Math 3A Homework 6 Solution (Fall 2010)

Section 4.6

61. Differentiating equations with respect to t gives

$$\frac{dN(t)}{dt} = N_0(\ln 2)2^t = (\ln 2)N(t),$$

which implies that $\frac{dN(t)}{dt}$ is proportional to $N(t)$.

62*. Differentiating equations with respect to t gives

$$\frac{dN(t)}{dt} = N_0(\ln 2)2^t = (\ln 2)N(t),$$

then the per capita growth rate is $\frac{1}{N(t)}\frac{dN(t)}{dt} = \ln 2$.

Section 4.7

9. $f(x) = \sqrt{x+1}, x \geq 0$

First $f'(x) = \frac{1}{2}\frac{1}{\sqrt{x+1}}$.

Next, set $y = \sqrt{x+1}$. Then solving this equation gives $x = y^2 - 1$. Interchanging x and y, we find

$$y = x^2 - 1.$$

Moreover, the range of $f(x)$ is $[1, \infty)$, so the inverse function is

$$f^{-1}(x) = x^2 - 1 \text{ for } x \geq 1.$$

Then

$$\frac{d}{dx}f^{-1}(x)|_{x=2} = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(3)} = 4.$$

10*. $f(x) = \sqrt{2+x^2}, x \geq 0$

First $f'(x) = \frac{1}{2}\frac{1}{\sqrt{2+x^2}}2x = \frac{x}{\sqrt{2+x^2}}$.

Next, set $y = \sqrt{2+x^2}, x \geq 0$. Then solving this equation gives $x = \sqrt{y^2-2}$. Interchanging x and y, we find

$$y = \sqrt{x^2-2}.$$

Moreover, the range of $f(x)$ is $[\sqrt{2}, \infty)$, so the inverse function is

$$f^{-1}(x) = \sqrt{x^2-2} \text{ for } x \geq \sqrt{2}.$$

Then

$$\frac{d}{dx}f^{-1}(x)|_{x=\sqrt{3}} = \frac{1}{f'(f^{-1}(\sqrt{3}))} = \frac{1}{f'(1)} = \sqrt{3}.$$

22*. Since $f(x) = \sin x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, we know the range of $f(x)$ is $[-1, 1]$.
Then $f'(x) = \cos x$ and we know the inverse function is $y = f^{-1}(x) = \arcsin x$, then $x = \sin y$. Moreover, we have $\sin^2 y + \cos^2 y = 1$, thus $\cos y = \sqrt{1 - \sin^2 y}$, since $\cos y \geq 0$, which is based on $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

$$\frac{d}{dx}\arcsin x = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}, \quad -1 < x < 1.$$

29. $f(x) = \ln(2x^3 - x)$

Applying the Chain rule gives

$$f'(x) = \frac{1}{2x^3 - x}\frac{d(2x^3 - x)}{dx} = \frac{6x^2 - 1}{2x^3 - x}.$$

30*. $f(x) = \ln(1 - x^3)$

Applying the Chain rule gives

$$f'(x) = \frac{1}{1 - x^3}\frac{d(1 - x^3)}{dx} = \frac{3x^2}{x^3 - 1}.$$

71. $y = x^{x^x}$

$$\ln y = x^x \ln x$$

Differentiating the equation gives

$$\frac{1}{y}\frac{dy}{dx} = x^{x-1} + x^x(\ln x + 1)\ln x,$$

from the textbook.
Then

$$\frac{dy}{dx} = x^{x^x}\{x^{x-1} + x^x(\ln x + 1)\ln x\}.$$

72*. $y = (x^x)^x$

$$\ln y = x \ln x^x = xx \ln x = x^2 \ln x$$

Differentiating the equation gives

$$\frac{1}{y}\frac{dy}{dx} = x + 2x\ln x.$$

Then

$$\frac{dy}{dx} = (x^x)^x(x + 2x\ln x).$$

Section 4.8

1. $f(x) = \sqrt{x}$

$$\sqrt{65} = f(65) \approx f(64) + f'(64)(65 - 64) = 8 + \frac{1}{2}\frac{1}{8}1 = 8.0625.$$

And the error is $|\sqrt{65} - 8.0625| = 2.42 \times 10^{-4}$.

2*. $f(x) = \sqrt{x}$

$$\sqrt{35} = f(35) \approx f(36) + f'(36)(35 - 36) = 6 - \frac{1}{2}\frac{1}{6}1 \approx 5.9167.$$

And the error is $|\sqrt{35} - 5.9167| = 5.87 \times 10^{-4}$.

12*. $f(x) = \frac{1}{1-x}$

We have $f'(x) = \frac{1}{(1-x)^2}$, and then

$$L(x) = f(0) + f'(0)(x - 0) = 1 + x.$$

13. $f(x) = \frac{2}{1+x}$

We have $f'(x) = -\frac{2}{(1+x)^2}$, and then

$$L(x) = f(1) + f'(1)(x - 1) = 1 - \frac{1}{2}(x - 1) = \frac{3}{2} - \frac{1}{2}x.$$

46*. $v(R) = cR^2$

$$\Delta v \approx v'(R)\Delta R = 2cR\Delta R$$

Then the percentage error $100|\frac{\Delta v}{v}|$ is related to the percentage error $100|\frac{\Delta R}{R}|$ as

$$100|\frac{\Delta v}{v}| \approx 100|\frac{v'(R)\Delta R}{v}| = 100|\frac{2cR\Delta R}{cR^2}| = 100|\frac{\Delta R}{R}|2.$$

If we require $100|\frac{\Delta R}{R}| = 5$, then $100|\frac{\Delta v}{v}| = 5 \times 2 = 10$.

Therefore, the accuracy of the speed is within 10%.

47. $N(L) = cL^{2.11}$ for some constant c

$$\Delta N \approx N'(L)\Delta L = 2.11cL^{1.11}\Delta L$$

Then the percentage error $100|\frac{\Delta N}{N}|$ is related to the percentage error $100|\frac{\Delta L}{L}|$ as

$$100|\frac{\Delta N}{N}| \approx 100|\frac{N'(L)\Delta L}{N}| = 100|\frac{2.11cL^{1.11}\Delta L}{cL^{2.11}}| = 100|\frac{\Delta L}{L}|2.11.$$

If we require $100|\frac{\Delta N}{N}| = 5$, then $100|\frac{\Delta L}{L}| = \frac{5}{2.11} = 2.37$.
Therefore, the accuracy of the speed is within 2.37%.

Section 5.1

21. $f(x) = -x^2$

Let

$$f'(x) = -2x = 0,$$

we have $x = 0$, and $f(x)$ has a local maximum at $x = 0$.

22* $f(x) = -(x+3)^2$

Let

$$f'(x) = -2(x+3) = 0,$$

we have $x = -3$, and $f(x)$ has a local maximum at $x = -3$.

31. $f(x) = |1 - |x||, \ -1 \leq x \leq 2$

$f(x)$ has local maximum 1 at $x = 0$ and 2;
$f(x)$ has local minimum 0 at $x = -1$ and 1;
$f(x)$ has global maximum 1 at $x = 0$ and 2;
$f(x)$ has global minimum 0 at $x = -1$ and 1.

32*. $f(x) = -||x| - 2|, \ -3 \leq x \leq 3$

$f(x)$ has local maximum 0 at $x = -2$ and 2;
$f(x)$ has local minimum -1 at $x = -3$ and 3, local minimum -2 at $x = 0$;
$f(x)$ has global maximum 0 at $x = -2$ and 2;
$f(x)$ has global minimum -2 at $x = 0$.