Homework 4

3.4, 1. Show that $-x^2 \leq x^2 \cos\frac{1}{x} \leq x^2$ holds for $x \neq 0$.
Solution: Since $-1 \leq \cos\frac{1}{x} \leq 1$, multiply all three parts by $x^2 > 0$, we get:

$$-x^2 \leq x^2 \cos\frac{1}{x} \leq x^2,$$

and since $\lim_{x\to 0} x^2 = \lim_{x\to 0}(-x^2) = 0$, then by Sandwich theorem, we get:

$$\lim_{x\to 0} x^2 \cos\frac{1}{x} = 0.$$

2b. Use the sandwich theorem to show that $\lim_{x\to 0} x \cos\frac{1}{x} = 0$.
Solution: Since $-1 \leq \cos\frac{1}{x} \leq 1$. If $x > 0$, multiply all three parts by $x > 0$, we get:

$$-x \leq x \cos\frac{1}{x} \leq x,$$

and since $\lim_{x\to 0+} x = \lim_{x\to 0+}(-x) = 0$, then by Sandwich theorem, we get:

$$\lim_{x\to 0+} x \cos\frac{1}{x} = 0.$$

If $x < 0$,multiply all three parts by $x < 0$, we get:

$$-x \geq x \cos\frac{1}{x} \geq x,$$

and since $\lim_{x\to 0-} x = \lim_{x\to 0-}(-x) = 0$, then by Sandwich theorem, we get:

$$\lim_{x\to 0-} x \cos\frac{1}{x} = 0.$$

Therefore, we get

$$\lim_{x\to 0} x \cos\frac{1}{x} = 0.$$

7.

$$\lim_{x\to 0} \frac{\sin 5x}{x} = \lim_{x\to 0} 5 \cdot \frac{\sin 5x}{5x} = 5 \lim_{x\to 0} \frac{\sin 5x}{5x} = 5.$$

8.

$$\lim_{x\to 0} \frac{\sin x}{-x} = -\lim_{x\to 0} \frac{\sin x}{x} = -1.$$

3.5, 5. Use the intermediate-value theorem to show that $e^{-x} = x$ has a solution in $(0, 1)$.
Solution: Let $f(x) = e^{-x} - x$, then $f(0) = 1 > 0$, and $f(1) = e^{-1} - 1 < 0$, thus by I-V theorem, there is c in $(0, 1)$, such that $f(c) = 0$, that is $e^{-c} = c$.

6. Use the intermediate-value theorem to show that $\cos x = x$ has a solution in $(0, 1)$.
Solution: Let $f(x) = \cos x - x$, then $f(0) = 1 > 0$, and $f(1) = \cos 1 - 1 < 0$, thus by I-V theorem, there is c in $(0, 1)$, such that $f(c) = 0$, that is $\cos c = c$.

13 Explain why a polynomial of degree 3 has at least one root.
Solution: Suppose $f(x) = ax^3 + bx^2 + cx + d$ is a polynomial of degree 3, where a, b, c, d are constants.
If $a > 0$ then, there is a number $t \gg 0$ big enough, such that $f(t) > 0$, and there is a number $s \ll 0$ small

enough, such that $f(s) < 0$, therefore by I-V theorem, we get there is l in (s,t), satisfies that $f(l) = 0$, that is $f(x)$ has at least one root l.
Similarly, if $a < 0$ then, there is a number $t \gg 0$ big enough, such that $f(t) < 0$, and there is a number $s \ll 0$ small enough, such that $f(s) > 0$, therefore by I-V theorem, we get there is l in (s,t), satisfies that $f(l) = 0$, that is $f(x)$ has at least one root l.

14 Explain why a polynomial of degree n, where n is an odd number has at least one root.
Solution: Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ is a polynomial of degree n, where $a_n, a_{n-1}, \cdots, a_0$ are constants.
Because n is odd, if $a_n > 0$ then, there is a number $t \gg 0$ big enough, such that $f(t) > 0$, and there is a number $s \ll 0$ small enough, such that $f(s) < 0$, therefore by I-V theorem, we get there is l in (s,t), satisfies that $f(l) = 0$, that is $f(x)$ has at least one root l.
Similarly, if $a_n < 0$ then, there is a number $t \gg 0$ big enough, such that $f(t) < 0$, and there is a number $s \ll 0$ small enough, such that $f(s) > 0$, therefore by I-V theorem, we get there is l in (s,t), satisfies that $f(l) = 0$, that is $f(x)$ has at least one root l.

15. Explain why $y = x^2 - 4$ has at least two roots.
Solution: Since $y(3) = 5 > 0$, and $y(1) = -3 < 0$, then by I-V theorem, there is a number c_1 in $(1,3)$, such that $y(c_1) = c_1^2 - 4 = 0$. Also since $y(-3) = 5 > 0$, and $y(-1) = -3 < 0$, then by I-V theorem, there is a number c_2 in $(-3,-1)$, such that $y(c_2) = c_2^2 - 4 = 0$, and because c_1 and c_2 are in disjoint sets respectively, so they are different, therefore $y = x^2 - 4$ has at least 2 roots $c_1,\ c_2$.

4.1, 25. Use the formal definition to find the derivative of $y = \sqrt{x}$, for $x > 0$.
Solution: Denote $f(x) = \sqrt{x}$, then by the formal definition of derivative, for $x > 0$, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

26. Use the formal definition to find the derivative of $f(x) = \frac{1}{x+1}$ for $x \neq -1$.
Solution: Denote $f(x) = \sqrt{x}$, then by the formal definition of derivative, for $x \neq -1$, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+1+h} - \frac{1}{x+1}}{h} = \lim_{h\to 0} \frac{\frac{x+1-(x+1+h)}{(x+1+h)(x+1)}}{h} \\ &= \lim_{h\to 0} \frac{-h}{h(x+1+h)(x+1)} = \lim_{h\to 0} \frac{-1}{(x+1+h)(x+1)} = \frac{-1}{(x+1)^2}. \end{aligned}$$

27. Find the equation of the tangent line to the curve $y = 3x^2$ at the point $(1,3)$.
Solution: $y'(x) = 6x$, then $y'(1) = 6$, which is the slope of the tangent line. Since the tangent line pass the point $(1,3)$, thus the equation of the tangent line is: $y - 3 = 6(x - 1)$, that is $y = 6x - 3$.

28. Find the equation of the tangent line to the curve $y = 2/x$ at the point $(2,1)$.
Solution: $y'(x) = -\frac{2}{x^2}$, then $y'(2) = -\frac{1}{2}$, which is the slope of the tangent line. Since the tangent line pass the point $(2,1)$, thus the equation of the tangent line is: $y - 1 = -\frac{1}{2}(x - 2)$, that is $y = -\frac{1}{2}x + 2$.

51. Which of the following statements is true:
(A) If $f(x)$ is continuous, then $f(x)$ is differentiable.
(B) If $f(x)$ is differentiable, then $f(x)$ is continuous.
Solution: (A) is false. Counterexample: $f(x) = x^{1/3}$ is continuous at $x = 0$, but not differentiable at $x = 0$.
(B) is true, the proof is in page 141-142.

52. Explain the relation between continuity and differentability.
Solution: If f is differentiable at $x = c$, then f is also continuous at $x = c$; but if f is continuous at $x = c$, then f need not be differentiable at $x = c$.

4.2, 7. Differentiate the function $g(s) = 5s^7 + 2s^3 - 5s$.
Solution: $g'(s) = 5 \cdot 7s^6 + 2 \cdot 3s^2 - 5 = 35s^6 + 6s^2 - 5$.

8.Differentiate the function $g(s) = 3 - 4s^2 - 4s^3$.
Solution: $g'(s) = -4 \cdot 2s - 4 \cdot 3s^2 = -8s - 12s^2$.

29. Differentiate $f(x) = rs^2x^3 - rx - s$ with respect to x. Assume that r and s are constant.
Solution:$f'(s) = rs^2 \cdot 3x^2 - r = 3rs^2x^2 - r$.

30 Differentiate $f(x) = \frac{r+x}{rs^2} - rsx + (r+s)x - rs$ with respect to x. Assume that r and s are nonzero constants.
Solution: $f'(x) = (\frac{r}{rs^2} - rs + \frac{x}{rs^2} - rsx + (r+s)x)' = \frac{1}{rs^2} - rs + r + s$.

81. Suppose that $P(x)$ is a polynomial of degree 4. Is $P'(x)$ a polynomial as well? If yes, what is the degree.
Solution: Yes. Because if $P(x) = ax^4 + bx^3 + cx^2 + dx + e$, where a, b, c, d, e are all constants. then $P'(x) = 4ax^3 + 3bx^2 + 2cx + d$, therefore degree of $P'(x)$ is 3.

82. Suppose that $P(x)$ is a polynomial of degree k. Is $P'(x)$ a polynomial as well? If yes, what is the degree?
Solution: Suppose $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, then $P'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1$, thus the degree of $P'(x)$ is $n-1$.

4.3, 31. Differentiate $f(x) = 2a(x^2 - a)^2 + a$ with respect to x. Assume that a is a positive constant.
Solution: Use the product rule to $f(x) = 2a(x^2 - a)(x^2 - a) + a$:

$$f'(x) = 2a \cdot 2x \cdot (x^2 - 2) + 2a \cdot 2x \cdot (x^2 - 2) = 8ax(x^2 - a) = 8ax^3 - 8a^2x.$$

32 Differentiate $f(x) = \frac{3(x-1)^2}{2+a}$ with respect to x. Assume that a is a positive constant.
Solution: Use the product rule to $f(x) = \frac{3}{2+a}(x-1)(x-1)$, we get:

$$f'(x) = \frac{3}{2+a}((x-1) + (x-1)) = \frac{6}{2+3}(x-1)$$

Differentiate the following functions:
67.$g(s) = \frac{s^{1/3}-1}{s^{2/3}-1}$.
Solution:

$$g'(s) = \frac{\frac{1}{3}s^{-2/3}(s^{2/3}-1) - \frac{2}{3}(s^{1/3}-1)s^{-1/3}}{(s^{2/3}-1)^2} = \frac{\frac{1}{3} - \frac{1}{3}s^{-2/3} - \frac{2}{3} + \frac{2}{3}s^{-1/3}}{(s^{2/3}-1)^2} = \frac{-\frac{1}{3} - \frac{1}{3}s^{-2/3} + \frac{2}{3}s^{-1/3}}{(s^{2/3}-1)^2}.$$

68.$gs = \frac{s^{1/7}-s^{2/7}}{s^{3/7}+s^{4/7}}$.
Solution:

$$\begin{aligned} g'(s) &= \frac{(\frac{1}{7}s^{-6/7} - \frac{2}{7}s^{-5/7})(s^{3/7} + s^{4/7}) - (s^{1/7} - s^{2/7})(\frac{3}{7}s^{-4/7} + \frac{4}{7}s^{-3/7})}{(s^{3/7} + s^{4/7})^2} \\ &= \frac{2}{7} \cdot \frac{-s^{-3/7} - s^{-2/7} + s^{-1/7}}{(s^{3/7} + s^{4/7})^2} \end{aligned}$$

Assume that $f(x)$ is differentiable. Find an expression for the derivative of y at $x = 2$, assuming that $f(2) = -1$ and $f'(2) = 1$:

84. $y = \frac{f(x)}{x^2+1}$.

Solution: $y'(x) = \frac{f'(x)(x^2+1)-2xf(x)}{(x^2+1)^2}$, then:

$$y'(2) = \frac{f'(2)(2^2+1) - 4f(2)}{(2^2+1)^2} = \frac{5+4}{25} = \frac{9}{25}.$$

85. $y = \frac{x^2+4f(x)}{f(x)}$

Solution:$y'(x) = \frac{(2x+4f'(x))f(x)-(x^2+4f(x))f'(x)}{(f(x)^2)}$, thus:

$$y'(2) = \frac{(4+4f'(2))f(2) - (2^2+4f(2))f'(2)}{(f(2)^2)} = \frac{8 \cdot (-1) - 0}{1} = -8.$$