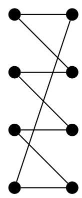
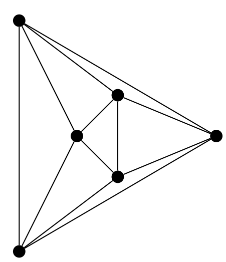
MIDTERM 1 SOLUTIONS

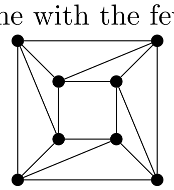
1. (*20 points*)

(**a**) Draw a bipartite graph with eight vertices where each vertex has degree 2.

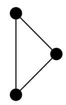
One example is: 

(**b**) Draw a connected planar graph where every vertex has degree 4.

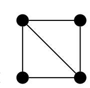
One example (the one with the fewest possible vertices) is: 

Another example is: 

(**c**) Draw a graph which has an Euler cycle that is also a Hamilton circuit.

Any cycle by itself, for instance 

(**d**) Draw a planar graph with a Hamiltonian circuit, an Euler trail, but no Euler cycle.

The easiest way is to take a cycle of length at least 4 and add exactly one diagonal: 

2. (*15 points*) For each of the following pairs of graphs, explain how you can be sure the pair is not isomorphic.

(**a**)

The graph on the left has 8 edges while the graph on the right has 9.

(**b**)

The graph on the left has vertices of degree 3 while the graph on the right does not.

(**c**)

The graph on the right has a triangle consisting of three vertices of degree 4, while the graph on the left does not.

3. (*15 points*) $G = (V, E)$ is a connected graph in which all vertices have even degree. Show that if we remove one edge from G, the graph remains connected.

One solution is: suppose we remove the edge (v, w) from G to give a new graph, G'. Then v and w have odd degree in G', and are the only vertices in G' with odd degree. Therefore v and w must belong to the same connected component in G'. But every vertex in G' has a path to one of v and w, so every vertex belongs to this component, so it is the only component in G', so G' is connected.

Another approach to the solution is: since G is connected and every vertex in G has even degree, G has an Euler cycle (using the theorem we proved in class), say $v_1 - v_2 - v_3 - \cdots - v_n$, and there is an edge from v_n to v_1. By rotating the cycle, we may assume that the edge from v_1 to v_n is the one we removed to get G'. But then the path from v_1 to v_1 remains a path going through every vertex in G', so in particular G' is still connected. (Indeed, this path must be an Euler trail, so we could use the theorem to notice that a graph with an Euler trail is connected.)

A common mistake was to notice that v and w have positive degree in G', and therefore are not isolated, but then failing to show that this meant the graph had to all be one piece. (Often these answers talked about v or w being "connected to the rest of the graph" without explaining what "the rest of the graph" means—the problem is that "the rest of the graph" could be two disconnected components.)

4. **(***15 points***)** Suppose $G = (V, E)$ is a connected graph with v vertices (that is, $|V| = v$). Let $L = \{w \in V \mid deg(w) \geq 11\}$: the set of vertices with degree at least 11.

(**a**) Suppose $|L| \geq v/2$. Show that $\sum_{w \in V} deg(w) \geq 6v$.

We want a lower bound on $\sum_{w \in V} deg(w)$—that is, what is the smallest possible value the sum of degrees could have. The sum of degrees will be small as possible if $|L| = v/2$ (the smallest possible value), for each $w \in L$, $deg(w) = 11$ (again, the smallest possible value), and for $w \notin L$, $deg(w) = 1$ (the smallest possible value—it has to be at least 1 since G is connected). In symbols,

$$\sum_{w \in V} deg(w) \geq \sum_{w \in L} deg(w) + \sum_{w \in V-L} deg(w) \geq (v/2)11 + (v/2)1 = 6v.$$

(**b**) Show that if G is planar, $|L| < v/2$.

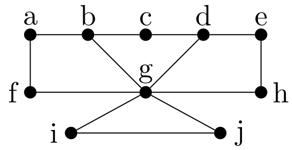
Using Euler's formula, we know that a planar graph must satisfy the inequality $e \leq 3v - 6$. If $|L| \geq v/2$ then $2e = \sum_{w \in V} deg(w) \geq 6v$, so

$$3v \leq e \leq 3v - 6.$$

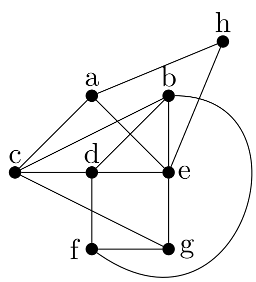
But this is impossible, so $|L| < v/2$.

5. (*15 points*) For each of the following, either write down a Hamilton circuit or show the graph does not have one.

(**a**)



One way to see there is no Hamilton circuit is the following. Since i and j each have degree 2, both adjacent edges would have to get used in any Hamilton circuit. But that means a circuit would contain $i-j-g$ as a subcircuit, and a Hamilton circuit cannot have a subcircuit which doesn't contain all vertices. So this graph cannot contain a Hamilton circuit.
Another way to see this is to notice that removing g leaves two connected components, violating a lemma we proved about Hamiltonian circuits.



(**b**)

$a-h-e-g-f-d-b-c$ is a Hamilton circuit.

6. (*20 points*) We want to prove that if G is a finite graph with no cycles then G can be 2-colored. (The case where G is empty is trivial, so we'll only worry about graphs with at least one vertex.) Let p_k be the statement "Every graph with k vertices and no cycles can be 2-colored."

(**a**) Prove p_1.

> The only graph with 1 vertex is the graph with a single vertex and no edges. This graph can be 2-colored by assigning either color to it.

(**b**) Suppose that p_k is true. Prove p_{k+1}. You may find the following fact useful: any finite graph with no cycles has a vertex of degree ≤ 1.

> Suppose p_k is true and that G is a graph with $k+2$ vertices and no cycles. Pick a vertex v in G which has degree ≤ 1. Then the graph $G - \{v\}$ has $k+1$ vertices and no cycles (because it is a subgraph of a graph with no cycles), and so p_k implies that $G - \{v\}$ can be 2-colored. Since v has at most one neighbor in $G - \{v\}$, we can extend a 2-coloring of $G - \{v\}$ to a 2-coloring of G by coloring v the opposite of its neighbor (if there is a neighbor) or giving it any color (if v is isolated).

(**c**) Prove that *any* finite graph with no cycles can be 2-colored.

> We have shown that p_0 is true and that whenever p_k is true, p_{k+1} is also true, so by mathematical induction, for all k, p_k is true.
>
> One student pointed out the following non-inductive argument: if a graph has no cycles, it has no odd cycles, so the graph is bipartite (by a result in the textbook). A bipartite graph can be 2-colored (by a result proved in class).