PRACTICE MIDTERM SOLUTIONS

Math 570

Name: ______________________________

ID: ______________________________

"I have adhered to the Penn Code of Academic Integrity in completing this exam."

Signature: ______________________________

Read all of the following information before starting the exam:

- Check your exam to make sure all pages are present.
- You may use any result proved in class or in the textbook in your arguments.
- Circle or otherwise indicate your final answers.
- Good luck!

1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

1. (*20 points*) One of the following is valid, and the other is not. Give a deduction of the valid one and prove that there is no deduction of the other.

$$\forall x\phi \to \exists x\phi, \qquad \exists x\phi \to \forall x\phi.$$

Solution: An example of a valid deduction of $\forall x\phi \to \exists x\phi$ is:

$$\begin{array}{ll} \forall x\phi \to \phi & (Q1, \text{since } \phi^x_x \text{ is the same as } \phi) \\ \phi \to \exists x\phi & (Q2) \\ \forall x \to \exists x\phi & (PC) \end{array}$$

If we don't want to use Q2 on grounds of its redundancy, we could instead give

$$\begin{array}{ll} \forall x\phi \to \phi & (Q1) \\ \forall x\neg\phi \to \neg\phi & (Q1) \\ \forall x\phi \to \exists x\neg\phi & (PC) \end{array}$$

(Because $\exists x\neg\phi$ is the same as $\neg\forall x\neg\phi$, so the propositional form is $\{A \to B, C \to \neg B\} \Rightarrow A \to \neg C$, which is a tautoloty.)
To see that $\nvdash \exists x\phi \to \forall x\phi$, we can use soundness and show $\nvDash \exists x\phi \to \forall x\phi$. When ϕ $x > 5$, the standard model $\mathbb{N}$ has $\mathbb{N} \nvDash \exists x\phi \to \forall x\phi$.

2. (*20 points*) Consider a language with a unary function symbol $\mathbf{g}$ and a binary predicate symbol $\mathbf{P}$. Give two examples of models which are not elementarily equivalent.

Solution: Let $\mathfrak{A}$ be the model with $|\mathfrak{A}| = \mathbb{N}$, $\mathbf{P}^{\mathfrak{A}} = \{0\}$, and $\mathbf{g}^{\mathfrak{A}}$ the identity function.
Let $\mathfrak{B}$ be the model with $|\mathfrak{B}| = \mathbb{N}$, $\mathbf{P}^{\mathfrak{B}} = \{0\}$, and $\mathbf{g}^{\mathfrak{B}}(n) = n + 1$.
Then $\mathfrak{A} \vDash \forall x \mathbf{g} x = x$ while $\mathfrak{B} \vdash \not\forall x \mathbf{g} x = x$.
(There are lots and lots of examples of pairs of structures which are not elementarily equivalent. A complete answer definitely needs two fully defined structures and an example of a formula satisfied by one but not the other.)

3. (*20 points*) Consider a language with a single constant symbol $\mathbf{e}$, a binary function symbol $\mathbf{f}$, and a unary predicate symbol $\mathbf{Q}$. Let $\mathfrak{A}$ be a model with:

- $|\mathfrak{A}| = \mathbb{Q}$,
- $\mathbf{e}^{\mathfrak{A}} = 0$,
- $\mathbf{f}^{\mathfrak{A}}(q, r) = q - r$,
- $\mathbf{Q}^{\mathfrak{A}} = \{1/z \mid z \in \mathbb{Z} \setminus 0\}$.

Let $s(v_i) = 1/i$.

(**a**) What is $\overline{s}(e)$?

Solution: 0

(**b**) What is $\overline{s}(\mathbf{f}v_2v_3)$?

Solution: 1/6

(**c**) Does $\models_{\mathfrak{A}} \mathbf{Q}fv_2v_3[s]$ hold?

Solution: Yes

(**d**) Does $\models_{\mathfrak{A}} \forall v_4(\mathbf{Q}v_4 \to \mathbf{Q}fev_4)[s]$ hold?

Solution: Yes: If $\mathfrak{A} \models \mathbf{Q}v_4[s']$ then $v_4 = 1/z$ for some z, so $\overline{s'}(fev_4) = -1/z = 1/(-z)$, so $\mathfrak{A} \models \mathbf{Q}fev_4[s']$.

4. (*20 points*) You have a computer terminal into which you can type 0's and 1's. At any point (that is, after you have typed some finite sequence), it may reject the input with an error. (For instance, you might type 0, then 1, and then 0, at which point it gives an error, so it rejects the sequence 010.) Note that if a sequence is rejected, all longer sequences are also rejected (since if you tried to type them, you'd get stopped along the way).

Suppose that for every possible length, there is at least one input of that which is not rejected. (And therefore all initial segments must also not be rejected.) Prove that there is a sequence of infinite length which is not rejected. (That is, every finite initial segment of this infinite sequence is not rejected.)

Solution: *The clue here is that to get from something about finite lengths to infinite lengths, the only tool we have around is the Compactness Theorem. In order to use it, we have to figure out how to encode this situation as a set of sentences so that any model of these sentences is describing an infinite sequence.*

We could take a language with infinitely many 1-ary predicate symbols, $A_1, A_2, \dots, A_n, \dots$ and a constant symbol c. Let us write $A_i^1 c$ to mean just $A_i c$ and $A_1^0 c$ to mean $\neg A_i c$. Whenever $b_1 \cdots b_n$ is rejected, Σ should contain the sentence

$$\neg(A_1^{b_1} c \wedge \cdots A_n^{b_n} c).$$

(For instance, if 010 is rejected, Σ contains $\neg(\neg A_1 c \wedge A_2 c \wedge A_3 c)$.)

For any finite subset $\Sigma_0 \subseteq \Sigma$, only finitely many predicate symbols $A_1, \dots, A_n$ appear. There is some sequence $b_1, \dots, b_n$ which is not rejected. Consider the structure $\mathfrak{B}$ where $|\mathfrak{B}|$ is a single element: $|\mathfrak{B}| = \{*\}$, $c^{\mathfrak{B}} = *$, and $* \in A_i^{\mathfrak{B}}$ exactly if $b_i = 1$. Since this path is not rejected and no initial segment is rejected, $\mathfrak{B}$ satisfies every formula in Σ_0.

So Σ is finitely satisfiable, and therefore satisfiable. Let $\mathfrak{B}$ satisfy Σ and take the sequence with $b_i = 1$ if and only if $\mathfrak{B} \vDash A_i c$. Then $b_1 b_2 \cdots$ is an infinite sequence which is not rejected.

5. (*20 points*) Consider a language with a single unary function symbol f, a single unary predicate symbol P, and $=$. Let Σ be the sentences:

- $\forall x f f x = x$,
- $\forall x P x \leftrightarrow \neg P f x$
- $\exists x_1 \exists x_2 x_1 \neq x_2$,
- $\exists x_1 \exists x_2 \exists x_3 x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3$,
- $\cdots$

(The last two sentences and $\cdots$ collectively say there are infinitely many elements.) Prove that $Cn\Sigma$ is complete.

Solution: Consider a countable model of Σ, $\mathfrak{A}$. It must contain infinitely many elements. Since $f^{\mathfrak{A}}$ is its own inverse, $f^{\mathfrak{A}}$ must be injective and surjective. So $f^{\mathfrak{A}}$ is a bijection and, by the second axiom, is a bijection between $P^{\mathfrak{A}}$ and $|\mathfrak{A}| \setminus P^{\mathfrak{A}}$.

This gives a complete description of a countable model of Σ: $f^{\mathfrak{A}}$ is a bijection between two disjoint infinite sets, $P^{\mathfrak{A}}$ and $|\mathfrak{A}| \setminus P^{\mathfrak{A}}$.

Therefore any two countable models of Σ are isomorphic: if $\mathfrak{A}$ and $\mathfrak{B}$ are models of Σ, pick any bijection $\pi : P^{\mathfrak{A}} \to P^{\mathfrak{B}}$ (which exists since both sets are countably infinite), and then for $a \in P^{\mathfrak{A}}$, define $\pi(f^{\mathfrak{A}}(a)) = f^{\mathfrak{B}}(\pi(a))$.

(We could also give an argument analogous to the one we saw in class, building an isomrophism between two arbitrary countable models by a back-and-forth argument.

Suppose $Cn\Sigma$ were not complete, so there is some ϕ so that both $\Sigma \cup \{\phi\}$ and $\Sigma \cup \{\neg\phi\}$ are consistent. Then, by completeness, there are models $\mathfrak{A}^+ \models \Sigma \cup \{\phi\}$ and $\mathfrak{B}^+ \models \Sigma \cup \{\neg\phi\}$. These must be infinite because they are models of Σ, so by DLS, there are countable models $\mathfrak{A} \prec \mathfrak{A}^+$ and $\mathfrak{B} \prec \mathfrak{B}^+$. We have just shown that $\mathfrak{A} \cong \mathfrak{B}$. This is a contradiction, since $\mathfrak{A} \models \phi$ and $\mathfrak{B} \models \neg\phi$.