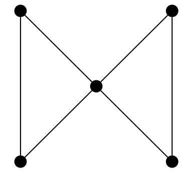
1. (*20 points*) You do not need to prove that the graphs have the specified properties; it suffices to draw the graphs.

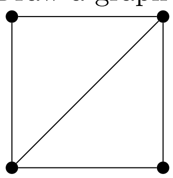
(**a**) Draw a graph with an Euler cycle but no Hamiltonian circuit.

`As with all these parts, there are many examples. Perhaps the simplest is:`

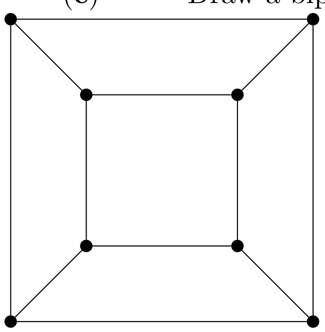


(**b**) Draw a graph with a Hamiltonian circuit but no Euler cycle.

`For example:`



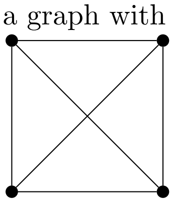
(**c**) Draw a bipartite planar graph where every vertex has degree 3.



Since $K_{3,3}$ is not planar, 4 vertices in each part is the smallest possible number. There's only one way to do this---4 vertices in each part, and each of the 4 vertices on the left omits a different one of the vertices on the right. Then it takes some moving vertices to see that the result is planar.

(**d**) Draw a graph with chromatic number 4 and a Hamiltonian circuit.

For example, K_4:



2. (*15 points*) For each of the following pairs of graphs, explain how you can be sure the pair is not isomorphic.

(**a**)

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These have the same degree sequence, but notice that left has a degree 1 adjacent
to a degree 3 vertex, while on the right the two degree 1 vertices are adjacent to
a degree 4 and a degree 5 vertex.
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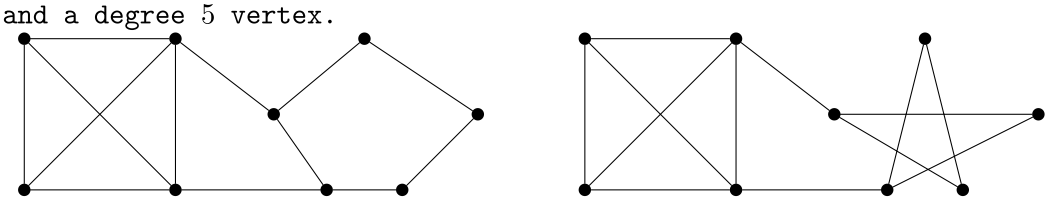
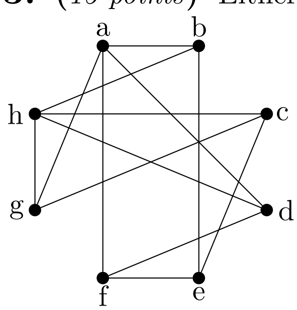
(**b**)



The graphs each contain only one copy of K_4, so we know an isomorphism must match those up. That means the two vertices which connect the copy of K_4 to the cycle of length 5 must be matched to each other in an isomorphism. But in one graph those are adjacent, and in the other they are not.
The most common mistake was only considering isomorphisms which made the two copies of K_4 isomorphic, without explaining why those were the only possible isomorphisms.

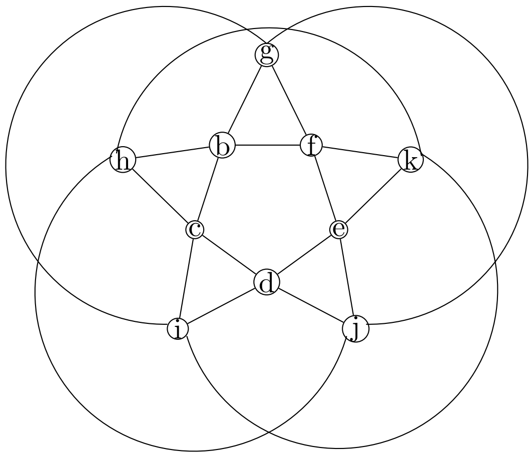
3. (*15 points*) Either redraw this graph so no lines cross or prove that it is non-planar.



The graph is non-planar because it contains a $K_{3,3}$ subdivision: take a, e, h to be one side and b, d, g to be the other: the edges (a, b), (a, d), (a, g), (e, b), (h, b), (h, d), and (h, g) are already present. For the remaining two, $e - f - d$ and $e - c - g$ give the needed paths.

4. (*15 points*) Prove that if G is a graph with 11 vertices, then either G or the complement $\overline{G}$ is non-planar. (The complement $\overline{G}$ is the graph with the same vertices as G, and where there is an edge in $\overline{G}$ between two vertices exactly when there is *not* an edge between them in G. Note that K_{11} has 55 edges.)
Suppose G is planar. Then $e \leq 3v - 6 = 3 \cdot 11 - 6 = 27$. Then means at most 27 of the 55 edges in K_{11} are in G, so the other ≥ 28 must be in $\overline{G}$. But then in $\overline{G}$, $28 \geq e > 3v - 6 = 27$, so $\overline{G}$ is non-planar.

5. (*15 points*) Show that the graph



has chromatic number 4.

We can exhibit a four coloring to show the chromatic number is ≤ 4; for example, color g, k, c red, b, e, i blue, f, h, d green, and j yellow.

To show that the chromatic number is > 3, we must show that it is impossible to color the graph with only 3 colors. One approach is brute force: if we have three colors, the triangle g, b, f must all get different colors. Without loss of generality, we color g red, b blue, and f green. We now consider two cases for the color of k and h.

First, suppose k is red. Then e must be blue, j must be green, d must be red, c must be green, and then h is a neighbor of b, of c, and of k, and there is no color left for h.

Otherwise, suppose k is blue. Then i is green and e is red, so c must be red, so d must be blue, so j must be green. But h is a neighbor of b, of c, and of j, so there is no color left for h again.

A slightly slicker argument, though probably harder to come up with on the spot, is to look at the inner pentagon $b - f - e - d - c$. We need three colors just for this pentagon; we must use two colors twice each and the third one once. Since the graph is symmetric, we may assume c, e are red, f, d are blue, and b is the single green vertex. Then both k and i must be green as well, which is impossible since they are adjacent.

6. (*20 points*) Prove by induction that if G is any graph with finitely many vertices, either G is connected or the complement $\overline{G}$ is connected. Let p_k be the statement that, for any graph G with k vertices, either G is connected or the complement $\overline{G}$ is connected.

(**a**) Prove p_1.

A graph with a single vertex is always connected, so if G has one vertex, G is connected.

(**b**) Assume that p_k is true and show that p_{k+1} is true. (Consider two cases: one where there is at least one vertex with $0 < deg(v) < k$, and one case where every vertex has either $deg(v) = 0$ or $deg(v) = k$.)

Let G be a graph with $k{+}1$ vertices. First, suppose there is some vertex v with $0 < deg(v) < k$. Consider the graph $G - \{v\}$; this graph has k vertices, so by IH, either $G{-}\{v\}$ or $\overline{G - \{v\}}$ is connected. If $G{-}\{v\}$ is connected, since $0 < deg(v)$, v has an edge with some vertex in $G{-}\{v\}$, and therefore G is connected. If $\overline{G - \{v\}}$

is connected then, since $deg(v) < k$, v has a non-edge with some vertex in $\overline{G - \{v\}}$, and therefore $\overline{G}$ is connected.

One common mistake was getting confused about the cases. If one case is where there is one vertex with intermediate degree then the other case is that every vertex has either degree 0 or degree k; the other case is *not* that either every vertex has degree 0 or every vertex has degree k---this includes (at least until an additional argument is made) the possibility that some vertices have degree 0 while others have degree k.

There was also, unsurprisingly, a lot of difficulty handing the inductive hypothesis. It's hard to prove this if you start with a k vertex graph and add a vertex to it (consider the case where G is connected, $\overline{G}$ is not, and you add an isolated vertex to G).

(c) Prove the whole statement: for any graph G with finitely many vertices, either G is connected or the complement $\overline{G}$ is connected.

Since p_1 is true and, for every k, p_k implies p_{k+1}, by induction, p_k is true for every k.