We wish to find
$$\int \frac{8x^3 + 2x^2 + 15x + 1}{(2x-1)(4x^2+9)} dx.$$

Long Division

This is not a proper fraction, since the degrees of both the top and bottom are 1. So our first step is long division. We need to find the value of the denominator, which is
$$(2x-1)(4x^2+9) = 8x^3 - 4x^2 + 18x - 9.$$

Long division gives
$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad\quad\quad 1 \\ \hline 8x^3 - 4x^2 + 18x - 9 & \; 8x^3 + 2x^2 + 15x \;\; + 1 \\ & - 8x^3 + 4x^2 - 18x \;\; + 9 \\ \hline & \quad\quad 6x^2 \;\; - 3x + 10 \end{array}$$

So
$$\frac{8x^3 + 2x^2 + 15x + 1}{(2x-1)(4x^2+9)} = x + \frac{6x^2 - 3x + 10}{(2x-1)(4x^2+9)}.$$

Partial Fractions

We need to turn $\frac{6x^2-3x+10}{(2x-1)(4x^2+9)}$ into its partial fractions form. We solve
$$\frac{6x^2 - 3x + 10}{(2x-1)(4x^2+9)} = \frac{A}{2x-1} + \frac{Bx+C}{4x^2+9}.$$

Multiplying both sides by $(2x-1)(4x^2+9)$ gives
$$6x^2-3x+10 = A(4x^2+9)+(Bx+C)(2x-1) = 4Ax^2+9A+2Bx^2-Bx+2Cx-C.$$

This turns into three equations:
$$6 = 4A + 2B \quad (1)$$
$$-3 = -B + 2C \quad (2)$$
$$10 = 9A - C \quad (3)$$

If we double the (3) equation, we get
$$20 = 18A - 2C \quad (4)$$

Adding (4) and (2), we get
$$17 = 18A - B \quad (5)$$

Doubling (5) gives
$$34 = 36A - 2B \quad (6)$$

Adding (6) and (1) gives

$$40 = 40A \tag{7}$$

So we have $A = 1$. Substituting into (1) gives

$$6 = 4 + 2B \tag{8}$$

so $B = 1$. Substituting into (2) gives

$$-3 = -1 + 2C \tag{9}$$

so $C = -1$.

Therefore
$$\frac{6x^2 - 3x + 10}{(2x-1)(4x^2+9)} = \frac{1}{2x-1} + \frac{x-1}{4x^2+9}.$$

Equating

Returning to our original integral, we have

$$\begin{aligned}\int \frac{8x^3 + 2x^2 + 15x + 1}{(2x-1)(4x^2+9)}dx &= \int 1 + \frac{1}{2x-1} + \frac{x-1}{4x^2+9}dx \\ &= x + \int \frac{1}{2x-1}dx + \int \frac{x}{4x^2+9}dx - \int \frac{1}{4x^2+9}dx\end{aligned}$$

Integrating $\frac{1}{2x-1}$

We make the substitution

$$\begin{aligned}u &= 2x - 1 \\ du &= 2\ dx\end{aligned}$$

so $dx = du/2$, so

$$\int \frac{1}{2x-1}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|2x-1| + C.$$

Integrating $\frac{x}{4x^2+9}$

We make the substitution

$$\begin{aligned}u &= 4x^2 + 9 \\ du &= 8\ dx\end{aligned}$$

so $dx = du/8$, so

$$\int \frac{x}{4x^2+9}dx = \frac{1}{8}\int \frac{1}{u}du = \frac{1}{8}\ln|u| + C = \frac{1}{8}\ln|4x^2+9| + C.$$

Integrating $\frac{1}{4x^2+9}$

We make the substitution

$$u = \frac{2}{3}x$$
$$du = \frac{2}{3}\ dx$$

so $dx = \frac{3}{2}dx$, so

$$\int \frac{1}{4x^2+9}dx \qquad = \frac{3}{2}\int \frac{1}{4(\frac{3}{2}u)^2+9}du \qquad = \frac{3}{2}\int \frac{1}{9u^2+9}du$$
$$= \frac{3}{2}\frac{1}{9}\int \frac{1}{u^2+1}dy \qquad = \frac{1}{6}\arctan u + C \qquad = \frac{1}{6}\arctan(4x^2+9) + C$$

Putting it Together

$$\int \frac{8x^3+2x^2+15x+1}{(2x-1)(4x^2+9)}dx = x + \int \frac{1}{2x-1}dx + \int \frac{x}{4x^2+9}dx - \int \frac{1}{4x^2+9}dx$$
$$= x + \frac{1}{2}\ln|2x-1| + \frac{1}{8}\ln|4x^2+9| - \frac{1}{6}\arctan(4x^2+9) + C.$$