MIDTERM 2

Math 3B
2/16/2011

Name: ________________________

Section: ________________________

Signature: ________________________

Read all of the following information before starting the exam:

- Check your exam to make sure all pages are present.
- You may use writing implements and a single 3"x5" notecard.
- NO CALCULATORS!
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Good luck!

1	25	
2	18	
3	17	
4	20	
5	14	
6	16	
Total	110	

1. (*25 points*) Find the following integrals.

(**a**)

$$\int_{-1/2}^{1/2} e^{\tan x} \sec^2 x \; dx$$

$u = \tan x, du = \sec^2 x \; dx$, so

$$\int_{-1/2}^{1/2} e^{\tan x} \sec^2 x \; dx = \int_{\tan(-1/2)}^{\tan(1/2)} e^u \; du = e^u \Big|_{\tan(-1/2)}^{\tan(1/2)} = e^{\tan(1/2)} - e^{\tan(-1/2)}$$

(**b**)

$$\int \frac{1}{(3x+5)^2+1} dx$$

$u = 3x + 5, du = 3 \; dx$, so

$$\int \frac{1}{(3x+5)^2+1} dx = \int \frac{1}{u^2+1} \frac{1}{3} du = \frac{1}{3} \arctan u + C = \frac{1}{3} \arctan(3x+5) + C.$$

(**c**)

$$\int \frac{x^5}{\sqrt[3]{x^2-7}} dx$$

$u = x^2 - 7, du = 2x \; dx, x^4 = (u-7)^2 = u^2 - 14u + 49$, so

$$\begin{aligned}
\int \frac{x^5}{\sqrt[3]{x^2-7}} dx &= \frac{1}{2} \int (u^2 - 14u + 49) u^{-1/3} du \\
&= \frac{1}{2} \int u^{5/3} - 14u^{2/3} + 49u^{-1/3} du \\
&= \frac{1}{2} \left[\frac{3}{8} u^{8/3} - 14 \frac{5}{3} u^{5/3} + 49 \frac{2}{3} u^{2/3} \right] + C \\
&= \frac{1}{2} \left[\frac{3}{8} (x^2-7)^{8/3} - 14 \frac{5}{3} (x^2-7)^{5/3} + 49 \frac{2}{3} (x^2-7)^{2/3} \right] + C
\end{aligned}$$

(**d**)

$$\int \frac{x^2}{x^2+1}dx$$

$$\int \frac{x^2}{x^2+1}dx = \int \frac{x^2+1-1}{x^2+1}dx = \int 1 - \frac{1}{x^2+1}dx = x - \arctan x + C.$$

(**e**)

$$\int \ln(x^2+1)dx$$

We try integration by parts:

$$u = \ln(x^2+1) \qquad v = x$$
$$du = \frac{2x}{x^2+1} \qquad dv = dx$$

So

$$\int \ln(x^2+1)dx = x\ln(x^2+1) - \int \frac{2x^2}{x^2+1}dx = x\ln(x^2+1) - 2x + 2\arctan x + C$$

because we can use part (d) to integrate $\int \frac{2x^2}{x^2+1}dx = 2\int \frac{x^2}{x^2+1}dx$.

2. (*18 points*) y_0 and y_1 are two different solutions to the equation $\frac{dy}{dt} = \tan(y+1)$.

(**a**) $y_0(0) = 1$. What is $\lim_{t\to\infty} y_0(t)$?

When $y_0(0) = 1$, $\tan(y_0 + 1) = \tan(1+1) = \tan 2$. $\tan\theta$ is negative when θ is more than $\pi/2$ and less than π, so $y_0'(0)$ is negative, so y_0 is decreasing. y_0 decreases towards $\pi/2 - 1$, going faster and faster as it gets close. However $\lim_{y\to(\pi/2-1)+} \tan(y+1)$ does not exist, so $\lim_{t\to\infty} y_0(t)$ does not exist.

(**b**) $y_1(0) = -1$. What is $\lim_{t\to\infty} y_1(t)$?

When $y_1(0) = -1$, $y_1'(0) = \tan(y_1(0)+1) = \tan 0 = 0$. So y_1 is constant, so $\lim_{t\to\infty} y_1(0) = -1$.

3. (*17 points*) Consider the differential equation $\frac{dy}{dt} = e^t y \ln y$.

(**a**) What is the general solution of this equation?

$$\frac{dy}{y\ln y} = e^t dt \text{ or } y = 0 \text{ or } y = 1$$

so

$$\ln|\ln y| = e^t + C \text{ or } y = 0 \text{ or } y = 1$$

so

$$y = e^{e^{e^t+C}} \text{ or } y = e^{-e^{e^t+C}} \text{ or } y = 0 \text{ or } y = 1.$$

(**b**) What is the particular solution of this equation such that $y(0) = 2$?

$$2 = e^{e^{e^0+C}} = e^{e^{1+C}}$$

so

$$C = \ln\ln 2 - 1.$$

Therefore $y = e^{e^{e^0 + \ln\ln 2 - 1}}$.

(**c**) What is the particular solution of this equation such that $y(0) = 1$?

This is an equilibrium, so $y = 1$.

4. (*20 points*) **(a)** Find the partial fraction decomposition for

$$\frac{1}{(x-1)^3(4x+1)(x^2+2)^2}.$$

You do not need to solve for the values!

$$\frac{1}{(x-1)^3(4x+1)(x^2+2)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{4x+1} + \frac{Ex+F}{x^2+2} + \frac{Gx+H}{(x^2+2)^2}.$$

(b) Find *and solve* the partial fraction decomposition for

$$\frac{7x}{(x+2)(2x-3)}$$

$$\frac{7x}{(x+2)(2x-3)} = \frac{A}{x+2} + \frac{B}{2x-3}$$

$$7x = 2Ax - 3A + Bx + 2B$$

$$7 = 2A + B, \quad 0 = -3A + 2B$$

Doubling the first equation, $14 = 4A + 2B$, and adding this to the second gives $14 = A$. Setting this in the first, $B = 7 - 2A = 7 - 28 = -21$.

(c) Solve

$$\int \frac{2}{(2x+1)^2} + \frac{3}{x^2+2x+4} dx$$

Setting $u = 2x+1, du = 2dx$ and $v = \frac{x+1}{\sqrt{3}}, dv = \frac{1}{\sqrt{3}}dx$, we have

$$\begin{aligned}
\int \frac{2}{(2x+1)^2} + \frac{3}{x^2+2x+4} dx &= \int \frac{1}{u^2} du + \int \frac{3}{(x+1)^2+3} dx \\
&= \frac{-1}{u} + \int \frac{1}{\frac{(x+1)^2}{3}+3} dx \\
&= \frac{-1}{2x+1} + \int \frac{1}{(\frac{x+1}{\sqrt{3}})^2+1} dx \\
&= \frac{-1}{2x+1} + \sqrt{3}\int \frac{1}{v^2+1} dx \\
&= \frac{-1}{2x+1} + \sqrt{3}\arctan v + C \\
&= \frac{-1}{2x+1} + \sqrt{3}\arctan \frac{x+1}{\sqrt{3}} + C
\end{aligned}$$

5. (*14 points*) Find the following integrals if they exist, otherwise state that they diverge.

(**a**)

$$\int_{-\infty}^{\infty} \frac{x}{1+x^4} dx$$

(Hint: use the substitution $u = x^2$.)

$u = x^2$, $du = 2x\ dx$, so

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x}{1+x^4} dx &= \int_{-\infty}^{0} \frac{x}{1+x^4} dx + \int_{0}^{\infty} \frac{x}{1+x^4} dx \\
&= \lim_{a \to -\infty} \int_{a}^{0} \frac{x}{1+x^4} dx + \lim_{b \to \infty} \int_{0}^{b} \frac{x}{1+x^4} dx \\
&= \lim_{a \to -\infty} \frac{1}{2} \int_{a^2}^{0} \frac{1}{1+u^2} dx + \lim_{b \to \infty} \frac{1}{2} \int_{0}^{b^2} \frac{1}{1+u^2} dx \\
&= \lim_{a \to -\infty} \frac{1}{2} \arctan u|_{a^2}^{0} + \lim_{b \to \infty} \frac{1}{2} \arctan u|_{0}^{b^2} \\
&= \frac{1}{2} \left[\lim_{a \to -\infty} \arctan 0 - \arctan a^2 + \lim_{b \to \infty} \arctan b^2 - \arctan 0 \right] \\
&= \frac{1}{2} \left[\lim_{a \to -\infty} - \arctan a^2 + \lim_{b \to \infty} \arctan b^2 \right] \\
&= \frac{1}{2} [-\pi/2 + \pi/2] \\
&= 0
\end{aligned}$$

Note that it's very important to handle the limits correctly in this problem. We have to remember that the lower bound of the first integral is $u = a^2$---in particular, it's always positive---and therefore goes to $\pi/2$.

(**b**)

$$\int_{1}^{\infty} \frac{3}{x^3}$$

$$\int_{1}^{\infty} \frac{3}{x^3} dx = \lim_{a \to \infty} \int_{1}^{a} \frac{3}{x^3} dx = \lim_{a \to \infty} \frac{-3}{2x^2} |_{1}^{a} = \lim_{a \to \infty} \frac{-3}{2a^2} - \frac{-3}{2} = \frac{3}{2}$$

6. (*16 points*) The function $\frac{e^t}{t}$ cannot be integrated in terms of functions you know. Ei is a new function defined by

$$\mathrm{Ei}(x) = \int_{1}^{x} \frac{e^t}{t} dt.$$

Evaluate $\int_e^x \frac{1}{\ln t} dt$ in terms of the function Ei (and other functions you know).

$u = \ln t, du = \frac{1}{t} dt$, and therefore $t = e^u$ and $e^u\ du = dt$, so

$$\begin{aligned}
\int_{e}^{x} \frac{1}{\ln t} dt &= \int_{1}^{\ln x} \frac{e^u}{u} du \\
&= \mathrm{Ei}(\ln x)
\end{aligned}$$