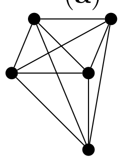
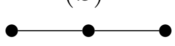
Practice Midterm Solutions

1. (*20 points*)
You do not need to prove that the graphs have the specified properties; it suffices to draw the graphs.

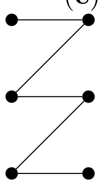
(**a**) Draw a non-planar graph with a Hamiltonian circuit.



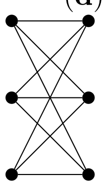
(**b**) Draw a planar graph with no Hamiltonian circuit.



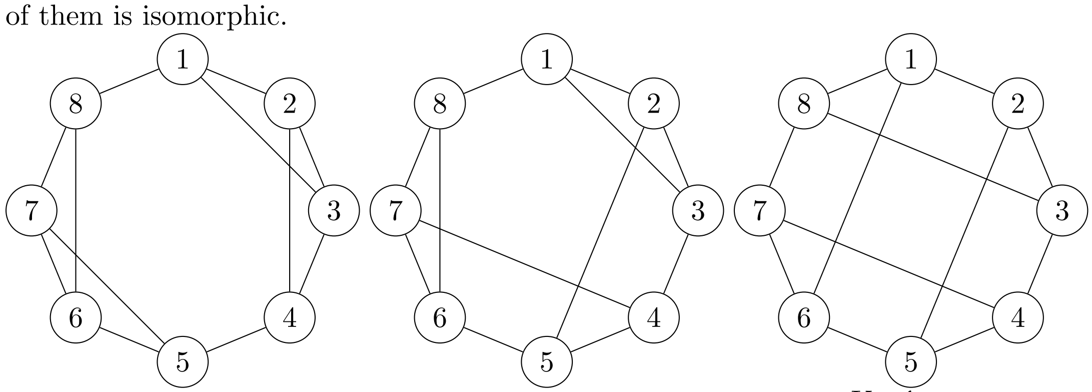
(**c**) Draw a bipartite graph with an Euler trail but no Euler cycle.



(**d**) Draw a bipartite, non-planar graph with a Hamiltonian circuit.

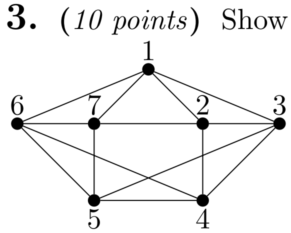


2. (*15 points*) All three of these graphs have 8 vertices, each of degree 3. Show that no pair of them is isomorphic.

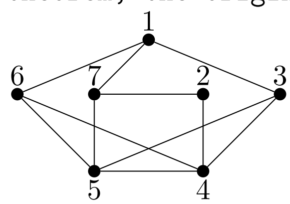


The first two graphs both have subgraphs isomorphic to K_3 (i.e. triangles); the third does not, so the third is not isomorphic to either of the others. The second graph has subgraphs isomorphic to C_4 (the circuit with four vertices---the ''square'') while the first does not, so the first two are not isomorphic.

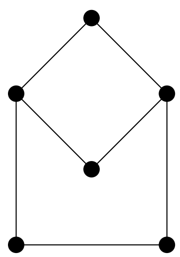
3. (*10 points*) Show that this graph is non-planar.



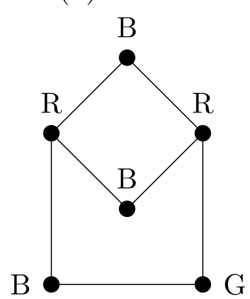
The following graph is a subgraph which is also a subdivision of $K_{3,3}$; by Kuratowski's theorem, the original graph is non-planar.



4. (*10 points*) Consider the following graph:



(**a**) Show that this graph can be colored using 3 colors.



(**b**) Show that this graph cannot be colored using 2 colors.

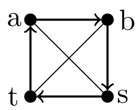
By theorems we have proven, a graph can be 2 colored iff it is bipartite, which happens iff it has no odd cycles. But there is a cycle of length 5, so this graph cannot be 2 colored.

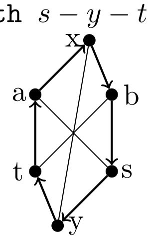
5. (*15 points*) $G = (V, E)$ is a graph with an Euler cycle. (In order for this to make sense, you may assume that $|V| \geq 3$.) Suppose I add some additional edges to obtain a new graph, $G' = (V, E \cup E')$. (So $E' \cap E = \emptyset$ and $|E'| > 0$; that is, I add at least one edge, and I only add new edges.) If G' also has an Euler cycle, prove that $|E'| \geq 3$. (That is, prove that if we add exactly 1 or exactly 2 edges, we can't get an Euler cycle. It matters that we're in a graph, not a multigraph—there are no loops or multiple edges.)

Since G has an Euler cycle, every vertex has even degree. Since G' also has an Euler cycle, each vertex in G' still has even degree. If $|E'| = 1$ then the two ends of the single new edge each see their degrees increase by 1, so they would have odd degree; that is impossible, so $|E'| > 1$. If $|E'| = 2$ then the total degree increases by 4; that must be spread across at least 3 vertices (it cannot be two vertices getting two new edges each, because G' does not have multiple edges). So $|E'| > 2$ as well, so $|E'| \geq 3$.

6. (*10 points*) Show that if n is even and $n \geq 4$ then there is a graph with n vertices such that every vertex has degree 3 and the graph has a Hamilton circuit.

When n is 4, this is K_4. Suppose that we have a graph G with n vertices where every vertex has degree 3 and the graph has a Hamiltonian circuit, and we want to construct a graph with $n+2$ vertices with the same property. Choose two edges in the Hamiltonian circuit, which do not share endpoints, say an edge from a to b and a second edge from s to t. (These exist because the graph is connected and has ≥ 4 vertices---the first and third edges in the Hamiltonian circuit always work.)
We define G' to contain the vertices of G plus two new vertices x, y. We delete the edge between a, b and the edge between s, t, and add edges $x - y$, $a - x$, $b - x$, $s - y$, $t - y$; each vertex has degree 3 in the new graph as well. The Hamiltonian circuit is like the old Hamiltonian circuit except that we replace $a-b$ with $a-x-b$ and $s-t$ with $s-y-t$.



becomes


7. (*20 points*) We want to prove that every finite connected graph has a connected *spanning* (containing every vertex) subgraph with no circuits. Let p_k be the statement "Every connected graph with k vertices has a spanning subgraph with no circuits."

(**a**) Prove p_1, that every connected graph with 1 vertex has a spanning subgraph with no circuits.

If a graph has 1 vertex, it is its own spanning subgraph.

(**b**) Suppose that p_k is true. Prove p_{k+1}.

Suppose every graph with k vertices has a spanning subgraph. Take a graph G with $k+1$ vertices, and pick any vertex v. The graph $G' = G \backslash \{v\}$ has connected components, and v must have at least one edge to each connected component (because the original graph was connected). By the inductive hypothesis, each component contains a spanning subgraph with no circuits. Combine these spanning subgraphs with the vertex v and exactly one edge from v to the spanning subgraph of each component. This is spanning (it contains every vertex other than v, and also v), connected, and has no circuits (each component of G' has no circuit in the spanning subgraph, and no circuit includes v because once we pass through v into any component, we can never leave that component again.[1]

(**c**) Using the previous two parts, prove that *any* finite connected graph has a spanning subgraph with no circuits.

By induction on k. The base case is the first part, the inductive case is the second part. So, by induction on k, every finite connected graph has a spanning subgraph with no circuits.

[1] I tweaked this problem to talk about circuits instead of cycles because I thought that made it slightly easier. If you solved it using cycles, the same basic argument applies; to see there are no cycles, notice that once you cross the bridge from any component to v, you can't get back to that component because there is only one such "bridge".