PRACTICE FINAL EXAM SOLUTIONS

Math 340
12/?/2018

Name: ______________________________

ID: ______________________________

"My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this"

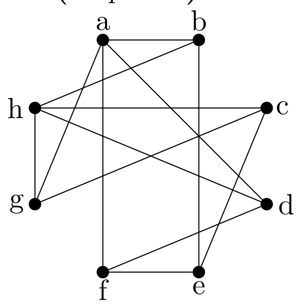
Signature: ______________________________

Read all of the following information before starting the exam:

- Check your exam to make sure all pages are present.
- Leave answers unsimplified—you may include factorials, $P(n,k)$, $\binom{n}{k}$, etc. in your answers. Do not leave unevaluated $\sum_{i\leq k}$ in your final answers.
- You may use writing implements and a single 3"x5" notecard.
- You may not use a calculator.
- You may use any result proved in class or in the textbook in your arguments.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Good luck!

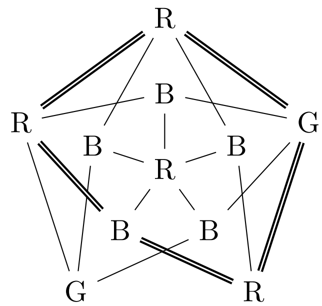
1	15	
2	10	
3	10	
4	15	
5	12	
6	12	
7	14	
Total	100	

1. (*10 points*) Either redraw this graph so no lines cross or prove that it is non-planar.



We can find a subdivision which is a copy of $K_{3,3}$: one side is a, h, e, the other side is b, d, g. $a-b, a-d, a-g, h-b, h-d, h-g, e-b$ are all edges of the graph, and $e-c-g$ and $e-f-d$ complete the copy of $K_{3,3}$.

2. (*15 points*) Show that the following graph has chromatic number exactly 3:



A 3 coloring is indicated above, so the chromatic number is ≤ 3.
An odd cycle is drawn above. A graph with an odd cycle cannot be 2-colored, so the chromatic number is > 2, and therefore exactly 3.

3. (*15 points*) (**a**) How may ways are there to rearrange the letters AEIOUCDVTTT so that none of the T's are adjacent.

There are 8 letters other than T's. We know that the arrangement will have the form

$$\_\_\_T\_\_\_T\_\_\_T\_\_\_$$

with an unknown number of other letters in each gap, but at least 1 in the middle two gaps. There are $\binom{6+4-1}{4-1} = \binom{9}{3}$ ways to decide how many non-T letters appear in each gap. There are then 8! ways to arrange them in those spaces, for a total of

$$\binom{9}{3}8!$$

arrangements.

(**b**) How may ways are there to rearrange the letters AEIOUBCDFGHTTT so that none of the T's are adjacent and the vowels appear in alphabetical order.

There are 11 letters other than T this time, so

$$\binom{12}{3}\frac{11!}{5!}:$$

we proceed as above and then forget the 5! ways to rearrange the vowels.

4. (*12 points*) Determine the number of integers between 1 and 1000 (including 1 and 1000) which are not divisible by 6, by 7, or by 8. ($1000/6 = 166.6\overline{6}$, $1000/7 = 142.8\ldots$, $1000/8 = 125$. You may need to do some additional division by hand.)

This calls for inclusion exclusion. We have three events: divisibility by 6, by 7, and by 8. There are 166 numbers divisible by 6, 142 divisible by 7, and 125 divisible by 8.

If a number is divisible by both 6 and 7, it is divisible by 42, there are $\lfloor 142/6 \rfloor = 23$ of these. If a number is divisible by 6 and 8, it is divisible by 24, and there are $\lfloor 125/3/rfloor = 41$ of these. If a number is divisible by 7 and 7, it is divisible by 56, and there are $\lfloor 125/7 \rfloor = 17$ of these.

If a number is divisible by 6, 7, and 8 then it is divisible by 168, and there are $\lfloor 17/3 \rfloor = 5$ of these.

So the numbers divisible by none of these are:

$$1000 - (166 + 142 + 125) + (23 + 41 + 17) - 5.$$

5. (*14 points*) Suppose we roll 8 six sided dice, each a different color, so there are 6^8 possible outcomes. In how many of these outcomes do all six numbers appear?
We can use inclusion-exclusion: the event A_i for i a number between 1 and 6 is that no die rolls an i. The number of outcomes is:

$$6^8 - \binom{6}{1}5^8 + \binom{6}{2}4^8 - \binom{6}{3}3^8 + \binom{6}{4}2^8 - \binom{6}{5}.$$

6. (*10 points*) We are interested in quaternary sequences (i.e. using the digits $\{0, 1, 2, 3\}$) which never contain either the subsequence 00 or the subsequence 02. Let s_n be the number of such sequences of length n.

(**a**) How many of these sequences are there of length 1?

4

(**b**) How many of these sequences are there of length 2?

14

(**c**) Verify that the numbers s_n satisfy the recurrence relation

$$s_n = 3s_{n-1} + 2s_{n-2}.$$

The quaternary sequences of length n containing neither 00 nor 02 consist of either:

- a digit other than 0 followed by a sequence of length $n-1$—there are $3s_{n-1}$ ways of doing this, or
- 01 or 03 followed by a sequence of length $n-2$—there are $2s_{n-2}$ ways of doing this.

(**d**) Solve the recurrence relation to give a formula for s_n.

The characteristic polynomial is $r^2 - 3r - 2 = 0$. Solutions are $r = 2$ and $r = 1$, so the general solution is $A2^n + B$. Setting $A + B = 4$ and $2A + B = 14$, we can solve for $A = 10$ and $B = -6$, so $s_n = 102^n - 6$.

7. (*10 points*) The game of Kayles is played with a row of n coins. Two players alternate, and on their turn, a player can either remove a single coin, or two adjacent coins. In this case, adjacent means that there are no holes (places where a coin was removed previously) between them. A player wins if they remove the final coin.
(For example, the first play wins when the game starts with three coins by first removing the middle coin; the remaining coins are not adjacent, so the second player can only remove one, and the first player wins by removing the other.)

What is the Grundy number of the game which begins with three coins?