Math 3B, Midterm 1 Solutions

1. (a) We partition the interval [1,3] into four equal subintervals, hence the length of each subinterval is $\Delta x_k = \frac{3-1}{4} = \frac{1}{2}$, and the endpoints of the subintervals are $1, 1.5, 2, 2.5, 3$. Therefore the midpoints are $c_1 = 1.25, c_2 = 1.75, c_3 = 2.25, c_4 = 2.75$. The Riemann sum approximation to the integral using midpoints of four subintervals is thus given by

$$\begin{aligned}\int_1^3 \sin(x)\,dx &\approx \sum_{k=1}^{4} \sin(c_k)\Delta x_k \\ &= \frac{1}{2}(\sin(1.25) + \sin(1.75) + \sin(2.25) + \sin(2.75)).\end{aligned}$$

   (b) We partition the interval [0,1] into three equal subintervals, hence the width of each subinterval is $\Delta x_k = \frac{1-0}{3} = \frac{1}{3}$, and the endpoints of the subintervals are $0, \frac{1}{3}, \frac{2}{3}, 1$. The left endpoints of the three subintervals are $c_1 = 0$, $c_2 = \frac{1}{3}$, $c_3 = \frac{2}{3}$. The Riemann sum approximation to the integral using left endpoints of three subintervals is thus given by

$$\begin{aligned}\int_0^1 f(x)\,dx &\approx \sum_{k=1}^{3} f(c_k)\Delta x_k \\ &= \frac{1}{3}(f(0) + f(\tfrac{1}{3}) + f(\tfrac{2}{3})).\end{aligned}$$

2. (a) $\int_2^2 g(x)\,dx = 0$ by one the properties of integration.

   (b) We have

$$\int_0^3 g(x)\,dx = \int_0^1 g(x)\,dx + \int_1^3 g(x)\,dx,$$

   by one of the properties of integration, hence

$$\int_1^3 g(x)\,dx = \int_0^3 g(x)\,dx - \int_0^1 g(x)\,dx = 5 - 1 = 4.$$

(c) No. When x is in the interval $[3,5]$, we have that $g(x) \geq 2$ by the information given, hence

$$\int_3^5 g(x)\,dx \geq \int_3^5 2\,dx = 2(5-3) = 4.$$

Thus it is inconsistent with the given information that the integal is equal to 3.

(d) Yes. To have $\int_0^5 g(x)\,dx = 13$, it is necessary and sufficient to have

$$\int_3^5 g(x)\,dx = 13 - \int_0^3 g(x)\,dx = 13 - 5 = 8.$$

The restriction $2 \leq g(x) \leq 4$ does not rule out such a possibility: If $g(x) = 4$ for $x \in [3,5]$, then $\int_3^5 g(x)\,dx = 8$.

(It was not necessary on the exam to give an explicit example of a function $g(x)$ with the given properties (including continuity), but here is one:

$$g(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ \frac{1}{2}(x-1)+1, & 1 \leq x \leq 2 \\ \frac{5}{2}(x-2)+\frac{3}{2}, & 2 \leq x \leq 3 \\ 4, & 3 \leq x \leq 5 \end{cases} .)$$

3. (a)

$$\int e^x + x^2\,dx = e^x + \frac{1}{3}x^3 + C.$$

(b) Integrating by means of a substitution,

$$\begin{aligned} \int \frac{e^{1/x}}{x^2}\,dx &= \left| \begin{array}{l} u = 1/x \\ du = -1/x^2 dx \end{array} \right| \\ &= \int -e^u\,du \\ &= -e^u + C \\ &= -e^{1/x} + C. \end{aligned}$$

(c)

$$\int 7x^4 + 5x^2\,dx = \frac{7}{5}x^5 + \frac{5}{3}x^3 + C$$

(d)

$$\int \sec^2 x\,dx = \tan x + C.$$

4. (a)

$$\int_0^1 e^x + x^2\,dx = e^x + \frac{1}{3}x^3\Big|_0^1 = (e^1 + \frac{1}{3}) - (e^0 + 0) = e - \frac{2}{3}$$

$$(\approx 2.052)$$

(b) The function $\frac{e^{1/x}}{x^2}$ is not continuous on the interval $[-1, 1]$ (it is discontinuous at the point $x = 0$).

(c)

$$\int_2^4 7x^4 + 5x^2\,dx = \frac{7}{5}x^5 + \frac{5}{3}x^3\Big|_2^4 = \left(\frac{7}{5}\cdot 4^5 + \frac{5}{3}\cdot 4^3\right) - \left(\frac{7}{5}\cdot 2^4 + \frac{5}{3}\cdot 2^3\right)$$

$$(= \frac{22658}{15} \approx 1504.533)$$

(d)

$$\int_0^{\pi/4} \sec^2 x\,dx = \tan x\Big|_0^{\pi/4} = \tan(\pi/4) - \tan(0)$$

$$(= 1 - 0 = 1).$$

5. (a) Since $f(x) = e^{-x^2}\sin x$ is an odd function (i.e. $f(-x) = -f(x)$) and the interval of integration is symmetric about 0, we have

$$\int_{-2}^2 e^{-x^2}\sin x\,dx = 0.$$

(b) The graph of $\sqrt{4 - x^2}$ is the upper half of a circle of radius 2 centered at the origin in $\mathbf{R}^2$. $\int_{-2}^2 \sqrt{4 - x^2}\,dx$ is therefore half the area of this circle, hence equal to $\frac{1}{2}\pi \cdot 2^2 = 2\pi$.

6. To find the points of intersection between the two curves we set $8-x^6 = 7x^3$ and solve for x:

$$\begin{aligned}
8 - x^6 &= 7x^3 \\
x^6 + 7x^3 - 8 &= 0 \\
(x^3)^2 + 7x^3 - 8 &= 0 \\
(x^3+8)(x^3-1) &= 0 \\
x^3 &= -8, 1 \\
x &= -2, 1.
\end{aligned}$$

Testing a point in the interval [-2,1] we see that when $x = 0$ we have $8 - x^6 = 8 > 0 = 7x^3$, thus the curve $y = 8 - x^6$ is above the curve $y = 7x^3$ on this interval. Thus the area contained between the two curves is

$$\begin{aligned}
\int_{-2}^{1} (8 - x^6) - 7x^3\,dx &= 8x - \frac{1}{7}x^7 - \frac{7}{4}x^4 \Big|_{-2}^{1} \\
&= \left(8 - \frac{1}{7} - \frac{7}{4}\right) - \left(8(-2) - \frac{1}{7}(-2)^7 - \frac{7}{4}(-2)^4\right) \\
&\left(= \frac{891}{28} \approx 31.821\right).
\end{aligned}$$