MIDTERM 2

Math 3A
11/17/2010

Name: ______________________

Section: ______________________

Signature: ______________________

Read all of the following information before starting the exam:

- Check your exam to make sure all pages are present.
- When you use a major theorem (like IVT, MVT, or the Extreme Value Theorem), make sure to note its use. (You do not need to explicitly mention the limit laws or the product, chain, etc. rules for derivatives.)
- You may use writing implements and a single 3"x5" notecard.
- NO CALCULATORS!
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Good luck!

1	20	
2	20	
3	15	
4	20	
5	15	
6	20	
Total	110	

1. (*20 points*) Find the following derivatives:

(a) $\frac{d}{dx}\cos(2x+1)$

$-2\mathtt{sin}(2\mathtt{x}+1)$

(b) $\frac{d^2}{dx^2}(x^7 - 4x^6 + 3x^2)$

$42x^5 - 120x^4 + 6$

(c) $\frac{d}{dx}e^{\sin\sqrt{\ln x}}$

$\frac{1}{2}e^{\sin(\sqrt{\ln x})}\cos\left(\sqrt{\ln x}\right)\frac{1}{\sqrt{\ln x}}\frac{1}{x}$

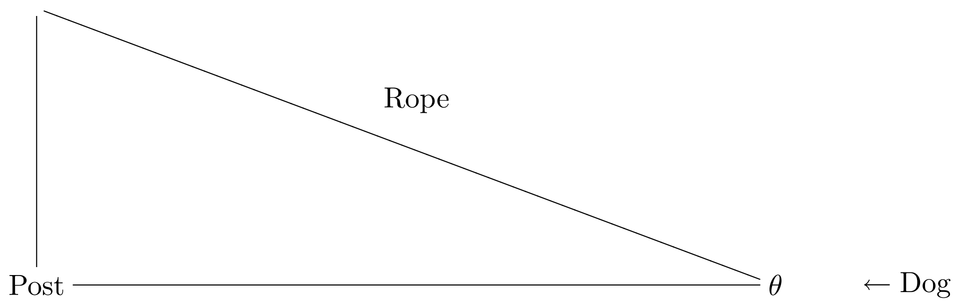
(**d**) $\frac{d}{dx}\frac{e^x(x^2+4+1)(x+4)^4}{(x-3)^2\tan x}$

$(1+\frac{2x}{x^2+4+1}+\frac{4}{x+4}-\frac{2}{x-3}-\sec^2 x\frac{1}{\tan x})\frac{e^x(x^2+4+1)(x+4^4)}{(x-3)^2\tan x}$

(**e**) $\frac{d^{78}}{dx^{78}}\sin 2x$

$-2^{78}\sin(2x)$

2. (*20 points*) A dog is tied to the top of a 1 foot tall post by a 50 foot piece of rope. (The rope is coiled on top of the post, and is slowly pulled off as needed, so you may assume that the rope always makes a straight line between the top of the post and the dog. Also, assume that the dog has 0 height, so that the part of the rope connected to the post is exactly one foot heigher than the end of the rope attached to the dog.) The dog begins running in a straight line at 20 feet per second.



(**a**) Write an equation giving a relationship between θ, the angle the rope makes with the ground, and the dog's distance from the post.

$\tan\theta = \frac{1}{s}$

(**b**) When the rope runs out, how quickly is the angle the rope makes with the ground changing? Do not leave unevaluated trig expressions in your answer.

$\tan\theta = \frac{1}{s}$

$\sec^2\theta \frac{d\theta}{dt} = -\frac{1}{s^2}\frac{ds}{dt}$

$\frac{d\theta}{dt} = -\frac{\cos^2\theta}{s^2}\frac{ds}{dt}$, $\cos\theta = \frac{s}{50}$, $\frac{ds}{dt} = 20$

$\frac{d\theta}{dt} = -\frac{1}{50^2}\frac{ds}{dt} = -\frac{1}{125}$

3. (*15 points*) Give an approximation of $(1.01)^{100}$ using a linearization.

Let $f(x) = x^{100}$, $a = 1$, $f'(x) = 100x^{99}$, so $f(1) = 1$, $f'(1) = 100$.
Therefore $L(x) = f(a) + f'(a)(x - a) = 1 + 100(x - 1)$
$1.01^{100} = f(1.01) \approx L(1.01) = 2$

4. (*20 points*) (**a**) Show that the equation $x^7 + 4x^5 + 8x^3 + 100x - 1000$ has at least one root.

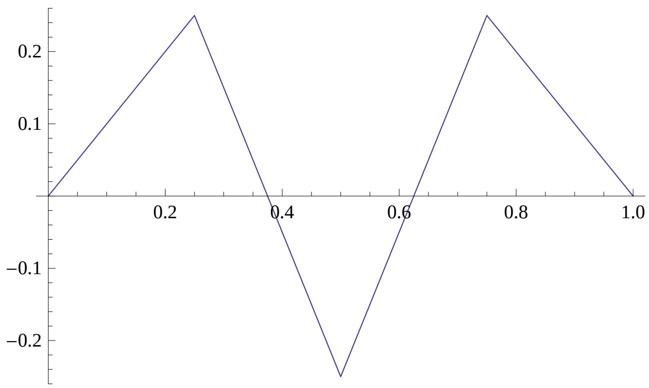
Since $\mathtt{f}(0) = -1000 < 0$, and $\mathtt{f}(10) = 10^7 + 4 \cdot 10^5 + 8 \cdot 10^3 > 0$, and $\mathtt{f}(x)$ is continuous on $[0, 10]$, thus by IVT, there is $a \in (0, 10)$, such that $f(a) = 0$. This proves $f(x)$ has at least one root.

(**b**) Show that the equation $x^7 + 4x^5 + 8x^3 + 100x - 1000$ has at most one root.

First notice that $f'(x) = 7x^6 + 20x^4 + 24x^2 + 100 > 0$, suppose there are at least two roots $a < b$, that is $f(a) = 0$, $f(b) = 0$, since $f(x)$ is continuous on $[a, b]$, and differentiable on (a, b) (because it is continuous and differentiable everywhere), thus by MVT, there exists c on (a, b) such that $f'(c) = 0$, which is contradiction. Therefore, there are only one root.

5. (*15 points*) Give (by sketching a graph) an example of a function f such that:

- f is defined and continuous on $[0, 1]$,
- $f(0) = f(1) = 0$,
- Neither 0 nor 1 is a global extremum of f,
- f has more local maxima in $(0, 1)$ than local minima,
- There is no c such that $f'(c) = 0$.



6. (*20 points*) The remainder of this exam concerns the function

$$f = \frac{(x+3)(x-2)^2}{(x+6)^2}.$$

The following information may be useful:

- $f'(x) = \frac{(x-2)(x+2)(x+18)}{(x+6)^3}$
- $f''(x) = \frac{32(7x+6)}{(x+6)^4}$
- The function f is equal to $x - 13 + \frac{112x+480}{(x+6)^2}$

(**a**) Identify the points where f is 0 or undefined, and the intervals where f is positive or negative.

$\mathtt{f(x)} > 0$ when $-6 < \mathtt{x} < -3$ $\mathtt{f(x)} < 0$ when $\mathtt{x} > -3$, and $\mathtt{x} < -6$

$\mathtt{f(x)}$ is UDF at $\mathtt{x} = -6$

(**b**) Identify the critical points of f and the intervals where f is increasing or decreasing.

$\mathtt{f'(x)} > 0$, and therefore f is increasing, when $\mathtt{x} < -18, -6 < \mathtt{x} < -2, \mathtt{and x} > 2$,

$\mathtt{f'(x)} < 0$, and therefore f is decreasing when $-18 < \mathtt{x} < -6$, and $-2 < \mathtt{x} < 2$.

(**c**) Identify the inflection points of f and the intervals where f is concave up or concave down.

$\mathtt{f''(x)} > 0$, and therefore f is concave up, when $\mathtt{x} > \frac{6}{7}$,

$\mathtt{f''(x)} < 0$, and therefore f is concave down, when $\mathtt{x} < -6$, $-6 < \mathtt{x} < -\frac{6}{7}$

(d) Determine $\lim_{x\to -6^+} f(x)$
$\lim_{\mathtt{x}\to -6^+} = -\infty$

(e) Determine $\lim_{x\to -6^-} f(x)$
$\lim_{\mathtt{x}\to -6^-} = -\infty$

(f) Determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty}[f(x)-(x-13)]$
$\lim_{\mathtt{x}\to\infty} \mathtt{f(x)} = \infty$, and $\lim_{\mathtt{x}\to\infty}[\mathtt{f(x)} - (\mathtt{x} - 13)] = 0$

(g) Determine $\lim_{x\to -\infty} f(x)$ and $\lim_{x\to -\infty}[f(x)-(x-13)]$
$\lim_{\mathtt{x}\to -\infty} \mathtt{f(x)} = -\infty$, and $\lim_{\mathtt{x}\to -\infty}[\mathtt{f(x)} - (\mathtt{x} - 13)] = 0$.

(h) Indicate all asymptotes (horizontal, vertical, or oblique) of f, and indicate which kind of asymptote each is.
The vertical asymptote is $\mathtt{x} = -6$, and the oblique asymptote is $\mathtt{y} = \mathtt{x} - 13$, and no horizontal asymptote.

(i) Sketch a graph of f. Be sure to label all zeros, critical points and inflection points with a dot and their x coordinate, and to indicate all asymptotes.