1 Reading

Read the proof of the Cantor-Schroeder-Bernstein Theorem and start trying to understand it.

2 Response

Do you understand the statement of the Cantor-Schroeder-Bernstein Theorem? Does each of the individual steps in the proof follow from what comes before? Do the steps successfully prove what the theorem claims?

Theorem 2.1 (Cantor-Schroeder-Bernstein). *Let A and B be functions and suppose that $f : A \to B$ and $g : B \to A$ are one-to-one functions. Then A and B have the same cardinality (i.e. there is a bijection between A and B).*

Proof. We define a sequence of sets, interleaved between subsets of A and subsets of B, by

$$B_1 = B \setminus f[A], A_n = g[B_n], B_{n+1} = f[A_n].$$

Observe that if $a \in \bigcup_{n\in\mathbb{N}} A_n$ then, by definition, there is a $b \in B$ with $g(b) = a$. Since g is one-to-one, this b is uniquely defined, so we will write $g^{-1}(a)$ for it.
We define $h : A \to B$ as follows:

$$h(a) = \begin{cases} g^{-1}(a) & \text{if } a \in \bigcup_{n\in\mathbb{N}} A_n \\ f(a) & \text{otherwise} \end{cases}$$

We claim that h is a bijection. First, we check that h is one-to-one. Suppose $a_1, a_2 \in A$ and $a_1 \neq a_2$. We have to consider cases depending on whether or not a_1 and a_2 belong to $\bigcup_{n\in\mathbb{N}} A_n$.
If a_1, a_2 are both in $\bigcup_{n\in\mathbb{N}} A_n$ then $h(a_1) = g^{-1}(a_1)$ and $h(a_2) = g^{-1}(a_2)$, and by the definition of the inverse (and the fact that g is a function), $h(a_1) \neq h(a_2)$.
If neither is in $\bigcup_{n\in\mathbb{N}} A_n$ then $h(a_1) = f(a_1) \neq f(a_2) = h(a_2)$ since f is one-to-one.
If one is in $\bigcup_{n\in\mathbb{N}} A_n$ and one is not, assume without loss of generality that $a_1 \in \bigcup_{n\in\mathbb{N}} A_n$ and $a_2 \notin \bigcup_{n\in\mathbb{N}} A_n$. Then $h(a_1) = g^{-1}(a_1) \in B_n$ for some n. If $n > 1$ then $g^{-1}(a_1) = f(a')$ for some $a' \in A_{n-1}$. Since $a_2 \notin \bigcup_{n\in\mathbb{N}} A_n$, $a' \neq a_2$, and since f is one-to-one, $h(a_2) = f(a_2) \neq f(a') = h(a_1)$. If $n = 1$ then $g^{-1}(a_1) \notin f[A]$, so $h(a_2) = f(a_2) \neq g^{-1}(a_1) = h(a_1)$.
Therefore h is one-to-one.
To see that h is onto, take any $b \in B$. If $b \in \bigcup_{n\in\mathbb{N}} B_n$ then $g(b) \in \bigcup_{n\in\mathbb{N}} A_n$, so $h(g(b)) = g^{-1}(g(b)) = b$. If $b \notin \bigcup_{n\in\mathbb{N}} B_n$ then $b \notin B_1$, so $b \in f[A]$, so there is an $a \in A$ with $f(a) = b$, and since f is one-to-one, there is exactly one such a. If $a \in \bigcup_{n\in\mathbb{N}} A_n$ then $a \in A_n$ for some n, so $b \in B_{n+1} \subseteq \bigcup_{n\in\mathbb{N}} B_n$, which is a contradiction. So $a \notin \bigcup_{n\in\mathbb{N}} A_n$, so $h(a) = f(a) = b$. □