1.1.28b

The strategy is for the second player to always end on a multiple of 4: if the first player plays 1, 2, or 3 pennies, the second player plays 3, 2, or 1 in response.

1.2.8

The fifth graph has vertices of degree 4, while the first four have all vertices of degree 4, so the fifth graph is not isomorphic to any of the others.
The fourth graph has an odd cycle (25-26-30-29-32-25) while the others are bipartite, so the fourth graph is not isomorphic to any of the others.
The first and second are isomorphic: 1-9, 2-16, 3-12, 4-13, 5-15, 6-10, 7-14, 8-11
The first and third are isomorphic (so the second and third are as well): 1-17, 2-18, 3-21, 4-22, 5-24, 6-19, 7-20, 8-21

1.2.14

a: Possible b: Not possible (odd number of vertices of odd degree) c: Possible

1.3.9

Since every vertex has degree p, p must divide $\sum_{v \in V} deg(v) = 2e$. Since p is odd, p does not divide 2, so p divides e.

1.3.16

The first one is bipartite: one piece is $abcd$ and the other is $efgh$
The second is not bipartite: $a - e - m - k - d$ is an odd cycle.

1.4.3a-g

a: non-planar: a $K_{3,3}$ configuration with a, f, e as one part and d, c, b as the other
b: non-planar: a $K_{3,3}$ configuration with c, i, f as one part and d, h, a as the other
c: non-planar: a $K_{3,3}$ configuration with b, d, h as one part and g, i, e as the other
d: non-planar: a $K_{3,3}$ configuration with a, d, g on one side and b, f, e on the other
e: planar
f: non-planar: a $K_{3,3}$ configuration with h, f, d on one side and e, g, b on the other
g: planar
h: non-planar: a $K_{3,3}$ configuration with a, f, d on one side and b, c, e on the other
i: non-planar: a $K_{3,3}$ configuration with a, e, c on one side and b, f, g on the other
j: non-planar: a $K_{3,3}$ configuration with a, e, c on one side and b, d, h on the other
k: non-planar: a $K_{3,3}$ configuration with a, c, e on one side and b, d, f on the other
l: Non-planar (compressing af, bg, ei, dj, and hc gives a K_5 minor)

1.4.6

The following cases are planar:

- $K_{n,1,1}$, $K_{1,n,1}$, or $K_{1,1,n}$ (place n vertices down the middle in a row and the other two on either side); also their subgraphs $K_{n,1,0}$, $K_{n,0,1}$, and so on.
- $K_{r,s,t}$ where $r \leq 2$, $s \leq 2$, and $t \leq 2$ (for $K_{2,2,2}$, the first two sets of vertices form a square, and we can put one of the remaining vertices inside and one outside, both connected to all four corners of the square; the remaining cases are subgraphs, so also planar).

To see that these are the only cases, suppose that $s + t \geq 3$ (for instance $K_{n,2,1}$); then the subgraph containing only edges between the first subset and the other two contains $K_{3,3}$ as a subgraph. (There are symmetric cases where $s \geq 3$ and $r + t \geq 3$ and so on.)
So for $K_{r,s,t}$ to be planar, either two of the three values are ≤ 1 (the first case above), or all three are < 3.

1.4.18

Suppose every vertex has degree at least 6. Then $2e = \sum_w \deg(w) \geq \sum_w 6 = 6v$. So $3v \leq e \leq 3v - 6$, which is impossible. Therefore some vertex must have degree < 6, so degree ≤ 5.
Let G be any planar graph. Take any connected component; this connected component has a vertex of degree at most 5 by the previous part, so G has a vertex of degree at most 5.

2.1.2

a: For K_1, it depends on the definition. K_2 does not. For $n \geq 3$, the complete graph has an Euler cycle exactly when n is odd: when n is even, the graph has all vertices with odd degree, while when n is odd, every vertex has even degree.
b: Yes: when $n = 2$.
c: When both r, s are even and ≥ 2.

2.2.16

We represent this as a graph which is a 5x5 grid—each desk is adjacent to the ones to the left or right, or the ones ahead and behind—and such a move is exactly a Hamilton circuit in this graph.
This graph is not Hamiltonian. One way to see this is to notice that it is bipartite: if we identify vertices by their coordinates (x, y) where $1 \leq x, y \leq 5$, a vertex (x, y) is only adjacent to $(x \pm 1, y)$ and $(x, y \pm 1)$, so vertices with $x + y$ even can only be adjacent to vertices with $x + y$ odd.
This is a bipartite graph with 25 vertices—12 in one part and 13 in the other. But a Hamilton circuit would have to alternate between the two parts, which is impossible because the two parts have different sizes.
A second way to see this is to use Grinberg's theorem. The standard planar embedding has 17 regions: 16 regions with 4 sides forming the grid, and one region with 12 sides forming the outside. A Hamilton circuit can't contain the outside, and has to contain some number n of the 16 regions in the grid. Grinberg's Theorem says

$$(10)(0 - 1) + 2(n - (16 - n)) = 0,$$

or

$$4n = 42.$$

But 42 is not divisible by 4, so this is not possible.

2.3.1a-d

a: The chromatic number is ≥ 3 because there is a triangle. A 3-coloring is to color a, d red, b, f, g blue, and c, e, h green.
b: A 4-coloring is to color a, f red, e, d blue, b, g green, and c yellow. To see that the chromatic number is > 3, suppose we tried to use 3 colors. Then g, b need to be different from a, e (which are different from each other), so b, e have the same color. Similarly, d, e and f, a have the same colors. Since a, e, g have different colors, b, f, d have different colors. But then c can't be any of these 3 colors, so we need 4.
c: There are odd cycles (like a-b-c-d-e-a), so this cannot be 2-colored. A 3-coloring is acf red, bdg blue, eh green.
d: This is a K_4 (abfg), so we need at least 4 colors. A 4-coloring is ac red, b blue, gd green, fe yellow