Bob take turns choosing a square to remove, together with the rectangle of squares *below* or *to the right*. Whichever player is forced to take the poisoned square loses.

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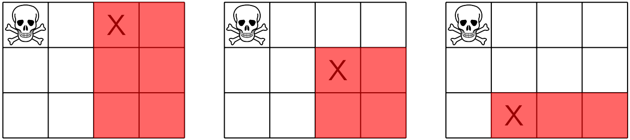
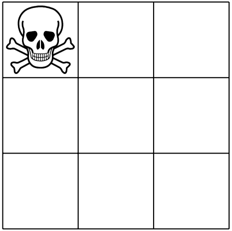


Like this   Or this   Or this.

The game is played between two players, who we'll call Alice and Bob. Alice and Bob take turns choosing a square to remove, together with the rectangle of squares *below* or *to the right*. Whichever player is forced to take the poisoned square loses. The bar of chocolate can be divided into many grids of many different sizes, so there are many possible games of Chomp.

Here's an example game, starting with a 3x3 board.



Alice's turn to play

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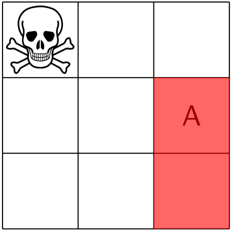

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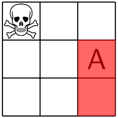

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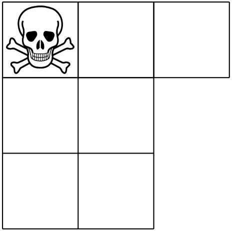

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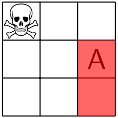

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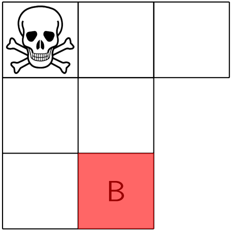

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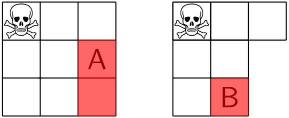

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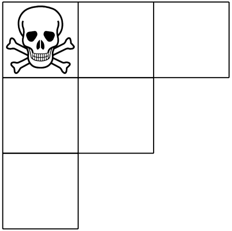

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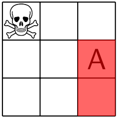

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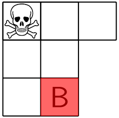

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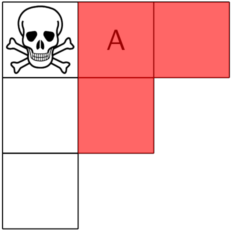

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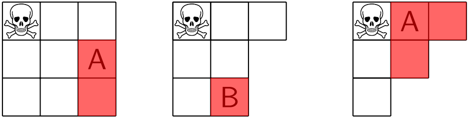

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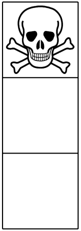

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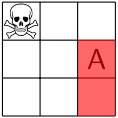

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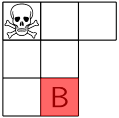

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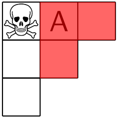

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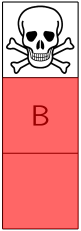

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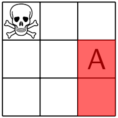

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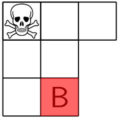

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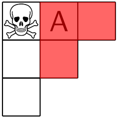

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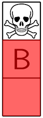

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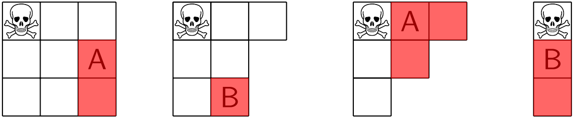

Alice's play Bob's play Alice's play Bob's play

Alice's play

Alice has to move, so she's forced to take the poison piece. Bob wins this time!

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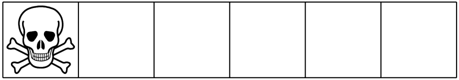

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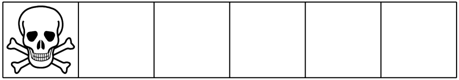

Alice's turn to play

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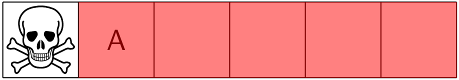

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Bob's turn to play

Alice just takes everything other than the poison, forcing Bob to lose on his very first move.

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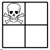

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Alice's play

Alice's first move will be to nibble the lower right corner.

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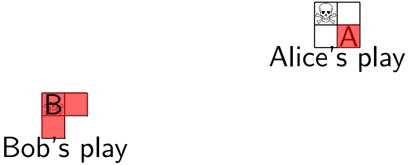

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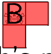
Alice's play



Bob's play

If Alice is lucky, Bob will eat the poison immediately, handing her the game. . .but that doesn't seem very likely.

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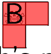
Alice's play



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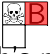

Bob's play

Instead, Bob might play like this.

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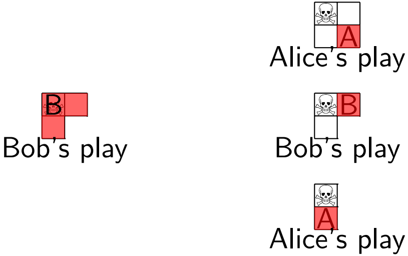

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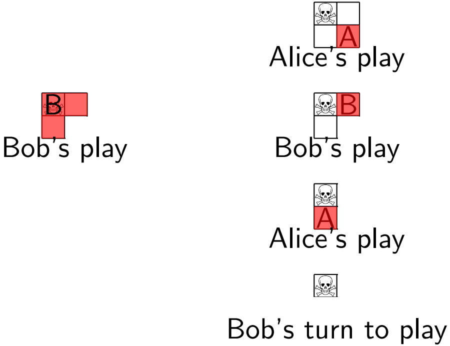

Instead, Bob might play like this. In which case Alice retaliates like this, and Bob is forced to eat the poison.

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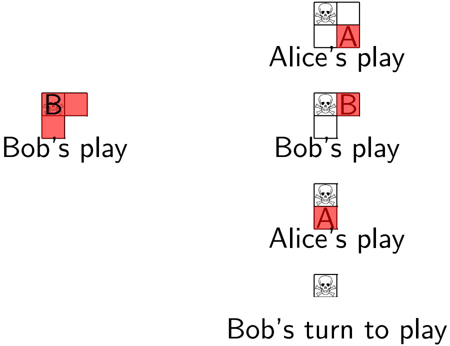

But Bob isn't out of options! Remember that Alice has to be able to defeat *anything* Bob does.

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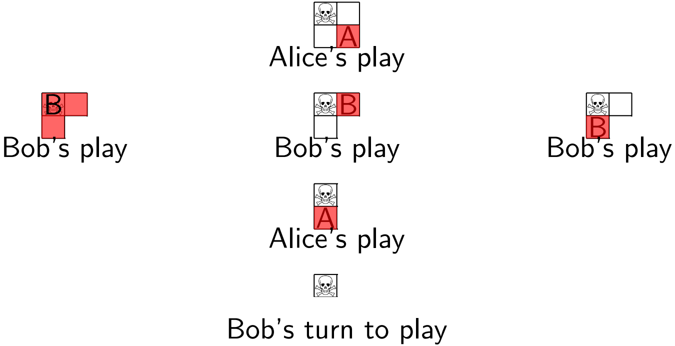

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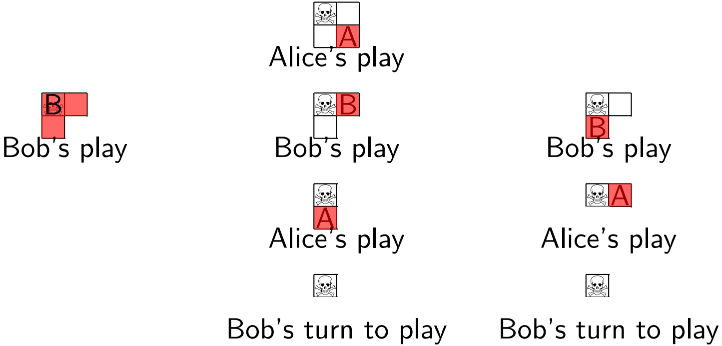
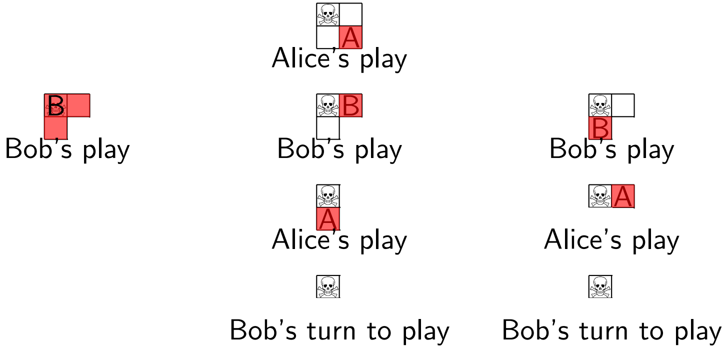


But Bob isn't out of options! Remember that Alice has to be able to defeat *anything* Bob does. What if, instead, Bob plays like this? Then Alice responds like this, and once again Bob has to take the poison.

Usually, however, Bob will get a chance to respond to Alice by making his own moves. In order to have a winning strategy, Alice has to be prepared for every eventuality. For instance, on a 2x2 board, Alice still has a winning strategy.



No matter what Bob does, Alice has a response. So Alice has a winning strategy: a rule for playing that guarantees her victory against any opponent.

In order to demonstrate that the player who went first had a winning strategy on a 2x2 board, we exhaustively listed all ways the game could play.

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That's fine for a very small game, but when we think about bigger boards, that's going to become unwieldy very quickly.

As mathematicians, we're prepared to accept less than that. We'll accept a *proof* that someone has a winning strategy—that is, a rigorous argument that the person can always win—in place of a giant book of moves.

Let's show that the player who goes first has a winning strategy in every game of Chomp on a *square* board bigger than 1x1.

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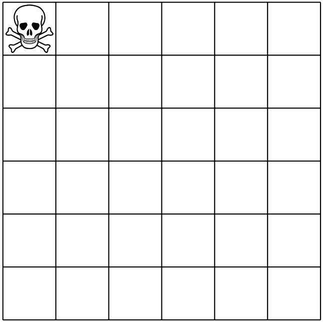

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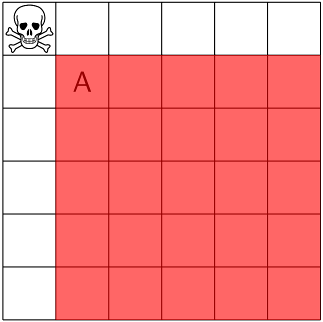

Alice's play

The first thing to do is to grab the space just below and right of the poison.

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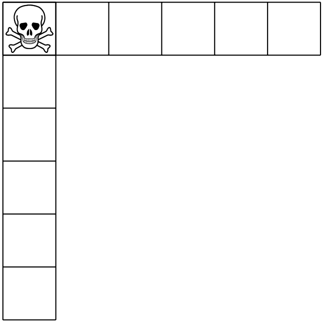

Bob's turn to play

The first thing to do is to grab the space just below and right of the poison. Bob is left with two long skinny legs to play in.

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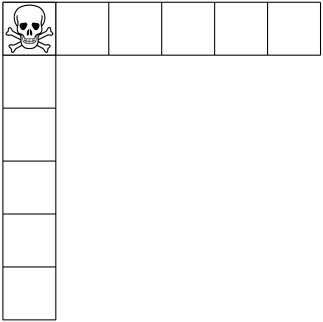

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The only way to affect both legs is to take the poison, so Bob has to pick one leg to play it.

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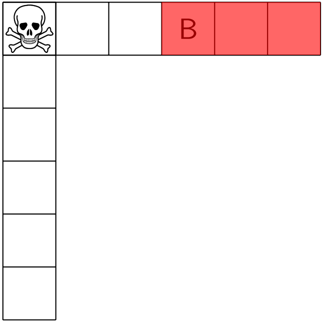

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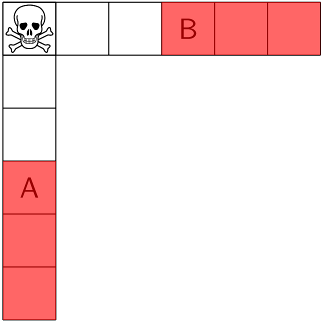

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The only way to affect both legs is to take the poison, so Bob has to pick one leg to play it. No matter what he does in one leg. . . Alice just copies it in the other leg.

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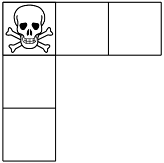

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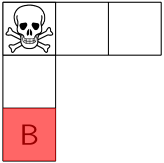

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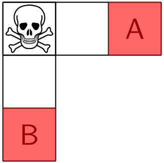

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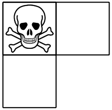

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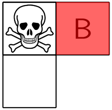

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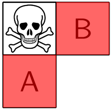

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Let's show that the player who goes first has a winning strategy in every game of Chomp on a *square* board bigger than 1x1.

Bob's turn to play

Alice ensures that both legs get used up at the same time, with Bob stuck taking the poisoned piece in the middle.

So Alice has a winning strategy on every square board.

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We didn't actually list all the possible moves, but we did something better: we gave a description of Alice's strategy, and (briefly) explained why it always works.

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But that doesn't exactly count.

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But that doesn't exactly count.
Other than that, the answer is no: there is no size board other that 1x1 where the second player has a winning strategy.

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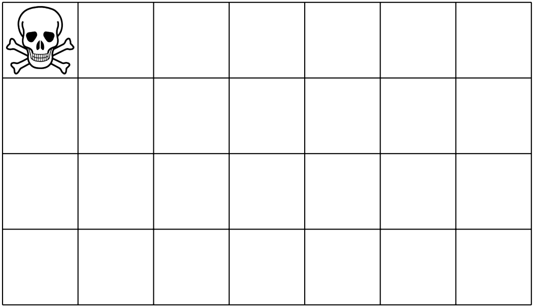
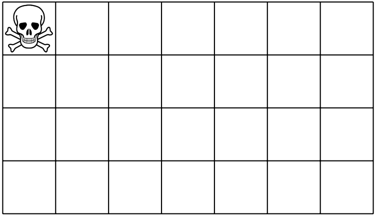
So suppose Bob announces that he has uncovered a masterful technique that allows him to win all games on, say, a 7x4 board, when he goes second.

(The same method will work with any other size board—except for the 1x1.)



Alice's turn to play

Alice agrees to play against Bob's marvelous strategy.



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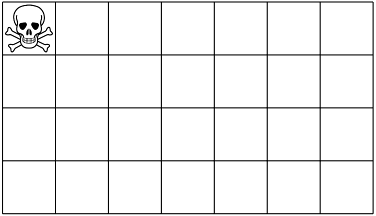

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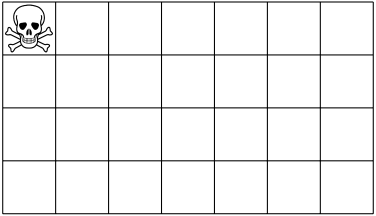
Alice agrees to play against Bob's marvelous strategy. But she knows a trick—she insists on playing two games at once!



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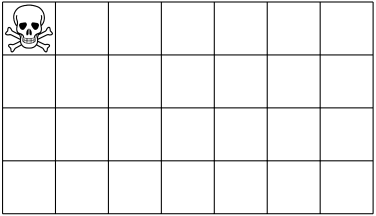
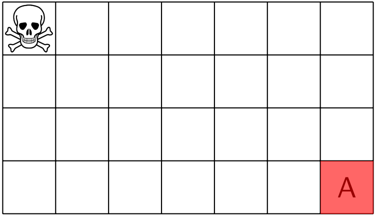


Alice's turn to play

Alice agrees to play against Bob's marvelous strategy. But she knows a trick—she insists on playing two games at once! After all, if Bob's strategy wins every game, it shouldn't be any harder for him to beat her in two games at once than it is to beat her in one.



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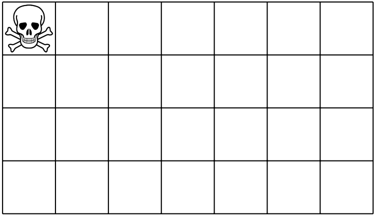
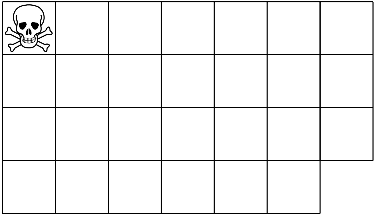


Alice's turn to play

Alice's first move is to nibble the lower right corner in *one* game.



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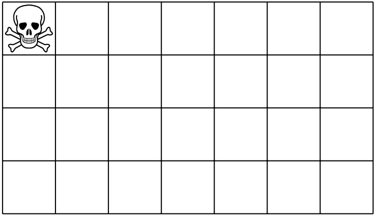
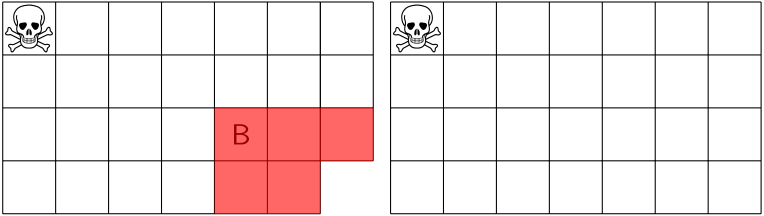


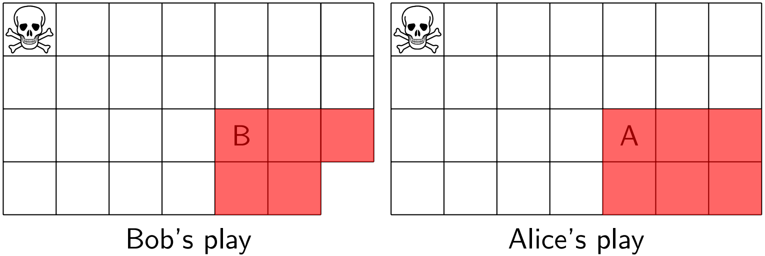
Alice's turn to play

Alice's first move is to nibble the lower right corner in *one* game. She tells Bob to go ahead and make his move in the first game, because she's still thinking about the second game.

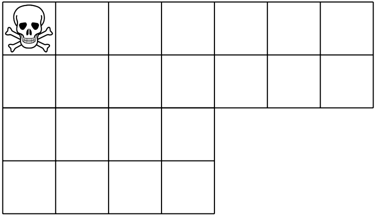


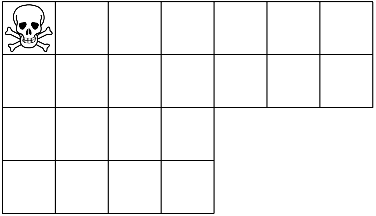
Bob's play Alice's turn to play

Alice's first move is to nibble the lower right corner in *one* game. She tells Bob to go ahead and make his move in the first game, because she's still thinking about the second game. Bob has his master strategy to use, so he ignores the second game and makes his "perfect response" to Alice's move in the first game.

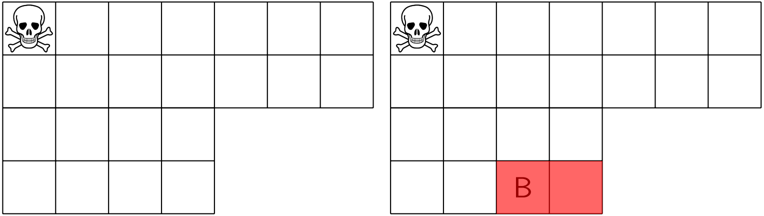


Alice thinks that move looks pretty good, so she makes the *same* move in the *second game*.





Alice thinks that move looks pretty good, so she makes the *same* move in the *second game*. Now she waits for Bob's response in the second game.



Alice's turn to play

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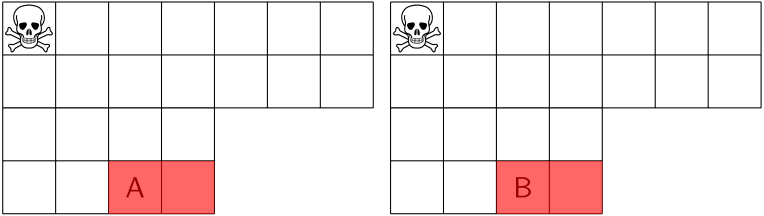
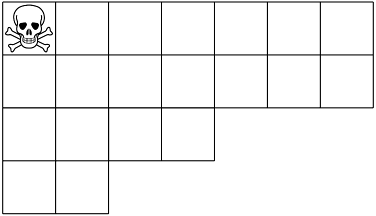
Alice thinks that move looks pretty good, so she makes the *same* move in the *second game*. Now she waits for Bob's response in the second game. And once she gets it,



Alice's play Bob's play

Alice thinks that move looks pretty good, so she makes the *same* move in the *second game*. Now she waits for Bob's response in the second game. And once she gets it,She copies it back to the first game.



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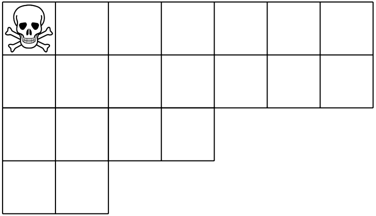
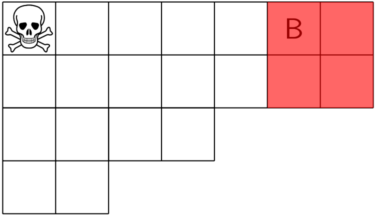


Alice's turn to play

Alice just keeps copying Bob's moves from one game to the other.



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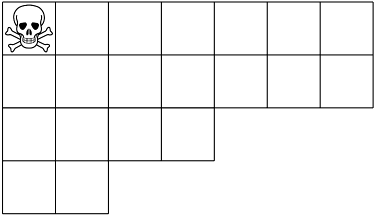

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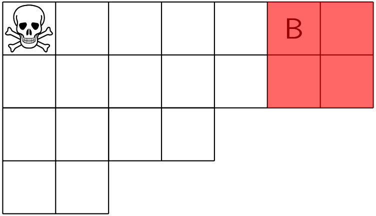

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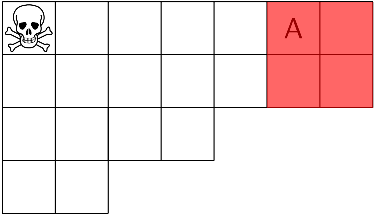

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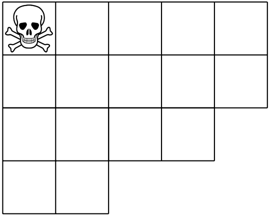

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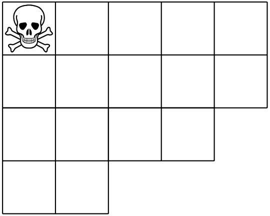

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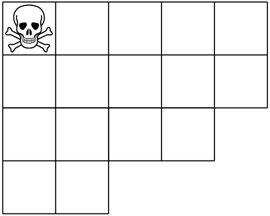

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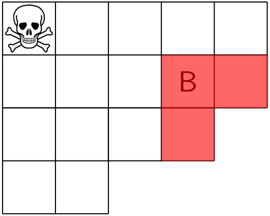

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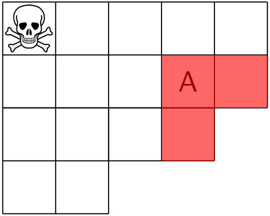

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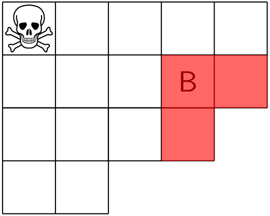

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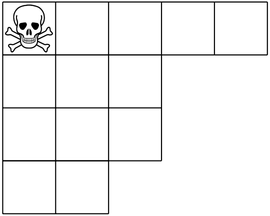

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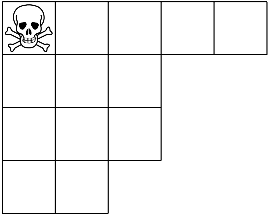

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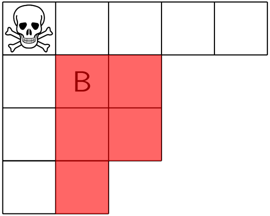

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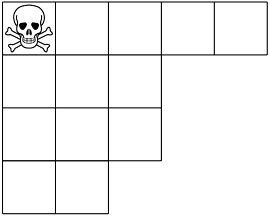
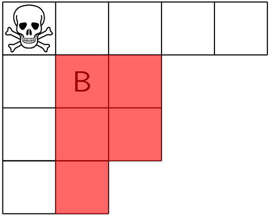


Alice's turn to play

Alice just keeps copying Bob's moves from one game to the other. Alice has tricked Bob into playing against his own strategy.



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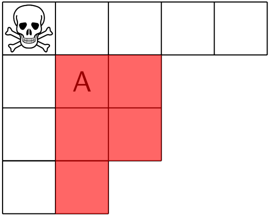

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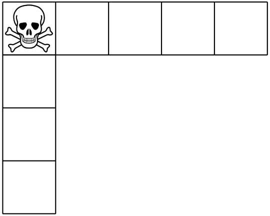

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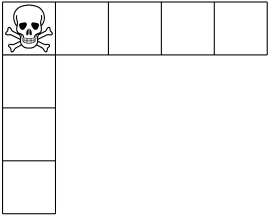

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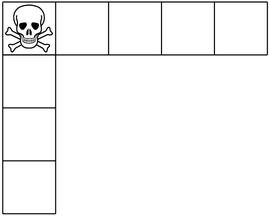

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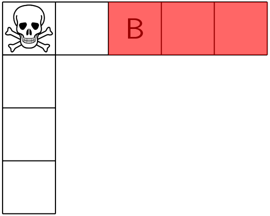

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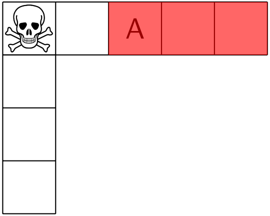

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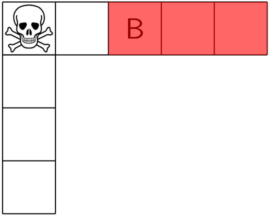

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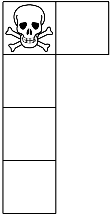

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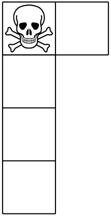

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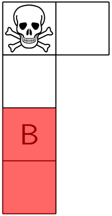

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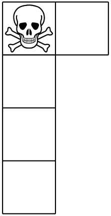

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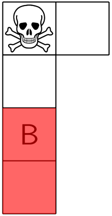

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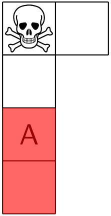

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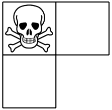

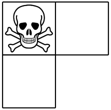

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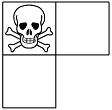

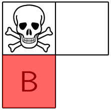

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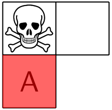

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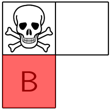

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So Bob's "winning strategy" doesn't always win after all.

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What is it?

We proved that there *is* a winning strategy, but we didn't find out what it was!

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If all we care about is whether or not we should spend our time trying to find a winning strategy for Bob, the difference may not matter. But if we're about to go play in the National Chomp Championships, knowing there is a winning strategy doesn't do us much good if we don't know what that strategy is.

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When they first appeared, non-effective proofs were *controversial*. Some mathematicians argued that non-effective proofs weren't proofs at all.

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But even if a non-effective proof convices us that something is true, sometimes we'd like an effective proof anyway.

Like in the case of Chomp:knowing that there exists some winning strategy in principle doesn't give us any clue how to win the game if we actually play it.

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In order to answer these sorts of questions, we need to pin down formally exactly what it means to be a proof, and what it means for a proof to be effective. This is a problem that involves both mathematics—to make the definition precise, and to use it to prove interesting things—and philosophy, to determine whether the formal notions we come up with actually encompass the informal ideas we started with.

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For instance, it turns out that we could mean several different things by an "effective" proof, and the answers to the questions above depend on which one we mean.

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Let's call this the *computability* interpretation of effectiveness.

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Sadly, there's a catch.

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- The game Chomp is finite, so there are only finitely many possible games, and it is possible to simply write down every possibility.
- These moves can be arranged in a "tree", listing all Alice's moves, then for each of her moves, all of Bob's possible responses, and so on. From this tree, it is possible to read off a winning strategy.

That fits the definition of what a computer can do, but it's not very satisfying in this case—the strategy it gives us is "play out all possible future games and pick a move which guarantees victory in all of them". Maybe we should change what counts as a method for purposes of being an effective proof.

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Without a formal definition, we can be happy when we have insightful strategies, but we can't hope to prove abstract theorems about all possible insightful proofs.

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Fortunately, there is a precise notion of what it means for a computer program to be efficient (actually, there are lots of such notions). Once we fill in which kind of efficiency we mean, we have a notion of effectiveness.

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As a mathematician, I mostly investigate the computability interpretation of effectiveness.

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- Taking a non-effective proof for which a computably effective proof should exist, and determining what the computably effective proof is (in a situation where either we don't care that the result is inefficient—for instance, because it's a theoretical result where there are no efficiency concerns—or where I suspect that the result will happen to be efficient, even though in general it doesn't have to be),

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- Actually, my favorite thing to do is the previous two together: find a theorem which I think should have a computably effective proof, but where the existing techniques don't work, and simultaneously figure out a general method for extracting a computably effective proof while applying it to the particular example,

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- Alternatively, sometimes I examine very complicated effective proofs, and look for non-effective proofs which are easier to understand. This can happen because effective proofs might require lots of calculations, while in a non-effective proof we can sometimes replace very big numbers with "infinity" or very small numbers with 0 (in a mathematically rigorous way) to give a simpler non-effective proof. In addition, sometimes it turns out that the non-effective proof can be generalized to new results more easily than the original proof.

The study of games like Chomp is part of the branch of mathematics known as *combinatorial game theory*.

Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy. *Winning ways for your mathematical plays*. Second. Natick, MA: A K Peters Ltd., 2001, pp. xx+276. ISBN: 1-56881-130-6.

Richard J. Nowakowski, ed. *Games of no chance*. Vol. 29. Mathematical Sciences Research Institute Publications. Papers from the Combinatorial Games Workshop held in Berkeley, CA, July 11–21, 1994. Cambridge: Cambridge University Press, 1996, pp. xii+537. ISBN: 0-521-57411-0.

Chomp itself has been studied in detail, but efficient or insightful winning strategies are known only for a few special cases.

- Eric J. Friedman and Adam Scott Landsberg. “Scaling, renormalization, and universality in combinatorial games: the geometry of Chomp”. In: *Combinatorial optimization and applications*. Vol. 4616. Lecture Notes in Comput. Sci. Berlin: Springer, 2007, pp. 200–207. DOI: 10.1007/978-3-540-73556-4_23. URL: http://dx.doi.org/10.1007/978-3-540-73556-4_23.
- Xinyu Sun. “Improvements on Chomp”. In: *Integers* 2 (2002), G1, 8. ISSN: 1867-0652.
- Doron Zeilberger. “Three-rowed Chomp”. In: *Adv. in Appl. Math.* 26.2 (2001), pp. 168–179. ISSN: 0196-8858. DOI: 10.1006/aama.2000.0714. URL: http://dx.doi.org/10.1006/aama.2000.0714.

Proof theory is a branch of logic focusing on the studying the intrinsic properties of proofs. Computably effective proofs play a central role in the field.

Samuel R. Buss. “An introduction to proof theory”. In: *Handbook of proof theory*. Vol. 137. Stud. Logic Found. Math. Amsterdam: North-Holland, 1998, pp. 1–78. DOI: 10.1016/S0049-237X(98)80016-5. URL: http://dx.doi.org/10.1016/S0049-237X(98)80016-5.

Jean-Yves Girard, Paul Taylor, and Yves Lafont. *Proofs and types*. Vol. 7. Cambridge Tracts in Theoretical Computer Science. Cambridge: Cambridge University Press, 1989, pp. xii+176. ISBN: 0-521-37181-3.

Computational complexity investigates different notions of efficient computation and their relationships.

Erwin Engeler. *Introduction to the theory of computation*. Computer Science and Applied Mathematics. New York: Academic Press, 1973, pp. viii+231.

Christos H. Papadimitriou. *Computational complexity*. Reading, MA: Addison-Wesley Publishing Company, 1994, pp. xvi+523. ISBN: 0-201-53082-1.

The End

Theorem

Every game of Chomp is determined: either the first player has a winning strategy, or the second player has a winning strategy.

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Unfortunately, this is a bit hard to prove, so we'll use a standard math trick: faced with a problem too hard to solve, make it harder.

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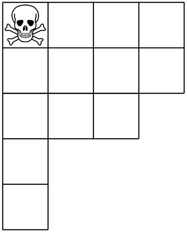
The game of Jagged Chomp is just like the game of Chomp, except the board doesn't have to be rectangular. So Jagged Chomp can be played on boards



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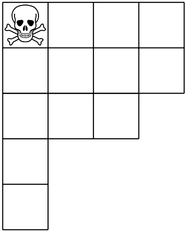

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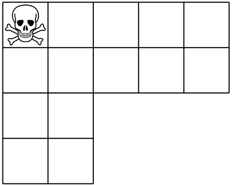

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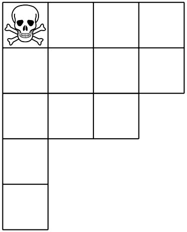

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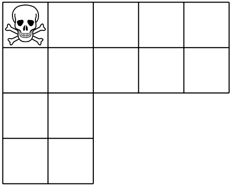

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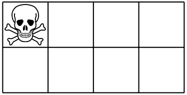

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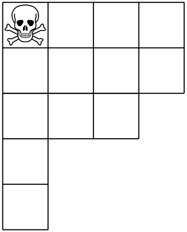

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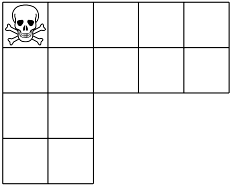

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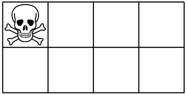

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Because the board *can* be rectangular, it just doesn't have to be.

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There's only one game whose board has only 1 square: the 1x1 game.

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There's only one game whose board has only 1 square: the 1x1 game. The 1x1 game is certainly determined: the second player always wins, just by waiting for the first player to make a move and immediately lose.

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Every game of Jagged Chomp is determined: either the first player has a winning strategy, or the second player has a winning strategy.

There must be some smallest size which is big enough for a board to be undetermined.

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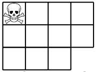
There must be some smallest size which is big enough for a board to be undetermined. Consider some undetermined board of minimal size. So all boards with fewer squares *are* determined.

Theorem

Every game of Jagged Chomp is determined: either the first player has a winning strategy, or the second player has a winning strategy.

For example, suppose this is the board.

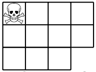


Alice's turn to play

Theorem

Every game of Jagged Chomp is determined: either the first player has a winning strategy, or the second player has a winning strategy.

For example, suppose this is the board. After Alice's move, what's left will be a smaller board.



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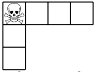
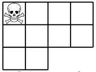


Bob's turn to play

or



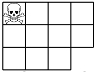
Bob's turn to play

or another option

Theorem

Every game of Jagged Chomp is determined: either the first player has a winning strategy, or the second player has a winning strategy.

For example, suppose this is the board. After Alice's move, what's left will be a smaller board. Alice will be the *second player* to move on these smaller boards.



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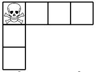

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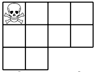

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For example, suppose this is the board. After Alice's move, what's left will be a smaller board. Alice will be the *second player* to move on these smaller boards.

Remember that we started with the *smallest* undetermined board.

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Every game of Jagged Chomp is determined: either the first player has a winning strategy, or the second player has a winning strategy.

Remember that we started with the *smallest* undetermined board. All the smaller boards *are* determined.

Theorem

Every game of Jagged Chomp is determined: either the first player has a winning strategy, or the second player has a winning strategy.

Remember that we started with the *smallest* undetermined board. All the smaller boards *are* determined. That means on each of these smaller boards, there is either a winning strategy for the first player or a winning strategy for the second player.

Theorem

Every game of Jagged Chomp is determined: either the first player has a winning strategy, or the second player has a winning strategy.

If the second player has a winning strategy on *one* of these smaller boards, Alice has a winning strategy.

Theorem

Every game of Jagged Chomp is determined: either the first player has a winning strategy, or the second player has a winning strategy.

If the second player has a winning strategy on *one* of these smaller boards, Alice has a winning strategy. She first makes the choice that leads to a board where the second player has a winning strategy, and then once there, uses that strategy to win.

Theorem

Every game of Jagged Chomp is determined: either the first player has a winning strategy, or the second player has a winning strategy.

If the second player *never* has a winning strategy on these smaller boards, the first player must have a winning strategy on all the smaller boards.

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Every game of Jagged Chomp is determined: either the first player has a winning strategy, or the second player has a winning strategy.

If the second player *never* has a winning strategy on these smaller boards, the first player must have a winning strategy on all the smaller boards. Then Bob *does* have a winning strategy on the original board

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Every game of Jagged Chomp is determined: either the first player has a winning strategy, or the second player has a winning strategy.

If the second player *never* has a winning strategy on these smaller boards, the first player must have a winning strategy on all the smaller boards. Then Bob *does* have a winning strategy on the original board: wait for Alice to pick a board, then use the winning strategy for the board Alice picks.

Theorem

Every game of Jagged Chomp is determined: either the first player has a winning strategy, or the second player has a winning strategy.

This is a contradiction: we started with a board that wasn't supposed to have a winning strategy, and showed that it had one anyway.

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This is a contradiction: we started with a board that wasn't supposed to have a winning strategy, and showed that it had one anyway. So *every* game of Jagged Chomp, and in particular every game of Chomp, is determined.