Math 3A — Midterm 1 Solutions

10/20/2010

Read all of the following information before starting the exam:

- Check your exam to make sure all pages are present.
- When you use a major theorem (like the indermediate value theorem or the sandwich theorem), make sure to note its use. (You do not need to explicitly mention the limit laws or the product, chain, etc. rules for derivatives.)
- You may use writing implements and a single 3"x5" notecard.
- NO CALCULATORS!
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Good luck!

1. (*24 points*) For each of the following limits, if the limit exists, say what value the limit converges to. If the limit does not exist, indicate this, and (if applicable) indicate whether the limit goes to positive or negative infinity. You may use any method for finding these limits. You should indicate what method you use for each problem, but need not show additional work.

(a) $\lim_{x\to 4} \frac{1}{x-3}$

Solution.
Since the function $f(x) = \frac{1}{x-3}$ is well-defined and continuous at 4, we can simply plug in $x = 4$ to compute the limit.

$$\lim_{x\to 4} \frac{1}{x-3} = \frac{1}{4-3} = \frac{1}{1} = 1$$

(b) $\lim_{z\to 3} \frac{z^2-2z-3}{z^2-9}$

Solution.
Observe that for $z \neq 3$ we have

$$\frac{z^2-2z-3}{z^2-9} = \frac{(z-3)(z+1)}{(z-3)(z+3)} = \frac{z+1}{z+3},$$

and therefore

$$\lim_{z\to 3} \frac{z^2-2z-3}{z^2-9} = \lim_{z\to 3} \frac{z+1}{z+3}.$$

The latter limit can be computed by plugging in $z = 3$, as the function $f(z) = \frac{z+1}{z+3}$ is well-defined and continuous at 3.

$$\lim_{z\to 3} \frac{z+1}{z+3} = \frac{3+1}{3+3} = \frac{2}{3}$$

(c) $\lim_{x\to 0} \sin|x|$

Solution.
Since the function $f(x) = \sin|x|$ is a composition of two continuous functions, $f = g \circ h$, $g(x) = \sin x$, $h(x) = |x|$, it is also well-defined and continuous at 0. Thus we can simply plug in $x = 0$ to compute the limit.

$$\lim_{x\to 0} \sin|x| = \sin|0| = \sin 0 = 0$$

(d) $\lim_{x\to 0} \frac{\sin 4x}{2x}$

Solution.

$$\lim_{x\to 0} \frac{\sin 4x}{2x} = \lim_{x\to 0} \left(2 \cdot \frac{\sin 4x}{4x}\right) = \text{using limit laws} = 2 \lim_{x\to 0} \frac{\sin 4x}{4x} = 2 \cdot 1 = 2$$

Above we have used the well-known limit $\lim_{t\to 0} \frac{\sin t}{t} = 1$.

(e) $\lim_{n\to\infty} n\cos(2\pi n)$

Solution.

Observe that $\cos(2\pi n) = 1$ for every positive integer n. Therefore

$$\lim_{n\to\infty} n\cos(2\pi n) = \lim_{n\to\infty} n = +\infty.$$

(f) $\lim_{x\to\infty} e^{-1/x}$

Solution.

Since $\lim_{x\to\infty}(-\frac{1}{x}) = 0$, and $\lim_{y\to 0} e^y = e^0 = 1$, we have

$$\lim_{x\to\infty} e^{-1/x} = 1.$$

(g) $\lim_{x\to 0^-}\ e^{-1/x}$

Solution.

Since $\lim_{x\to 0^-}(-\frac{1}{x}) = +\infty$, and $\lim_{y\to+\infty} e^y = +\infty$, we have

$$\lim_{x\to 0^-} e^{-1/x} = +\infty.$$

(h) $\lim_{x\to 0^+}\ e^{-1/x}$

Solution.

Since $\lim_{x\to 0^+}(-\frac{1}{x}) = -\infty$, and $\lim_{y\to-\infty} e^y = 0$, we have

$$\lim_{x\to 0^+} e^{-1/x} = 0.$$

2. (*14 points*) Use the sandwich theorem to find the following limit:

$$\lim_{x\to-\infty} \frac{\sin(2x)}{4x^2+7}$$

Solution.
One always has

$$-1 \le \sin(2x) \le 1.$$

Dividing this inequality by $4x^2+7$ (which is positive), we obtain

$$\frac{-1}{4x^2+7} \le \frac{\sin(2x)}{4x^2+7} \le \frac{1}{4x^2+7}$$

Since

$$\lim_{x\to-\infty} \frac{1}{4x^2+7} = \lim_{x\to-\infty} \frac{\frac{1}{x^2}}{4+\frac{7}{x^2}} = \text{using limit laws} = \frac{0^2}{4+7\cdot 0^2} = 0$$

and

$$\lim_{x\to-\infty} \frac{-1}{4x^2+7} = -\lim_{x\to-\infty} \frac{1}{4x^2+7} = 0,$$

the sandwich theorem gives

$$\lim_{x\to-\infty} \frac{\sin(2x)}{4x^2+7} = 0.$$

3. (*12 points*) What value of a makes the function f below continuous everywhere?

$$f(x) = \begin{cases} x^2+1 & \text{if } x > 2 \\ x+a & \text{if } x \le 2 \end{cases}$$

Solution.

The function is clearly continuous on $x < 2$ and $x > 2$ for any choice of a. Therefore we have to find a for which the function is also continuous at 2.

$$\begin{aligned} &f(2) = 2 + a \\ &\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x+a) = 2 + a \\ &\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^2+1) = 2^2 + 1 = 5 \end{aligned}$$

All three of these quantities have to be equal, and so $2 + a = 5$, which gives $a = 3$.

4. (*10 points*) One day, you leave your house, which is 900 meters from Bruin Plaza, early in the morning when it is $60°F$. By the time you reach your classroom 30 minutes later, which is 70 meters from Bruin Plaza, the temperature has risen to $80°F$. Show that there is some time during your walk when the temperature outside (in degrees Fahrenheit) is the same as your distance from Bruin Plaza (in meters). (Hint: use the Intermediate Value Theorem.)
Solution.
Denote

$$\begin{aligned} f(t) &= \text{ our current distance from Bruin Plaza (in meters) at time } t \\ g(t) &= \text{ outside temperature (in degrees Fahrenheit) at time } t \end{aligned}$$

Let us also assume that time t is measured in minutes, and that we start measuring time at the moment we leave the house. Then the information given in the problem can be written as:

$$\begin{aligned} &f(0) = 900, \quad f(30) = 70 \\ &g(0) = 60, \quad\ \ g(30) = 80 \end{aligned}$$

Define a new function

$$h(t) = f(t) - g(t)$$

Since f and g are continuous (by physical arguments), the function h is also continuous, as a difference of two continuous functions. Furthermore

$$\begin{aligned} h(0) &= 900 - 60 = 840 > 0, \\ h(30) &= 70 - 80 = -10 < 0, \end{aligned}$$

so by the Intermediate Value Theorem there exists t_0 between 0 and 30 such that $h(t_0) = 0$, i.e. $f(t_0) = g(t_0)$. This is exactly what we had to prove.

5. (*16 points*) Find the following derivatives using any method.

(a) $\frac{d}{dx}(x^3 + 2x + 1)$

Solution.

$$\frac{d}{dx}(x^3 + 2x + 1) = 3x^2 + 2$$

(b) $\frac{d}{dx}(x^8 + 9x^7 - 3x^6 + 14x^5 + 8x + 4 - \frac{7}{x} - \frac{4}{x^2})$

Solution.

$$\begin{aligned} &\frac{d}{dx}(x^8 + 9x^7 - 3x^6 + 14x^5 + 8x + 4 - 7x^{-1} - 4x^{-2}) \\ &= 8x^7 + 63x^6 - 18x^5 + 70x^4 + 8 + 7x^{-2} + 8x^{-3} \end{aligned}$$

(c) $\frac{d}{dy}\frac{y}{y^2+1}$

Solution.

We use the quotient rule.

$$\frac{d}{dy}\frac{y}{y^2+1} = \frac{1 \cdot (y^2+1) - y \cdot 2y}{(y^2+1)^2} = \frac{1-y^2}{(y^2+1)^2}$$

(d) Find $\frac{dy}{dx}$ when $y^2 + 2y = x$

Solution.

Differentiating both sides

$$\frac{d}{dx}(y^2 + 2y) = \frac{d}{dx}(x)$$

and using the chain rule: $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$, we get

$$2y\frac{dy}{dx} + 2\frac{dy}{dx} = 1.$$

We can factor out $\frac{dy}{dx}$ to obtain

$$\frac{dy}{dx} = \frac{1}{2y+2}.$$

Alternative solution.

We can solve the quadratic equation $y^2 + 2y - x = 0$ for y to get

$$y = \frac{-2 \pm \sqrt{4+4x}}{2} = -1 \pm \sqrt{x+1}.$$

Now we differentiate directly:

$$\frac{dy}{dx} = \frac{d}{dx}\Big(-1 \pm (x+1)^{1/2}\Big) = \pm\frac{1}{2}(x+1)^{-1/2}$$

6. (*8 points*) Find the tangent line to the following curves when $x = 0$:

(a) $y = x^3 + 2x + 1$

Solution.
We first compute:

$$y(0) = 0^3 + 2 \cdot 0 + 1 = 1$$
$$y'(x) = 3x^2 + 2$$
$$y'(0) = 3 \cdot 0^2 + 2 = 2$$

The equation of the tangent line at $x = 0$ is

$$y - y(0) = y'(0)(x - 0),$$

and by plugging in the numbers we obtain

$$y - 1 = 2(x - 0),$$

i.e.

$$y = 2x + 1.$$

(b) $y = \frac{x}{x^2+1}$

Solution.
We first compute using the quotient rule or problem 5 (c):

$$y(0) = \frac{0}{0^2 + 1} = 0$$
$$y'(x) = \frac{1 - x^2}{(x^2 + 1)^2}$$
$$y'(0) = \frac{1 - 0^2}{(0^2 + 1)^2} = 1$$

The equation of the tangent line at $x = 0$ is

$$y - y(0) = y'(0)(x - 0),$$

and by plugging in the numbers we obtain

$$y - 0 = 1(x - 0),$$

i.e.

$$y = x.$$

7. (*12 points*) Find $\frac{d}{dx}\frac{1}{\sqrt{x+1}}$ using *the definition of the derivative.*

Solution.

$$\begin{aligned}
\frac{d}{dx}\frac{1}{\sqrt{x+1}} &= \lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h+1}}-\frac{1}{\sqrt{x+1}}}{h}\\
&= \lim_{h\to 0}\frac{\sqrt{x+1}-\sqrt{x+h+1}}{h\,\sqrt{x+h+1}\,\sqrt{x+1}}\\
&= \lim_{h\to 0}\frac{\sqrt{x+1}-\sqrt{x+h+1}}{h\,\sqrt{x+h+1}\,\sqrt{x+1}}\cdot\frac{\sqrt{x+1}+\sqrt{x+h+1}}{\sqrt{x+1}+\sqrt{x+h+1}}\\
&= \lim_{h\to 0}\frac{(\sqrt{x+1})^2-(\sqrt{x+h+1})^2}{h\,\sqrt{x+h+1}\,\sqrt{x+1}\,(\sqrt{x+1}+\sqrt{x+h+1})}\\
&= \lim_{h\to 0}\frac{(x+1)-(x+h+1)}{h\,\sqrt{x+h+1}\,\sqrt{x+1}\,(\sqrt{x+1}+\sqrt{x+h+1})}\\
&= \lim_{h\to 0}\frac{-h}{h\,\sqrt{x+h+1}\,\sqrt{x+1}\,(\sqrt{x+1}+\sqrt{x+h+1})}\\
&= \lim_{h\to 0}\frac{-1}{\sqrt{x+h+1}\,\sqrt{x+1}\,(\sqrt{x+1}+\sqrt{x+h+1})}\\
&= \frac{-1}{\sqrt{x+0+1}\,\sqrt{x+1}\,(\sqrt{x+1}+\sqrt{x+0+1})}\\
&= \frac{-1}{2(\sqrt{x+1})^3}
\end{aligned}$$

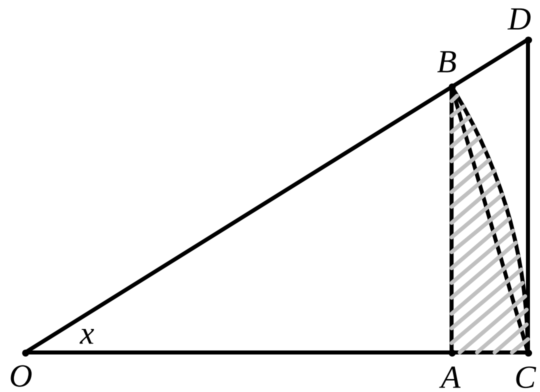
8. (*14 points*)

(a) In the proof that $\lim_{x\to 0} \frac{\sin x}{x} = 0$, we use a diagram to geometrically show relationships between certain quantities. Use a similar diagram to show that when x is a small positive number,

$$\frac{1}{2}\sin x(1-\cos x) \leq \frac{1}{2}(x - \sin x\cos x) \leq \frac{1}{2}(1-\cos x)(\sin x + \tan x).$$

(Hint: the formula for the area of a trapezoid is $\frac{1}{2} \times \text{base} \times (\text{height1} + \text{height2})$.)

Solution.



Observe that in the above picture we have

$$OB = OC = 1, \quad OA = \cos x, \quad AB = \sin x, \quad CD = \tan x.$$

We compute the following areas:

$$\begin{aligned}
&\text{area(triangle } ABC) = \frac{1}{2}AC \cdot AB = \frac{1}{2}(1-\cos x)\sin x \\
&\text{area(shaded shape } ABC) = \text{area(sector } OBC) - \text{area(triangle } OAB) \\
&\qquad = \frac{x}{2\pi}\text{area(full circle)} - \frac{1}{2}OA \cdot AB = \frac{x}{2\pi} \cdot \pi 1^2 - \frac{1}{2}\cos x \sin x \\
&\qquad = \frac{1}{2}(x - \sin x \cos x) \\
&\text{area(trapezoid } ABDC) = \frac{1}{2}AC(AB + CD) = \frac{1}{2}(1-\cos x)(\sin x + \tan x)
\end{aligned}$$

Therefore, the inequality follows from a geometrically evident one:

$$\text{area(triangle } ABC) \leq \text{area(shaded shape } ABC) \leq \text{area(trapezoid } ABDC).$$

(b) Use the inequality above to show that $\lim_{x\to 0^+} \frac{x-\sin x}{1-\cos x} = 0$. (Note that you don't need to complete the first part to do this part.)

Solution.
Let us divide the inequality by $\frac{1}{2}(1-\cos x)$:

$$\sin x \leq \frac{x - \sin x \cos x}{1-\cos x} \leq \sin x + \tan x.$$

We then subtract $\sin x$ from each part to obtain:

$$0 \leq \frac{x - \sin x \cos x}{1-\cos x} - \sin x \leq \tan x.$$

Since

$$\frac{x - \sin x \cos x}{1-\cos x} - \sin x = \frac{x - \sin x \cos x - \sin x(1-\cos x)}{1-\cos x} = \frac{x-\sin x}{1-\cos x}$$

we have

$$0 \leq \frac{x-\sin x}{1-\cos x} \leq \tan x.$$

Since $\lim_{x\to 0^+} \tan x = 0$ and $\lim_{x\to 0^+} 0 = 0$, the sandwich theorem gives $\lim_{x\to 0^+} \frac{x-\sin x}{1-\cos x} = 0$.

V.K.