PRACTICE FINAL EXAM SOLUTIONS

Math 340
12/15/2017

Name: ____________________

ID: ____________________

"My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this"

Signature: ____________________

Read all of the following information before starting the exam:

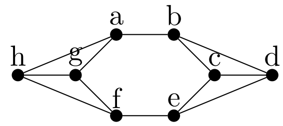
- Check your exam to make sure all pages are present.
- Leave answers unsimplified—you may include factorials, $P(n, k)$, $\binom{n}{k}$, etc. in your answers. Do not leave unevaluated $\sum_{i \leq k}$ in your final answers.
- You may use writing implements and a single 3"x5" notecard.
- You may not use a calculator.
- You may use any result proved in class or in the textbook in your arguments.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Good luck!

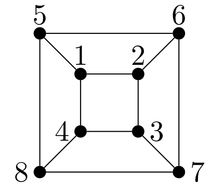
1	15	
2	10	
3	10	
4	15	
5	12	
6	12	
7	14	
Total	100	

Name: ______________________

1. (*15 points*) For each of the following pairs of graphs, explain how you can be sure the pair is not isomorphic. (All the graphs have eight vertices and every vertex has degree 3.)

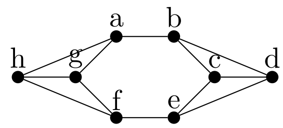
(**a**)

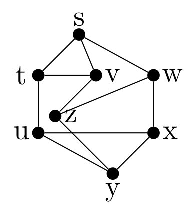




The graph on the right has a copy of C_4 (the vertices 1-2-3-4-1); the graph on the left does not. (The graph on the left has cycles of length 3, like $b - c - e - d - b$, but they have an extra diagonal.)

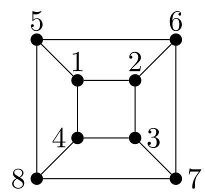
(**b**)

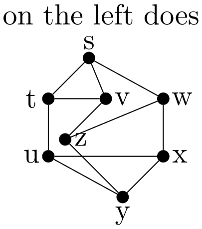




The graph on the right has a copy of C_4 (the vertices s-w-z-v-s); the graph on the left does not.

(**c**)





The graph on the right has a triangle (s-v-t) while the graph on the left does not.

2. (*10 points*) Suppose G is a connected graph such that every vertex has degree at least 5 and which contains no triangles.

(**a**) If $\mathbf{e}$ is the number of edges and $\mathbf{v}$ is the number of vertices, give an upper bound for $\mathbf{v}$ in terms of $\mathbf{e}$. (That is, find something to go in the blank in $\mathbf{v} \le$ ___.)

Since every vertex has degree at least 5, $v \le \frac{2}{5}e$.

(**b**) Suppose that we are able to draw G in the plane and that $\mathbf{r}$ is the number of regions in this drawing. Give an upper bound on $\mathbf{r}$ in terms of $\mathbf{e}$.

Each region has at least 4 edges, so $4r \le 2e$.

(**c**) Use Euler's formula, $\mathbf{r} = \mathbf{e} - \mathbf{v} + 2$, to show that G is not planar.

$\mathbf{e} + 2 = \mathbf{r} + \mathbf{v} \le \frac{1}{2}\mathbf{e} + \frac{2}{5}\mathbf{e} \le \frac{9}{10}\mathbf{e}$. But this would require that $\mathbf{e} \le -20$, which is clearly nonsense, so G must not be planar.

3. (*15 points*) (**a**) How many ways are there to rearrange the letters in the word PLATYPUS so that the vowels A and U are not adjacent?

There are $8!/2$ total rearrangements—rearranging the 8 letters and dividing by 2 because the letters P are indistinguishable. There are $2 \cdot 7!.2$ arrangements where the A and U are adjacent (place AU as a single letter, then order it either of 2 ways), so $(8! - 2 \cdot 7!)/2$.

(**b**) How many ways are there to rearrange the letters in the word PLATYPUS so that the vowels A and U are not adjacent and the letter Y comes after the letter A?

There are $\binom{8}{2}\binom{6}{2}4!$ ways to arrange the letters so that A comes before Y (choose 2 places to put A, Y, place A in the earlier one, then place the two P's, and then place the other four letters). There are $\binom{8}{2}\binom{5}{2}2!2$ ways to arrange the letters so that A comes before Y and A is adjacent to U: first place AU as a single letter earlier than Y, then order AU.
So $\binom{8}{2}\binom{6}{2}4! - \binom{8}{2}\binom{5}{2}3!2$.

4. (*14 points*) Suppose we make a 9 digit ternery sequence (i.e. using the digits $\{0, 1, 2\}$). How many sequences are there where the subsequence 012 (those three values in a row in that order) never appears? (Hint: use inclusion-exclusion, with the set A_i being the number of sequences where the subsequence 012 does appear, starting in the i-th position; for instance, A_1 is the number of 9 digit sequences which start with 012.)

Using inclusion exclusion as suggested, we have $|U| = 3^9$, $|A_i| = 3^6$ (where i goes from 1 to 7), so $S_1 = 7 \cdot 3^6$.
$|A_i \cap A_j| = 0$ if i and j are within two, and $|A_i \cap A_j| = 3^3$ otherwise. There are 10 pairs of the second kind ($A_1 \cap A_4, \ldots, A_1 \cap A_7, A_2 \cap A_5, \ldots, A_2 \cap A_7$, and so on), so $S_1 = 10 \cdot 3^3$.
$S_3 = 1$ (the only way for 012 to appear three times is 012012012, which is $A_1 \cap A_4 \cap A_7$), so

$$3^9 - 7 \cdot 3^6 + 10 \cdot 3^3 - 1.$$

5. (*12 points*) There are 6 place settings around a circular table. At each setting is a bowl of ice cream, chosen from one of 31 available flavors.

(**a**) If each place setting is distinct, how many possible arrangements of ice cream flavors are there?

31^6

(**b**) If we treat rotations as symmetric, how many possible arrangements are there? (So if we rotate all the ice creams one step clockwise, that counts as the same arrangement.)

We can't quite divide by 6, because some cases (for instance, where all the places got the same flavor) were only counted once, while others (like where all six got different flavors) were counted up to six times.
We have to distinguish some cases to deal with the different ways a circle could be symmetric.
If all six place settings get the same, all places are symmetric; there are 31 ways to do this. We already counted these correctly.
If we use two distinct ice creams alternating, there are $31 \cdot 30$ ways, which we double counted above. If we use three ice creams, at least two distinct, there are $31^3 - 31$ ways, which we triple counted above.
The remaining ones were counted six times. So we get:

$$\frac{31^6 - 31 - 2(31 \cdot 30) - 3(31^3 - 31)}{6} + (31^3 - 31) + 31 \cdot 30 + 31.$$

6. (*10 points*) We are interested in ternery sequences (i.e. using the digits $\{0, 1, 2\}$) which never contain two consecutive 0's. (So 121, 010, and 011 are allowed sequences, but 100 is not.) Let s_n be the number of such sequences of length n.

(**a**) How many of these sequences are there of length 1?

3

(**b**) How many of these sequences are there of length 2?

8

(**c**) Verify that the numbers s_n satisfy the recurrence relation

$$s_n = 2(s_{n-1} + s_{n-2}).$$

Using the addition rule, s_n is the sum of:

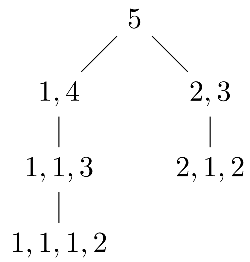
- sequences starting with a 1 or 2 followed by one of the s_{n-1} sequences of length $n-1$ ($2s_{n-1}$ ways in total), and
- sequences starting with a 01 or 02 followed by one of the s_{n-2} sequences of length $n-2$ ($2s_{n-2}$ ways in total).

(**d**) Solve the recurrence relation to give a formula for s_n.

The characteristic equation is $r^2 - 2r - 2 = 0$, so $r = \frac{2 \pm \sqrt{4+8}}{2}$, so $r = 1 \pm \sqrt{3}$. So $s_n = A(1+\sqrt{3})^n + B(1-\sqrt{3})^n$. We have $A + B = 1$ (sequences of length 0) and $A(1+\sqrt{3}) + B(1-\sqrt{3}) = 3$, so $B = 1 - A$ and $2\sqrt{3}A = 2 + \sqrt{3}$, so $A = \frac{2+\sqrt{3}}{2\sqrt{3}}$ and $B = 1 - \frac{2+\sqrt{3}}{2\sqrt{3}} = \frac{\sqrt{3}-2}{2\sqrt{3}}$. Therefore

$$s_n = \frac{2+\sqrt{3}}{2\sqrt{3}}(1+\sqrt{3})^n + \frac{\sqrt{3}-2}{2\sqrt{3}}(1-\sqrt{3})^n.$$

7. (*10 points*) Grundy's Game is, like Nim, a two-player game played with several heaps, where each heap has some number of pennies. In Grundy's Game, the only available move is splitting an existing heap into two smaller heaps which have two different sizes. The game ends when all heaps have size 1 or 2, and therefore cannot be further split; the player who made the last split wins.
What is the Grundy number (nimber) of the game starting with a single heap with five pennies?



The leaf $1, 1, 1, 2$ has Grundy number 0, so $1, 1, 3$ has Grundy number 1, so $1, 4$ has Grundy number 0.
The leaf $2, 1, 2$ has Grundy number 0, so $2, 3$ has Grundy number 1.
So the state with a single heap of 5 has Grundy number 2.