# MIDTERM

Math 570

**Name:** __________________________________

**ID:** __________________________________

“I have adhered to the Penn Code of Academic Integrity in completing this exam.”

**Signature:** __________________________________

**Read all of the following information before starting the exam:**

- Check your exam to make sure all pages are present.
- You may use any result proved in class or in the textbook in your arguments.
- Circle or otherwise indicate your final answers.
- Good luck!

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|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 20 | |
| 5 | 20 | |
| Total | 100 | |

**1.** (*20 points*) One of the following is valid, and the other is not. Give a deduction of the valid one and prove that there is no deduction of the other.

$$\forall x \exists y \phi \to \exists x \forall y \phi, \qquad \exists x \forall y \phi \to \forall x \exists y \phi.$$

**Solution:** There's a mistake in this problem, and as written, both statements are invalid. For both, we can take $\phi$ to be $Pxy$ where $P$ is a binary relation symbol. For a counterexample to the first, take $|\mathfrak{A}| = \mathbb{N}$ and $P^{\mathfrak{A}} = \{(n,m) \mid n < m\}$. Then $\mathfrak{A} \models \forall x \exists y Pxy$ (for each $x$, there is a $y$ with $x < y$), but $\mathfrak{A} \not\models \exists x \forall y Pxy$ (there is no $x$ so that, for all $y$, $Pxy$—for any $x$, take $y = x$). For a counterexample to the second, take $|\mathfrak{B}| = \mathbb{N}$ and $P^{\mathfrak{B}} = \{(n,m) \mid n = 0\}$. Then $\mathfrak{A} \models \exists x \forall y Pxy$ (take $x$ to be 0) but $\mathfrak{A} \models \forall x \exists y Pxy$ (if $x$ is anything other than $x$).

I'd meant to switch the variables:

$$\forall x \exists y \phi \to \exists y \forall x \phi, \qquad \exists x \forall y \phi \to \forall y \exists x \phi.$$

Then the first statement is not valid, and the second one is. For a deduction of the second, take:

- $\forall x \phi \to \phi$ (Q1, since $\phi^x_x$ is $\phi$)
- $\phi \to \exists x \phi$ (Q2, since $\phi^x_x$ is $\phi$)
- $\forall y \phi \to \exists x \phi$ (PC)
- $\exists x \forall y \phi \to \exists x \phi$ (by QR rule)
- $\exists x \forall y \phi \to \forall y \exists x \phi$ (by QR rule)

**2.** (*20 points*) Consider a language with a unary relation symbol $\mathbf{P}$.

(**a**) Give two examples of structures in this language which are not elementarily equivalent.

**Solution:** There are lots of examples, but the easiest is something like $|\mathfrak{A}| = |\mathfrak{B}| = \mathbb{N}$, $\mathbf{P}^{\mathfrak{A}} = \mathbb{N}$, and $\mathbf{P}^{\mathfrak{B}} = \emptyset$, so $\mathfrak{A} \vDash \forall x \mathbf{P} x$ while $\mathfrak{B} \vDash \neg \forall x \mathbf{P} x$.

(**b**) Give two examples of structures in this language which are elementarily equivalent, but are not isomorphic.

**Solution:** $|\mathfrak{A}| = \mathbb{N}, \mathbf{P}^{\mathfrak{A}} = \mathbb{N}$, while $|\mathfrak{B}| = \mathbb{R}, \mathbf{P}^{\mathfrak{B}} = \mathbb{R}$.

It's hard to make examples which elementarily equivalent but not isomorphic here without using different infinite cardinalities—if either $\mathbf{P}^{\mathfrak{A}}$ or $|\mathfrak{A}| \setminus \mathbf{P}^{\mathfrak{A}}$ is finite, this size can be expressed by a sentence, and any structure where both sets are countable will be isomorphic to any other such structure.

**3.** (*20 points*) Consider a language with a single constant symbol $\mathbf{d}$, a unary function symbol $\mathbf{f}$, and a unary predicate symbol $\mathbf{Q}$. Let $\mathfrak{A}$ be a model with:

- $|\mathfrak{A}| = \mathbb{R}$,
- $\mathbf{d}^{\mathfrak{A}} = \pi$,
- $\mathbf{f}^{\mathfrak{A}}(r) = r^2$,
- $\mathbf{Q}^{\mathfrak{A}} = \mathbb{N}$.

Let $s(v_i) = i$.

(**a**) What is $\overline{s}(\mathbf{f}d)$?

**Solution:** $\pi^2$

(**b**) What is $\overline{s}(\mathbf{ff}v_2)$?

**Solution:** 16

(**c**) Does $\mathfrak{A} \vDash \mathbf{Qff}v_2[s]$ hold?

**Solution:** Yes (16 is a natural number)

(**d**) Does $\mathfrak{A} \vDash \forall x \exists y(\mathbf{Q}x \to (x \neq y \land \mathbf{Qf}y))[s]$?

**Solution:** Yes—for instance, if we interpret $y$ with $\sqrt{2}$, for any $x$ such that $\mathbf{Q}x$ holds, we will have $x \neq y$ and $\mathbf{Qf}y$. (This is a somewhat unnatural sentence—the goal was something that isn't easy to recognize, but whose truth is easy to check once you interpret the symbols.)

**4.** (*20 points*) Recall that when $(G, +)$ is an abelian group, an element $g \in G$ *has finite order* if $\underbrace{g + g + \cdots + g}_{n \text{ times}} = 0$ for some $n \in \mathbb{N}$. We say $G$ is torsion if every element of $G$ has finite order.

Let $T_{\text{Abel}}$ be the set of axioms for abelian groups. Show that there is no set $\Sigma$ so that $G \vDash T_{\text{Abel}} \cup \Sigma$ if and only if $G$ is a torsion group. (There are other ways to do this, but if you don't remember much group theory, it may be helpful to remember the cyclic group with $n$ elements, $\mathbb{Z}/n\mathbb{Z}$, which is a torsion group containing an element $[1]_{\mathbb{Z}/n\mathbb{Z}}$ of order exactly $n$.)

**Solution:**

Suppose there were some set $\Sigma$ such that every torsion group satisfies $\Sigma \cup T_{Abel}$. Extend the language by a constant $c$, and consider the sentences $\phi_n$, $\underbrace{c + \cdots + c}_{n \text{ times}} \neq 0$. Let $\Gamma = T_{Abel} \cup \Sigma \cup \{\phi_n\}$.

For any finite $\Gamma_0 \subseteq \Gamma$, there is some maximal $n$ so that $\phi_n \in \Gamma_0$. Then $\mathbb{Z}/n\mathbb{Z}$, with $c$ interpreted by $[1]_{\mathbb{Z}/(n+1)\mathbb{Z}}$, satisfies $\Gamma_0$.

Therefore $\Gamma$ is finitely satisfiable, and therefore satisfiable. Therefore there is a model $\mathfrak{G}$ of $\Gamma$, which must be an Abelian group with satisfies $\Sigma$, but $c^{\mathfrak{G}}$ does not have finite order, because $\phi_n$ is true in $\mathfrak{G}$ for every $n$. Therefore if every torsion group satisfies $\Sigma$, there is also a non-torsion group satisfying $\Sigma$.

Everyone knew this was a compactness argument, but some people had trouble putting together the argument carefully. In particular, I'm not sure it's possible to prove this, at least not easily, without expanding the language with a constant.

**5.** (*20 points*) Consider a language with a single binary relation **E**. Let $\Sigma$ be the sentences:

- $\forall x\, Exx$,
- $\forall x\, \forall y\, Exy \rightarrow Eyx$,
- $\forall x\, \forall y\, \forall z\, (Exy \wedge Eyz) \rightarrow Exz$,
- $\exists x\, \exists y\, \neg Exy \wedge \forall z (Exz \vee Eyz)$,
- $\forall x\, \exists y_1\, Exy_1 \wedge x \neq y_1$,
- $\forall x\, \exists y_1\, \exists y_2\, Exy_1 \wedge Exy_2 \wedge x \neq y_1 \wedge x \neq y_2 \wedge y_1 \neq y_2$,
- $\forall x\, \exists y_1\, \exists y_2\, \exists y_3\, Exy_1 \wedge Exy_2 \wedge Exy_3 \wedge x \neq y_1 \wedge x \neq y_2 \wedge y_1 \neq y_2 \wedge x \neq y_3 \wedge y_1 \neq y_3 \wedge y_2 \neq y_3$,
- $\cdots$

The first three axioms say that $E$ is an equivalence relation. The fourth says that $E$ has exactly two equivalence classes. The final three together with $\cdots$ say that each equivalence class is infinite.

Prove that $Cn\Sigma = \{\phi \mid \Sigma \vdash \phi\}$ is complete.

**Solution:** First, we show that all countable models of this theory are isomorphic. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are two countable models. The equivalence classes of $E^{\mathfrak{A}}$ partition $|\mathfrak{A}| = A_1 \cup A_2$ and the equivalence classes of $E^{\mathfrak{B}}$ partition $|\mathfrak{B}| = B_1 \cup B_2$. The axioms of $\Sigma$ ensure that $A_1, A_2, B_1, B_2$ are all infinite, and since both structures are countable, all four sets are countably infinite.

Fix bijections $\pi_1 : A_1 \rightarrow B_1$ and $\pi_2 : A_2 \rightarrow B_2$. Then $\pi = \pi_1 \cup \pi_2$ is a bijection between $|\mathfrak{A}|$ and $|\mathfrak{B}|$. $\pi$ is an isomorphism, since $(a, a') \in E^{\mathfrak{A}}$ if and only if $a, a'$ are from the same part $A_i$, if and only if $\pi(a), \pi(a')$ are from the same part $B_i$, if and only if $(\pi(a), \pi(a')) \in E^{\mathfrak{B}}$.

On to the main claim. Suppose $Cn\Sigma$ were not complete. Then there would be some $\phi$ so that both $\Sigma \cup \{\phi\}$ and $\Sigma \cup \{\neg\phi\}$ are consistent. Then, by completeness, there would be structures $\mathfrak{A} \models \Sigma \cup \{\phi\}$ and $\mathfrak{B} \models \Sigma \cup \{\neg\phi\}$. Both structures must be infinite, since they satisfy $\Sigma$, so by DLS, there are countably infinite structures $\mathfrak{A}' \prec \mathfrak{A}$ and $\mathfrak{B}' \prec \mathfrak{B}$.

But then $\mathfrak{A}' \cong \mathfrak{B}'$, $\mathfrak{A}' \models \phi$, and $\mathfrak{B}' \models \neg\phi$, which is a contradiction.