What is Tame Hypergraph Regularity About?

Henry Towsner

Introduction

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If you aren't sure how to do that, you can probably find a command named something like "Full Screen" or "Presentation" in the "View" menu.

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"Tame hypergraph regularity" was inspired by the discovery of "tame graph regularity", so I should explain that first.

Tame Graph Regularity: The idea

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- A pair is good if their sum is even.
- A pair is good if x is smaller than y.
- In advance, we flip a coin for each pair, and make a big table of the results as a reference. A pair is good if the coin we flipped for that pair came up heads.

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It's important that the the number of questions is much smaller than the sizes of X and Y, since if we had enough to pin down x and y exactly, the game wouldn't be very interesting.

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The question we're interested in is:

> *Which choices of the good pairs let us do well at this game? That is, when can we, with a small number of questions, figure out if a pair is good most of the time?*

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The answers to those questions are enough for us to figure out if x, y is a good pair. If x and y are both even or both odd, it's a good pair. If one is even and one is not, it's not a good pair.

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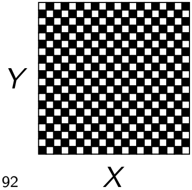
So in this case, we each ask one question and then together we know the answer with complete certainty.

Here's a graphical representation of what's going on.



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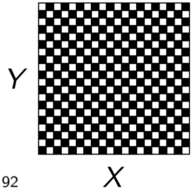

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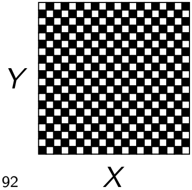

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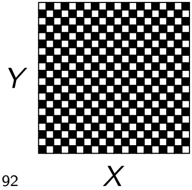

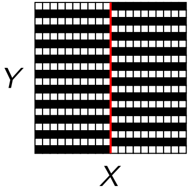

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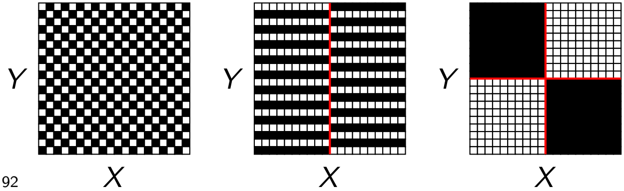
Meanwhile, I ask whether y is even, and again we reorganize the grid based on the answer: we put the even y's on top and the odd ones on the bottom and separate them with a red line, because we know which side of the red line we're on.



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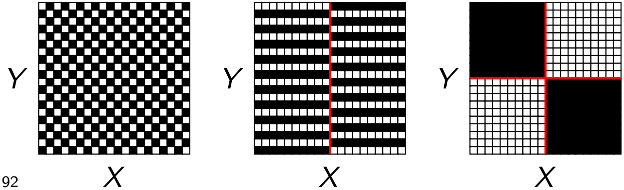
Meanwhile, I ask whether y is even, and again we reorganize the grid based on the answer: we put the even y's on top and the odd ones on the bottom and separate them with a red line, because we know which side of the red line we're on. The fact that we can win this game is reflected in the fact that (after rearranging the rows and columns to reflect our questions), the red lines divide the possible pairs into rectangles each of which is either all black (all good pairs) or all white (no good pairs).



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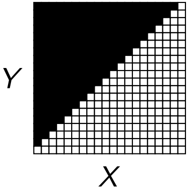
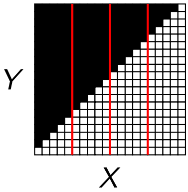
If we're unlucky, though, x and y belong to the same quarter, and then we're not sure if the pair is good. But that only happens a quarter of the time. So in this case, *most* of the time we know the answer after asking a few questions.

Here's the picture for $x < y$.



Here's the picture for $x < y$.

The answers to your two questions divide the X axis into four quarters. (As it happens, the columns are already in the right positions this time.)



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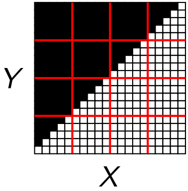

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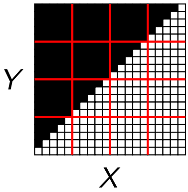

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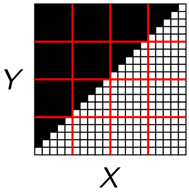
The red lines, which separate the axes based on the information you have after asking your questions, divide *most* of the picture into rectangles which are either all black or all white. But there are still the areas along the diagonal which are mixed.



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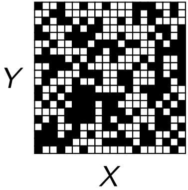
About half the pairs are good, and after asking a bunch of questions, we'll still probably think there's about a fifty percent chance that we're dealing with a good pair.

The random table might look something like this.



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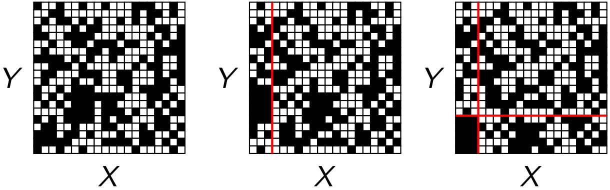

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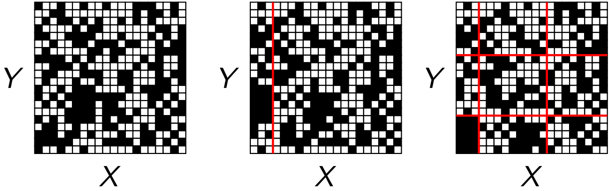
But if we ask questions that meaningfully divide up the region, we're left with recangles that are still a jumbled mix of good and bad pairs.



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Let's talk about precisely how we define these "paradigms" and what the theorems characterizing them say.

Tame Graph Regularity: Approximable rules

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Instead of talking about "good pairs", we'll use the standard terminology and call them *edges*. So formally, a graph is three things, (X, Y, E) where X and Y are sets and $E \subseteq X \times Y$ is the set of edges.

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- $\mathcal{R}_{<}$: X and Y are the set of numbers up to n for some n, and the edges are where $x < y$,
- $\mathcal{R}_{\mathrm{rand}}$: For each n, we will generate a random graph where X and Y are both the set of numbers up to n, and we determine which pairs are edges by flipping a separate coin for each pair.

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A rule $\mathcal{R}$ is *approximable* if for every $\epsilon > 0$, there is a number N so that for any graph (X, Y, E) in $\mathcal{R}$, if $x \in X$ and $y \in Y$ are chosen randomly and we ask N questions about x and y separately, the probability that we correctly guess whether the pair is an edge is at least $1 - \epsilon$.

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A rule $\mathcal{R}$ is *approximable* if for every $\epsilon > 0$, there is a number N so that for any graph (X, Y, E) in $\mathcal{R}$, if $x \in X$ and $y \in Y$ are chosen randomly and we ask N questions about x and y separately, the probability that we correctly guess whether the pair is an edge is at least $1 - \epsilon$.

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$\mathcal{R}$ is approximable exactly when this game is worth playing anyway: no matter how stingy the payout is, and no matter how hard the house tries to pick a rule from $\mathcal{R}$ that makes our life difficult, we can pick a number N which is big enough so that we'll make money on average.

The name “approximable” isn’t standard. The notion has appeared in various papers under various names, most of which are kind of ad hoc, and no widely accepted name has emerged, so “approximable” will do for us.

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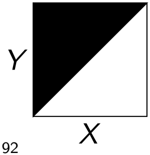

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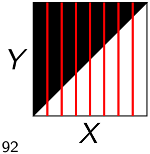

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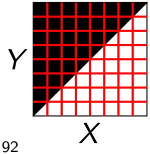

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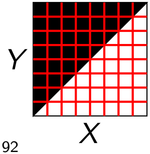

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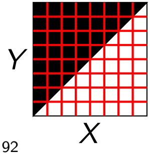
$7/8$ of the time, x and y belong to different octiles, and we know for sure if the pair is an edge. The remaining $1/8$ of the time, we have to risk a $50/50$ guess. But in total, we get the right answer $15/16$ of the time—about 93%, which is good enough to make money on average even if we only win \$1.10 for every correct guess.



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So approximability is a property that distinguishes rules "like $x < y$" from "like a random graph".

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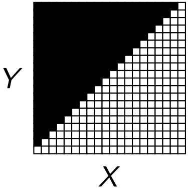
That is, if we pick two rows and two columns in



the subgrid we get never looks like (even if we reorder the rows and columns to try to make them match).

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The name "VC dimension" is the standard one. (VC stands for "Vapnik–Chernovenkis", the people who first discovered it.) It's a useful notion that's been rediscovered several times in several fields.

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Theorem (Lovász-Szegedy)

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Finite VC dimension is such a well-studied notion, and many other equivalences are known, so approximability is one more way that finite VC dimension distinguishes simple sets of graphs from complicated ones.

A small confession: I'm glossing over a technicality here.

To actually get the equivalence, we have to clarify that the house is allowed to pick the probability distribution on X and Y (and tell us at the same time the house tells us which graph (X, Y, E) has been picked)

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I'll continue to ignore this detail below.

Tame Graph Regularity: Unary error

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To really illustrate the interesting property that makes $\mathcal{R}_{\mathrm{even}}$ nicer than $\mathcal{R}_{<}$, it will be helpful to have a new example that's going to be nice in the same way as $\mathcal{R}_{\mathrm{even}}$, but better illustrates what sort of behavior is allowed.

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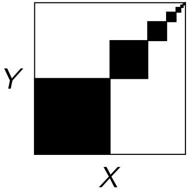

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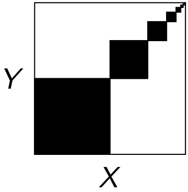

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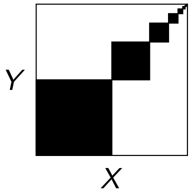

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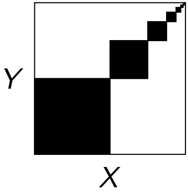

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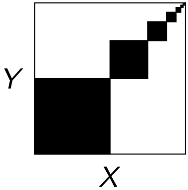

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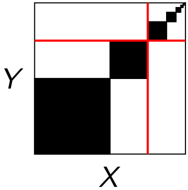

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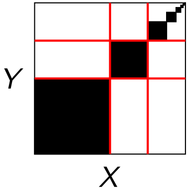

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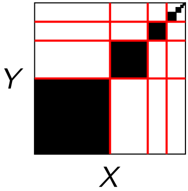

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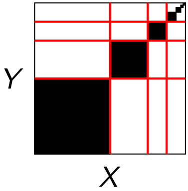

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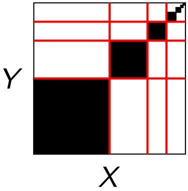
Most of the time, we know whether x, y is a good pair, and a small fraction of the time—when all four of our questions got answered "no"—we don't. Or we could notice that  is omitted and invoke the theorem characterizing approximability.



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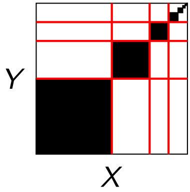

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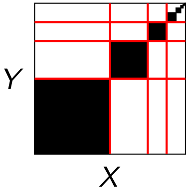
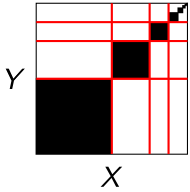


Like with $\mathcal{R}_<$, a small fraction of the time we'll get this wrong. But we can tell *before* we compare answers whether that's in danger of happening.

The only times we get the answer wrong are when we end up in the upper right corner. So if you give up every time your answers tell you we're in the far right segment, and I give up every time my answers tell me we're in the topmost segment, then we only give up a small fraction of the time, and whenever we haven't given up, we can be confident we have the right answer.



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We say $\mathcal{R}$ *admits unary error* if we can win this harder game. The word "unary" here refers to the fact that we have to decide whether to give up individually, based on one person's information.

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In general, though, the definition allows either of us to give up unilaterally (as long as we don't do it often).

Another confession: in the full definition, we don't actually have to get it exactly perfect even when we don't give up, we just have to be pretty confident we know what the answer is. This only comes up in more complicated examples, so we'll continue glossing over this detail.

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Definition

The *half-graph of size k* is the $k \times k$ graph whose edges are where $x < y$. A graph is *k-stable* if it omits the half-graph of size k. $\mathcal{R}$ is stable if there is some k so that every graph in $\mathcal{R}$ is k-stable.

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To be approximable, a graph must omit *some* small subgraph. To admit unary error, it turns out it has to omit a *particular* small subgraph.

Definition

The *half-graph of size k* is the $k \times k$ graph whose edges are where $x < y$. A graph is *k-stable* if it omits the half-graph of size k. $\mathcal{R}$ is stable if there is some k so that every graph in $\mathcal{R}$ is k-stable.

As we already noticed, $\mathcal{R}_{\text{digit}}$ is 2-stable: it omits the half-graph of size 2, .

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The second theorem about tame graph regularity was:

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An Aside: What is regularity?

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"Regularity" is a reference to Szemerédi's Regularity Lemma. Roughly speaking, in the context of the game we've been describing, it just says: "eventually you should stop asking questions and guess".

Szemerédi's Regularity Lemma is what we can say about any graph, not just approximable ones. What could we hope to do with a graph where we definitely can't win this gambling game?

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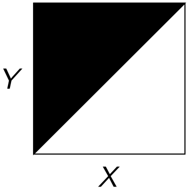

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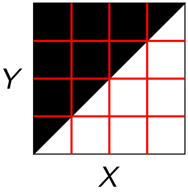
For instance, on a graph from $\mathcal{R}_<$, initially all we can say is that half the pairs are edges. After asking a few questions, we're probably in a situation where we can say either all or none of the pairs are edges, and there's a small chance the pair on the diagonal and we still only know that there's a fifty percent chance it's an edge.



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There's a term for what happened with $\mathcal{R}_{\mathrm{rand}}$: a graph is *quasirandom* if asking a few questions doesn't change the chance that we're looking at an edge. We started out thinking there was probability p that the pair is an edge, and after asking a few questions, we still (probably) think the chance is pretty close to p that the pair is an edge.

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When we generate a graph by flipping coins, the result is quasirandom. Technically the graph with all edges and the graph with no edges are quasirandom, too: if you're initially certain there's an edge, nothing you learn by asking a few questions will change that certainty.

So Szemerédi's Regularity Lemma says that if we ask enough questions about *any* graph—not necessarily an approximable one—we probably end up in a quasirandom rectangle (at which point we might as well stop asking questions).

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Szemerédi's Regularity Lemma became an important tool in parts of graph theory, but it had some limitations: the number of questions you need to ask is huge (a tower of exponents, $2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$, that gets taller as you ask for that "probably end up" to get closer to probability 1), and you only know that you're *probably* in a quasirandom rectangle.

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Tame graph regularity was motivated by showing that for "tame graphs"—graphs which fell on the nice side of a dividing line—these limitations went away, and that for "wild graphs"—graphs on the complex side of a dividing line—these limitations were unavoidable.

Tame Hypergraph Regularity: A new game

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We don't know as much about what the dividing lines are for sets of triples. So the goal in tame hypergraph regularity is to find them: we'll figure out what the right analogs of things like approximability and admiting unary error are, and then use those to help identify the dividing lines.

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- $\mathcal{R}^3_{sum}$: x, y, z is an edge when $x + y + z < n$. (This is actually pretty analogous to our $x < y$ example—if you flip the y axis, $x < y$ is a lot like $x + y < n$.)

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- $\mathcal{R}^3_{rand}$: We flip a coin for each triple and record the result in a big table.
- $\mathcal{R}^3_{rand\ pair}$ We flip a coin for each *pair* and record the result in a big table. x, y, z is an edge if an odd number of the pairs (x, y), (x, z), and (y, z) came up heads.

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But it would still be a meaningful game if we were each allowed to ask questions about one of the pairs: you could ask about x and y together, I could ask about x and z together, and our friend could ask about y and z together. It's still meaningful because none of us can ask about all three at once, so no one can just ask "is the triple an edge?"

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Within each of those versions, we have further variations based on what kind of error they admit: that is, whether we can identify the tricky cases where we aren't sure about the answer with less information than we need to actually determine the answer.

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- There are two notions of "approximable", one for each version of the game, and we understand them pretty well.
- There are notions of "admitting unary error" and "admitting binary error". One we sort-of understand, but it's more complicated that we expected, and the other we really don't understand yet.
- There's a totally new notion that mixes admitting error coming from unary questions with asking binary questions.

The picture so far looks like this:

Before we investigate those, let's get our terminology right.

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Definition

A 3-*graph* consists of three sets, X, Y, and Z, and a set E of triples. We call E the *edges* of the 3-graph.

Technically this might be called a "tripartite 3-graph". Sometimes these are also called *hypergraphs* or 3-*regular hypergraphs*. What we're calling edges might be called *hyperedges* or 3-*edges*.

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Definition

A *rule for triples* is a set of 3-graphs.

We'll always name rules for triples with a superscript, like $\mathcal{R}^3$, so we don't confuse them with rules for pairs.

Tame Hypergraph Regularity: Approximable rules

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Definition

A rule for triples, $\mathcal{R}$, is *unary approximable* if for every $\epsilon > 0$, there is a number N so that for any 3-graph (X, Y, Z, E) in $\mathcal{R}$, if $x \in X$, $y \in Y$, $z \in Z$ are chosen randomly and we ask N questions about each of x, y, and z separately, we can guess whether the triple is an edge and get it right at least $1 - \epsilon$ of the time.

Definition

A rule for triples $\mathcal{R}$ is *unary approximable* if for every $\epsilon > 0$, there is a number N so that for any 3-graph (X, Y, Z, E) in $\mathcal{R}$ and any weights on X, Y, Z, if $x \in X$, $y \in Y$, $z \in Z$ are chosen randomly according to the weights and we ask N questions about each of x, y, and z separately, we can guess whether the triple is an edge and get it right at least $1 - \epsilon$ of the time.

That is, a rule is unary approximable if, when you, me, and a friend team up and we each ask questions about one value, we're usually able to guess if the triple is an edge by combining the information we learned.

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A small fraction of the time, we'll be unsure: maybe you and I learn that $n/4 \leq x, y < 3n/8$ while our friend learns that $3n/8 \leq z < n/2$. Then all we know is that $x + y + z$ is between $7n/8$ and $5n/4$, so we know whether $x + y + z < n$.

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But the times we're unsure are a small fraction, so $\mathcal{R}^3_{sum}$ is unary approximable.

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This rule is also unary approximable, for basically the same reason: each player pins their value to a small interval, and most of the time that's enough to be certain of the comparison.

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If our rule is $\mathcal{R}^3_{rand\ pair}$, so we flip a coin for each pair and decide a triple is an edge when an odd number of the pairs got heads, it's still not unary approximable: for instance, nothing we ask about x and y helps much for figuring out if our coin came up heads for the pair x, y.

Here's another example, $\mathcal{R}^3_{fun}$. We'll let X and Y be the numbers up to n, but Z will be the set of *functions* from X to Y, and x, y, z is an edge when $z(x) < y$.

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This is also not unary approximable. It's a bit harder to see why, but a good start is trying to come up with questions our friend could be asking about z, and noticing that unless z happens to be a really nice function, you can't learn much about $z(x)$ by asking about x and about z separately.

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When E is a set of triples, for any $u \in X \cup Y \cup Z$, the *slice* corresponding to u, E_u, is the set of pairs (v, w) so that $(u, v, w) \in E$.

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That is, we fix any one of the values, and ask which pairs give us an edge when taken together with our fixed value. I'm using the letters u, v, w to emphasize that this is symmetric: we can fix an X value and look at pairs from $Y \times Z$, or fix a Y value and look at pairs from $X \times Z$, or fix a Z value and look at pairs from $Y \times Z$.

Definition

When $\mathcal{R}^3$ is a rule for triples, $\mathcal{R}^3$ has *finite slicewise VC dimension* if there is an n so that, for every (X, Y, Z, E) in $\mathcal{R}^3$ and every $u \in X \cup Y \cup Z$, E_u omits some $n \times n$ subgraph.

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Another way to say this is that a rule for triples gives us a rule for pairs—take *every* slice from *every* 3-graph in $\mathcal{R}^3$.

Definition

When $\mathcal{R}^3$ is a rule for triples, $\mathcal{R}^3$ has *finite slicewise VC dimension* if there is an n so that, for every (X, Y, Z, E) in $\mathcal{R}^3$ and every $u \in X \cup Y \cup Z$, E_u omits some $n \times n$ subgraph.

Another way to say this is that a rule for triples gives us a rule for pairs—take *every* slice from *every* 3-graph in $\mathcal{R}^3$. Saying $\mathcal{R}^3$ has finite slicewise VC dimension us exactly saying that this rule consisting of all the slices has finite VC dimension.

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On the other hand, the example $\mathcal{R}^3_{fun}$, where Z is the functions from X to Y and the edges are where $z(x) < y$, does not have finite slicewise VC dimension.

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We'll let the n values in X, 1 through n, be our columns. Then for each row of our graph, choose a z which gives exactly that row: to make the j-th row, choose z_j so that $z_j(i) = 1$ when (i,j) is an edge and $z_j(i) = y$ when (i,j) is not an edge.

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Then $1, \dots, n$ and $z_1, \dots, z_n$ are a copy of the graph we wanted. Since we can do that for any graph, E_y contains every $n \times n$ graph.

As those examples suggest, slicewise VC dimension is the notion we need to characterize unary approximations.

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Theorem

A rule for triples is unary approximable if and only if it has finite slicewise VC dimension.

What about the other version of the game, where we get to ask questions about two values at once?

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Definition

A rule for triples, $\mathcal{R}$, is *binary approximable* if for every $\epsilon > 0$, there is a number N so that for any 3-graph (X, Y, Z, E) in $\mathcal{R}$, if $x \in X$, $y \in Y$, $z \in Z$ are chosen randomly and we ask N questions about each of the pairs (x, y), (x, z), and (y, z) separately, we can guess whether the triple is an edge and get it right at least $1 - \epsilon$ of the time.

It’s easier for us to win the binary game than the unary game, It's easier for us to win the binary game than the unary game, because when you get to ask questions about the pair (x, y), you could choose to only ask about x, and similarly for the other two players.

It's easier for us to win the binary game than the unary game, because when you get to ask questions about the pair (x, y), you could choose to only ask about x, and similarly for the other two players. The binary game just gives us more options, and therefore more rules will be binary approximable.

For instance, recall $\mathcal{R}^3_{rand\ pair}$, the rule where we make a random table by flipping a coin for each pair, and x, y, z is an edge when an odd number of the pairs got heads.

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That isn't unary approximable, but it's easy to give a binary approximation, since we can each just outright ask whether the coin flipped for our pair came up heads.

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One of the other players asks questions about y to pin y down to a small interval as well. Then we compare our answers and most of the time the intervals don't overlap, so we know for sure whether $z(x) < y$.

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This rule isn't binary approximable: no matter how many questions about pairs we ask, it doesn't help us figure out if the coin came up heads for the particular triple we're dealing with.

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Definition
A rule for triples $\mathcal{R}$ has *finite VC_2 dimension* if there is some n so that every 3-graph in $\mathcal{R}$ omits some $n \times n \times n$ sub-3-graph.

Consider the rule $\mathcal{R}^3_{rand\ pair}$ where we flip a coin for each pair, and x, y, z is an edge if an odd number of the coins for pairs came up heads.

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So $c_{x,y} + c_{x,z} + c_{y,z}$ must be odd while $c_{x',y'} + c_{x',z'} + c_{y',z'}$ must be even.

If we add up the seven combinations we know are odd, we get

$$\begin{aligned}&(c_{x,y}+c_{x,z}+c_{y,z})+(c_{x',y}+c_{x',z}+c_{y,z})+(c_{x,y'}+c_{x,z}+c_{y',z})\\+&(c_{x,y}+c_{x,z'}+c_{y,z'})+(c_{x',y'}+c_{x',z}+c_{y',z})+(c_{x,y'}+c_{x,z'}+c_{y',z'})\\+&(c_{x',y}+c_{x',z'}+c_{y,z'})\end{aligned}$$

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which must also be odd. But this exactly tells us that an odd number of the pairs (x',y'), (x',z'), and (y',z') must have come up heads: if 7 of the 8 edges are present, the eighth must be as well.

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Theorem (Chernikov-Towsner)

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Theorem (Chernikov-Towsner)

A rule for triples is binary approximable if and only if it has finite VC_2 dimension.

This wasn't the first result about tame hypergraph regularity, but it was the first result that came from the perspective here: we had conjectured that *something* should be equivalent to binary approximable, guessed that it would be finite VC_2 dimension, and then looked for the proof.

Tame Hypergraph Regularity: Admitting unary error

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We will say that $\mathcal{R}^3$ *admits unary error* if, for any $\epsilon > 0$, there is an N so that after we have each asked N questions about one of the variables, we are each allowed to give up ϵ of the time, and any time none of us gives up, we are able to get the right answer by combining our information.

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Once again, the exact definition gives us slightly more leeway to get the answer wrong, but we'll ignore this extra complication.

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Definition
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One implication works, using basically the same arguments that worked for pairs.

Theorem

If $\mathcal{R}^3$ admits unary error then $\mathcal{R}^3$ is slicewise stable.

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The slices E_u are all stable—roughly speaking, what happens to a triple u, v, w is determined by the first coordinate at which v or w differs from u, which is too restrictive to be unstable.

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But this example doesn't admit unary error. The idea is that the natural questions for us to ask are about the initial digits of our value: does it start with a 0, is the second digit a 1, and so on. But those questions will always leave boxes where x, y, z all share the same initial digits, and no one looking at just one of x, y, or z on its own can tell if that's going to happen.

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It's more work, of course, to show that there aren't some cleverer questions which do better.

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Another way to get graphs from 3-graphs is to view triples as being pairs where one element happens to itself be a pair: instead of triples from $X \times Y \times Z$, we have pairs from X and $Y \times Z$, or from Y and $X \times Z$, or from Z and $X \times Y$.

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Definition

A rule for triples, $\mathcal{R}^3$, is *partitionwise stable* if there is a k so that, for every (X, Y, Z, E) in $\mathcal{R}^3$, all three graphs $(X, Y \times Z, E)$, $(Y, X \times Z, E)$, and $(Z, X \times Y, E)$ are k-stable.

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Partitionwise stability is a stronger property: any partitionwise stable rule is slicewise stable, but the example above was slicewise stable without being partitionwise stable.

Theorem
$\mathcal{R}^3$ is partitionwise stable exactly when $\mathcal{R}^3$ admits unary error.

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It turns out that the subtleties we've been ignoring create some complications here: when we get into the details, there are two definitions which are equivalent for graphs, but lead to distinct generalizations for 3-graphs. One is equivalent to partitionwise stability while the other ends up being strictly between partitionwise and slicewise stability.

Tame Hypergraph Regularity: Admitting binary error

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We can guess what admitting binary error should mean: $\mathcal{R}$ *admits binary error* if, for any $\epsilon > 0$, there is an N so that after we have each asked N questions about two of the variables, we are each allowed to give up ϵ of the time, and any time none of us gives up, we are able to get the right answer by combining our information.

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This is exactly like our definition of admitting unary error, except that the players ask questions about two variables at a time.

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One way to see the resemblence is to think about the pictures. The half-graph looks like a square divided along the diagonal, while a drawing a of $\mathcal{R}^3_{sum}$ would look like a cube divided along the diagonal.

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One way to see the resemblence is to think about the pictures. The half-graph looks like a square divided along the diagonal, while a drawing a of $\mathcal{R}^3_{sum}$ would look like a cube divided along the diagonal.

Indeed, $\mathcal{R}^3_{sum}$ does not admit binary error.

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x, y, z is an edge if the first 1 appears before the first 2.

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Some intuition starts with the observation that if you know x and y, even in full detail, there's always a z which gives us an edge and a z which gives us a non-edge: knowing two coordinates, even in their entirety, doesn't help predict what's going to happen.

More generally, instead of having sequences of digits in $\{0, 1, 2\}$, we could have digits $\{0, 1, 2, \ldots, k\}$ for some $k \geq 2$.

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Furthermore, $\mathcal{R}^3_{sum}$ and $\mathcal{R}^3_{GS_3}$ are *distinct* obstacles to admitting binary error: neither is contained in the other.

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Theorem

If $\mathcal{R}^3$ is contained in both $\mathcal{R}^3_{sum}$ and $\mathcal{R}^3_{GS_3}$ then $\mathcal{R}^3$ admits binary error.

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There's some hope for a different kind of characterization, perhaps with "partial 3-graphs" (where some triples are not committed to being either edges or non-edges). There is a characterization involving some fairly complicated "trees".

Tame Hypergraph Regularity: Admitting linear error

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There's a notion that encompasses all these examples, but is still more restrictive than being merely binary approximable.

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All our variations amount to tweaking two parameters of this general game: which stages we ask questions at, and when we decide whether to give up.

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Our new notion comes from playing both stages, and requiring that we decide whether to give up after sharing the unary information but before asking binary question.

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If we can win this game, we can say the 3-graph is *admits linear error*.

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But for our purposes, it’s important to distinguish *unary* error, which comes from *one player’s* unary information, and *linear* error, which lets us use *everyone’s* unary information.

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Theorem (Terry–Wolf)

$\mathcal{R}^3$ admits linear error if and only if there is some n so that $\mathcal{R}^3$ does not contain the $\mathcal{R}^3_{fun}$ example of size n.

Putting these together, these properties look like this: