1. Propositional Logic

1.1. **Basic Definitions.**

Definition 1.1. The alphabet of propositional logic consists of

- Infinitely many *propositional variables* $p_0, p_1, \ldots$,
- The *logical connectives* $\wedge, \vee, \rightarrow, \bot$, and
- Parentheses (and).

We usually write $p, q, r, \ldots$ for propositional variables. $\bot$ is pronounced "bottom".

Definition 1.2. The formulas of propositional logic are given inductively by:

- Every propositional variable is a formula,
- $\bot$ is a formula,
- If ϕ and ψ are formulas then so are $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$.

We omit parentheses whenever they are not needed for clarity. We write $\neg\phi$ as an abbreviation for $\phi \rightarrow \bot$. We occasionally use $\phi \leftrightarrow \psi$ as an abbreviation for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$. To limit the number of inductive cases we need to write, we sometimes use $\circledast$ to mean "any of $\wedge, \vee, \rightarrow$".

The propositional variables together with $\bot$ are collectively called *atomic formulas.*

1.2. **Deductions.**

We want to study proofs of statements in propositional logic. Naturally, in order to do this we will introduce a completely formal definition of a proof. To help distinguish between ordinary mathematical proofs, written in (perhaps slightly stylized) natural language, and our formal notion, we will call the formal objects "deductions".

Following standard usage, we will write $\vdash \phi$ to mean "there is a deduction of ϕ (in some particular formal system)". We'll often indicate the formal system in question either with a subscript or by writing it on the left of the turnstile: $\vdash_c \phi$ or $\mathbf{P}_c \vdash \phi$.

We will ultimately work exclusively in the system known as the *sequent calculus*, which turns out to be very well suited to proof theory. However, to help motivate the system, we briefly discuss a few better known systems.

Probably the simplest family of formal systems to describe are *Hilbert systems.* In a typical Hilbert system, a deduction is a list of formulas in which each formula is either an axiom from some fixed set of axioms, or an application of modus ponens to two previous elements on the list. A typical axiom might be

$$\phi \rightarrow (\psi \rightarrow (\phi \wedge \psi)),$$

and so a deduction might consist of steps like

$$\begin{array}{ll} & \vdots \\ \text{(n)} & \phi \end{array}$$

$$\begin{array}{rl} & \vdots \\ \text{(n')} & \phi \to (\psi \to (\phi \wedge \psi)) \\ \text{(n'+1)} & \psi \to (\phi \wedge \psi) \\ & \vdots \\ \text{(n'')} & \psi \\ \text{(n''+1)} & \phi \wedge \psi \end{array}$$

The linear structure of of Hilbert-style deductions, and the very simple list of cases (each step can be only an axiom or an instance of modus ponens) makes it very easy to prove some theorems about Hilbert systems. However these systems are very far removed from ordinary mathematics, and they don't expose very much of the structure we will be interested in studying, and as a result are poorly suited to proof-theoretic work.

The second major family of formal systems are *natural deduction systems*. These were introduced by Gentzen in part to more closely resemble ordinary mathematical reasoning. These systems typically have relatively few axioms, and more rules, and also have a non-linear structure. One of the key features is that the rules tend to be organized into neat groups which help provide some meaning to the connectives. A common set-up is to have two rules for each connective, an *introduction* rule and an *elimination* rule. For instance, the $\wedge$ introduction rule states "given a deduction of ϕ and a deduction of ψ, deduce $\phi \wedge \psi$".

A standard way of writing such a rule is

$$\wedge I \; \frac{\phi \qquad \psi}{\phi \wedge \psi}$$

The formulas above the line are the *premises* of the rule and the formula below the line is the *conclusion*. The label simply states which rule is being used. Note the non-linearity of this rule: we have two distinct deductions, one deduction of ϕ and one deduction of ψ, which are combined by this rule into a single deduction.

The corresponding elimination rules might be easy to guess:

$$\wedge E_1 \; \frac{\phi \wedge \psi}{\phi} \quad \text{and} \quad \wedge E_2 \; \frac{\phi \wedge \psi}{\psi}$$

These rules have the pleasant feature of corresponding to how we actually work with conjunction: in an ordinary English proof of $\phi \wedge \psi$, we would expect to first prove ϕ, then prove ψ, and then note that their conjunction follows. And we would use $\phi \wedge \psi$ at some later stage of a proof by observing that, since $\phi \wedge \psi$ is true, whichever of the conjuncts we need must also be true.

This can help motivate rules for the other connectives. The elimination rule for $\to$ is unsurprising:

$$\rightarrow E\ \frac{\phi \qquad \phi \rightarrow \psi}{\psi}$$

The introduction rule is harder. In English, a proof of an implication would read something like: "Assume ϕ. By various reasoning, we conclude ψ. Therefore $\phi \rightarrow \psi$." We need to incorporate the idea of reasoning under assumptions.

This leads us to the notion of a *sequent*. For the moment (we will modify the definition slightly in the next section) a sequent consists of a set of assumptions Γ, together with a conclusion ϕ, and is written

$$\Gamma \Rightarrow \phi.$$

Instead of deducing formulas, we'll want to deduce sequents; $\vdash \Gamma \Rightarrow \phi$ means "there is a deduction of ϕ from the assumptions Γ". The rules of natural deduction should really be rules about sequents, so the four rules already mentioned should be:

$$\wedge I\ \frac{\Gamma \Rightarrow \phi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \wedge \psi} \qquad \wedge E_1\ \frac{\Gamma \Rightarrow \phi \wedge \psi}{\Gamma \Rightarrow \phi}$$

$$\wedge E_2\ \frac{\Gamma \Rightarrow \phi \wedge \psi}{\Gamma \Rightarrow \psi} \qquad \rightarrow E\ \frac{\Gamma \Rightarrow \phi \qquad \Gamma \Rightarrow \phi \rightarrow \psi}{\Gamma \Rightarrow \psi}$$

(Actually, in the rules with multiple premises, we'll want to consider the possibility that the two sub-derivations have different sets of assumptions, but we'll ignore that complication for now.)

This gives us a natural choice for an introduction rule for $\rightarrow$:

$$\rightarrow I\ \frac{\Gamma \cup \{\phi\} \Rightarrow \psi}{\Gamma \Rightarrow \phi \rightarrow \psi}$$

In plain English, "if we can deduce ψ from the assumptions Γ and ϕ, then we can also deduce $\phi \rightarrow \psi$ from just Γ".

This notion of reasoning under assumptions also suggests what an axiom might be:

$$\frac{}{\phi \Rightarrow \phi}$$

(The line with nothing above it represents an axiom—from no premises at all, we can conclude $\phi \Rightarrow \phi$.) In English, "from the assumption ϕ, we can conclude ϕ".

1.3. **The Sequent Calculus.** Our chosen system, however, is the sequent calculus. The sequent calculus seems a bit strange at first, and gives up some of the "naturalness" of natural deduction, but it will pay us back by being the system which makes the structural features of deductions most explicit. Since our main interest will be studying the formal properties of different deductions, this will be a worthwhile trade-off.

The sequent calculus falls naturally out of an effort to symmetrize natural deduction. In natural deduction, the left and right sides of the sequent behave very differently: there can be many assumptions, but only one consequence, and while rules can add or remove formulas from the assumptions, they can only *modify* the conclusion.

In the sequent calculus, we will allow both sides of a sequent to be sets of formulas (although we will later study what happens when we put back the restriction that the right side have at most one formula). What should we mean by the sequent

$$\Gamma \Rightarrow \Sigma?$$

It turns out that the right choice is

> If *all* the assumptions in Γ are true then *some* conclusion in Σ is true.

In other words we interpret the left side of the sequent *conjunctively*, and the right side *disjunctively*. (The reader might have been inclined to interpret both sides of the sequent conjunctively; the choice to interpret the right side disjunctively will ultimately be supported be the fact that it creates a convenient *duality* between assumptions and conclusions.)

Definition 1.3. A *sequent* is a pair of sets of formulas, written

$$\Gamma \Rightarrow \Sigma,$$

such that Σ is finite.

Often one or both sets are small, and we list the elements without set braces: $\phi, \psi \Rightarrow \gamma, \delta$ or $\Gamma \Rightarrow \phi$. We will also use juxtaposition to abbreviate union; that is

$$\Gamma\Delta \Rightarrow \Sigma\Upsilon$$

abbreviates $\Gamma \cup \Delta \Rightarrow \Sigma \cup \Upsilon$ and similarly

$$\Gamma, \phi, \psi \Rightarrow \Sigma, \gamma$$

abbreviates $\Gamma \cup \{\phi, \psi\} \Rightarrow \Sigma \cup \{\gamma\}$. When Γ is empty, we simply write $\Rightarrow \Sigma$, or (when it is clear from context that we are discussing a sequent) sometimes just Σ.

It is quite important to pay attention to the definition of Γ and Σ as *sets*: they do not have an order, and they do not distinguish multiple copies of the same element. For instance, if $\phi \in \Gamma$ we may still write $\Gamma, \phi \Rightarrow \Sigma$.

When $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ is finite and $\Sigma = \{\sigma_1, \dots, \sigma_k\}$, we will write $\bigwedge \Gamma \rightarrow \bigvee \Sigma$ for the formula $(\gamma_1 \wedge \cdots \wedge \gamma_n) \rightarrow (\sigma_1 \vee \cdots \vee \sigma_n)$.

We refer to $\Rightarrow$ as *metalanguage implication* to distinguish it from $\rightarrow$.

Before we introduce deductions, we need one more notion: an *inference rule*, or simply a *rule*. An inference rule represents a single step in a deduction; it says that from the truth its premises we may immediately infer the truth of its conclusion. (More precisely, an inference rule will say that if we have deductions of all its premises, we also have a deduction of its conclusion.) For instance, we expect an inference rule which says that if we know both ϕ and ψ then we also know $\phi \wedge \psi$.

A rule is written like this:

$$\textbf{Name}\ \frac{\Gamma_0 \Rightarrow \Delta_0 \qquad \cdots \qquad \Gamma_n \Rightarrow \Delta_n}{\Sigma \Rightarrow \Upsilon}$$

This rule indicates that if we have deductions of all the sequents $\Gamma_i \Rightarrow \Delta_i$ then we also have a deduction of $\Sigma \Rightarrow \Upsilon$.

Definition 1.4. Let R be a set of rules. We define $R \vdash \Sigma \Rightarrow \Upsilon$ inductively by:

- If for every $i \leq n$, $R \vdash \Gamma_i \Rightarrow \Delta_i$, and the rule above belongs to R, then $R \vdash \Sigma \Rightarrow \Upsilon$.

We will omit a particular set of rules R if it is clear from context.

Our most important collection of inference rules for now will be classical propositional logic, which we will call $\mathbf{P}_c$. First we have a *structural* rule—a rule with no real logical content, but only included to make sequents behave properly.

$$\mathrm{W}\ \frac{\Gamma \Rightarrow \Sigma}{\Gamma\Gamma' \Rightarrow \Sigma\Sigma'}$$

W stands for "weakening"—the sequent $\Gamma\Gamma' \Rightarrow \Sigma\Sigma'$ is weaker than the sequent $\Gamma \Rightarrow \Sigma$, so if we can deduce the latter, surely we can deduce the former.

$\mathbf{P}_c$ will include two axioms (rules with no premises):

$$\mathrm{Ax}\ \frac{}{\phi \Rightarrow \phi} \text{ where } \phi \text{ is atomic.}$$

$$\mathrm{L}\bot\ \frac{}{\bot \Rightarrow \emptyset}$$

$\mathbf{P}_c$ includes inference rules for each connective, neatly paired:

$$\mathrm{L}\wedge\ \frac{\Gamma, \phi_i \Rightarrow \Sigma}{\Gamma, \phi_0 \wedge \phi_1 \Rightarrow \Sigma} \qquad \mathrm{R}\wedge\ \frac{\Gamma \Rightarrow \Sigma, \phi_0 \qquad \Gamma' \Rightarrow \Sigma', \phi_1}{\Gamma\Gamma' \Rightarrow \Sigma\Sigma', \phi_0 \wedge \phi_1}$$

$$\mathrm{L}\vee\ \frac{\Gamma, \phi_0 \Rightarrow \Sigma \qquad \Gamma', \phi_1 \Rightarrow \Sigma'}{\Gamma\Gamma', \phi_0 \vee \phi_1 \Rightarrow \Sigma\Sigma'} \qquad \mathrm{R}\vee\ \frac{\Gamma \Rightarrow \Sigma, \phi_i}{\Gamma \Rightarrow \Sigma, \phi_0 \vee \phi_1}$$

$$\mathrm{L}\rightarrow\ \frac{\Gamma \Rightarrow \Sigma, \phi \qquad \Gamma', \psi \Rightarrow \Sigma'}{\Gamma\Gamma', \phi \rightarrow \psi \Rightarrow \Sigma\Sigma'} \qquad \mathrm{R}\rightarrow\ \frac{\Gamma, \phi \Rightarrow \Sigma, \psi}{\Gamma \Rightarrow \Sigma, \phi \rightarrow \psi}$$

If we think of $\bot$ as a normal (but "0-ary") connective, L$\bot$ is the appropriate left rule, and there is no corresponding right rule (as befits $\bot$). Ax can be thought of as simultaneously a left side and right side rule.

Finally, the *cut rule* is

$$\textbf{Cut}\ \dfrac{\Gamma \Rightarrow \Sigma, \phi \qquad \Gamma', \phi \Rightarrow \Sigma'}{\Gamma\Gamma' \Rightarrow \Sigma\Sigma'}$$

These nine rules collectively are the system $\mathbf{P}_c$. Each of these rules other than **Cut** has a distinguished formula in the conclusion; we call this the *main formula* of that inference rule.

Example 1.5. $\mathbf{P}_c \vdash (p \wedge q) \rightarrow (p \wedge q)$

$$\mathrm{R}\!\rightarrow \dfrac{\mathrm{R}\wedge \dfrac{\mathrm{L}\wedge \dfrac{\mathrm{Ax}\ \dfrac{}{p \Rightarrow q}}{p \wedge q \Rightarrow p} \qquad \mathrm{L}\wedge \dfrac{\mathrm{Ax}\ \dfrac{}{q \Rightarrow q}}{p \wedge q \Rightarrow q}}{p \wedge q \Rightarrow p \wedge q}}{\Rightarrow (p \wedge q) \rightarrow (p \wedge q)}$$

The fact that Ax is limited to atomic formulas will make it easier to prove things about deductions, but harder to explicitly write out deductions. Later we'll prove the following lemma, but for the moment, let's take it for granted:

Lemma 1.6. $\mathbf{P}_c \vdash \phi \Rightarrow \phi$ *for any formula* ϕ*.*

We'll adopt the following convention when using lemmas like this: we'll use a double line to indicate multiple inference rules given by some established lemma. For instance, we'll write

$$\overline{\overline{\phi \Rightarrow \phi}}$$

to indicate some deduction of the sequent $\phi \Rightarrow \phi$ where we don't want to write out the entire deduction. We will *never* write

$$\overline{\phi \Rightarrow \phi}$$

with a single line unless ϕ is an atomic formula: a single line will always mean exactly one inference rule. (It's very important to be careful about this point, because we're mostly going to be interested in treating deductions as formal, purely syntactic objects.)

Example 1.7. $\mathbf{P}_c \vdash (\phi \rightarrow (\phi \rightarrow \psi)) \rightarrow (\phi \rightarrow \psi)$

$$\mathrm{R}\!\rightarrow \dfrac{\mathrm{R}\!\rightarrow \dfrac{\mathrm{L}\!\rightarrow \dfrac{\overline{\overline{\phi \Rightarrow \phi}} \qquad \mathrm{L}\!\rightarrow \dfrac{\overline{\overline{\phi \Rightarrow \phi}} \qquad \overline{\overline{\psi \Rightarrow \psi}}}{\phi, \phi \rightarrow \psi \Rightarrow \psi}}{\phi, \phi \rightarrow (\phi \rightarrow \psi) \Rightarrow \psi}}{\phi \rightarrow (\phi \rightarrow \psi) \Rightarrow (\phi \rightarrow \psi)}}{\Rightarrow (\phi \rightarrow (\phi \rightarrow \psi)) \rightarrow (\phi \rightarrow \psi)}$$

Example 1.8 (Pierce's Law)**.** $\mathbf{P}_c \vdash ((\phi \rightarrow \psi) \rightarrow \phi) \rightarrow \phi$

$$\mathrm{R}\to\ \dfrac{\mathrm{L}\to\ \dfrac{\mathrm{R}\to\ \dfrac{\overline{\overline{\phi \Rightarrow \psi, \phi}}}{\Rightarrow \phi \to \psi, \phi} \qquad \overline{\overline{\phi \Rightarrow \phi}}}{(\phi \to \psi) \to \phi \Rightarrow \phi}}{\Rightarrow ((\phi \to \psi) \to \phi) \to \phi}$$

We will not maintain the practice of always labeling our inference rules. In fact, quite the opposite—we will usually only include the label when the rule is particularly hard to recognize.

Example 1.9 (Excluded Middle)**.** $\mathbf{P}_c \vdash \phi \vee \neg\phi$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\overline{\phi \Rightarrow \phi}}}{\phi \Rightarrow \phi, \bot}}{\Rightarrow \phi, \neg\phi}}{\Rightarrow \phi, \phi \vee (\neg\phi)}}{\Rightarrow \phi \vee (\neg\phi)}$$

This last example brings up the possibility that we will sometimes want to rewrite a sequent from one line to the next without any inference rules between. We'll denote this with a dotted line. For instance:

$$\begin{array}{c} \Rightarrow \phi \to \bot \\ \cdots\cdots\cdots \\ \Rightarrow \neg\phi \end{array}$$

The dotted line will always mean that the sequents above and below are *formally identical*: it is only a convenience for the reader that we separate them. (Thus our convention is: single line means one rule, double line means many rules, dotted line means no rules.)

Under this convention, we might write the last deduction as:

Example 1.10 (Excluded Middle again)**.**

$$\dfrac{\dfrac{\begin{array}{c} \dfrac{\overline{\overline{\phi \Rightarrow \phi, \bot}}}{\Rightarrow \phi, \phi \to \bot} \\ \cdots\cdots\cdots\cdots \\ \Rightarrow \phi, \neg\phi \end{array}}{\Rightarrow \phi, \phi \vee (\neg\phi)}}{\Rightarrow \phi \vee (\neg\phi)}$$

We now prove the lemma we promised earlier:

Lemma 1.11. $\mathbf{P}_c \vdash \phi \Rightarrow \phi$ *for any formula* ϕ.

Proof. By induction on formulas. If ϕ is a atomic, this is simply an application of the axiom.

If ϕ is $\phi_0 \wedge \phi_1$ then we have

$$\dfrac{\dfrac{\overline{\overline{\phi_0 \Rightarrow \phi_0}}}{\phi_0 \wedge \phi_1 \Rightarrow \phi_0} \qquad \dfrac{\overline{\overline{\phi_1 \Rightarrow \phi_1}}}{\phi_0 \wedge \phi_1 \Rightarrow \phi_1}}{\phi_0 \wedge \phi_1 \Rightarrow \phi_0 \wedge \phi_1}$$

If ϕ is $\phi_0 \vee \phi_1$ then we have

$$\dfrac{\dfrac{\overline{\overline{\phi_0 \Rightarrow \phi_0}}}{\phi_0 \Rightarrow \phi_0 \vee \phi_1} \qquad \dfrac{\overline{\overline{\phi_1 \Rightarrow \phi_1}}}{\phi_1 \Rightarrow \phi_0 \vee \phi_1}}{\phi_0 \vee \phi_1 \Rightarrow \phi_0 \vee \phi_1}$$

If ϕ is $\phi_0 \to \phi_1$ then we have

$$\dfrac{\dfrac{\overline{\overline{\phi_0 \Rightarrow \phi_0}} \qquad \overline{\overline{\phi_1 \Rightarrow \phi_1}}}{\phi_0 \to \phi_1, \phi_0 \Rightarrow \phi_1}}{\phi_0 \to \phi_1 \Rightarrow \phi_0 \to \phi_1} \qquad \square$$

An important deduction that often confuses people at first is the following:

$$\mathrm{R}\vee \, \dfrac{\Gamma \Rightarrow \Sigma, \phi_0 \vee \phi_1, \phi_1}{\Gamma \Rightarrow \Sigma, \phi_0 \vee \phi_1}$$

To go from the first line to the second, we replaced ϕ_1 with $\phi_0 \vee \phi_1$, as permitted by the R$\vee$ rule. But the right side of a sequent is a *set*, which means it can only contain one copy of the formula $\phi_0 \vee \phi_1$. So it seems like the formula "disappeared". This feature takes some getting used to. To make arguments like these easier to follow, we will sometimes used our dotted line convention:

$$\mathrm{R}\vee \, \begin{array}{c} \Gamma \Rightarrow \Sigma, \phi_0 \vee \phi_1, \phi_1 \\ \hline \Gamma \Rightarrow \Sigma, \phi_0 \vee \phi_1, \phi_0 \vee \phi_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \Gamma \Rightarrow \Sigma, \phi_0 \vee \phi_1 \end{array}$$

Note that the second and third lines are literally the same sequent, being written in two different ways.

We note a few frequently used deductions:

$$\begin{array}{c} \Gamma \Rightarrow \Sigma, \phi_0, \phi_1 \\ \hline\hline \Gamma \Rightarrow \Sigma, \phi_0 \vee \phi_1 \end{array}$$

$$\begin{array}{c} \Gamma, \phi_0, \phi_1 \Rightarrow \Sigma \\ \hline\hline \Gamma, \phi_0 \wedge \phi_1 \Rightarrow \Sigma \end{array}$$

$$\begin{array}{c} \Gamma \Rightarrow \phi, \Sigma \\ \hline\hline \Gamma, \neg\phi \Rightarrow \Sigma \end{array}$$

$$\begin{array}{c} \Gamma, \phi \Rightarrow \Sigma \\ \hline\hline \Gamma \Rightarrow \neg\phi, \Sigma \end{array}$$

Definition 1.12. If R is a set of rules and I is any rule, we say I is *admissible* over R if whenever $R + I \vdash \Gamma \Rightarrow \Sigma$, already $R \vdash \Gamma \Rightarrow \Sigma$.

The examples just given are all examples of admissible rules: we could decide to work, not in $\mathbf{P}_c$, but in some expansion of $\mathbf{P}'_c$ in which, say

$$\mathrm{L}\neg \frac{\Gamma \Rightarrow \phi, \Sigma}{\Gamma, \neg\phi \Rightarrow \Sigma}$$

was an actual rule. The new rule is admissible: we could take any deduction in $\mathbf{P}'_c$ and replace each instance of the L¬ rule with a short deduction in $\mathbf{P}_c$, giving a deduction of the same thing in $\mathbf{P}_c$. One might ask if all admissible rules are like this: if I is admissible, is it always because I is an abbreviation for some fixed list of steps? We will see below that the answer is no; in fact, we'll prove that the rule **Cut** is actually admissible over the other rules of $\mathbf{P}_c$: we will introduce a system $\mathbf{P}_c^{cf}$ ("cf" stands for "cut-free"), which consists of the rules of $\mathbf{P}_c$ other than **Cut**, and prove *cut-elimination*—that every proof in $\mathbf{P}_c$ can be converted to one in $\mathbf{P}_c^{cf}$—and *speed-up*—that there are sequences which have short deductions in $\mathbf{P}_c$, but have only very long deductions in $\mathbf{P}_c^{cf}$. Among other things, this will tell us that even though the cut rule is admissible, it does not abbreviate some fixed deduction; in fact, the only way to eliminate the cut rule is to make very substantial structural changes to our proof.

1.4. **Variants on Rules.** We have taken sequents to be sets, meaning we don't pay attention to the order formulas appear in or how many times a formula appears. Some people take sequents to be multisets (which do count the number of times a formula appears) or sequences (which also track the order formulas appear in). One then needs to add *contraction* rules, which combine multiple copies of a formula into one copy, and *exchange* rules, which alter the order of formulas. These are considered structural rules, like our weakening rule.

If we omit or restrict some of the structural rules we obtain *substructural* logics. The most important substructural logic is Linear Logic; one interpretation of Linear Logic is that formulas represent resources. (In propositional logic, our default interpretation is that formulas represent *propositions*—things that can be true or false.) So in Linear Logic, a sequent $\Gamma \Rightarrow \Sigma$ could be interpreted to say "the collection of resources Γ can be converted into one of the resources in Σ". As might be expected from this interpretation, it is important for sequents to be multisets, since there is a real difference between having one copy of a resource and having multiple copies. Furthermore, the contraction rule is limited (just because we can deduce $\phi, \phi \Rightarrow \psi$, we wouldn't expect to deduce $\phi \Rightarrow \psi$—being able to buy a ψ for two dollars doesn't mean we can also buy a ψ for one dollar). For example, in linear logic, $\wedge$ is replaced by two connectives, $\otimes$, which represents having both resources, and &, which represents having the choice between the two resources. The corresponding sequent calculus rules are

$$\frac{\Gamma, \phi_0, \phi_1 \Rightarrow \Sigma}{\Gamma, \phi_0 \otimes \phi_1 \Rightarrow \Sigma} \qquad \frac{\Gamma \Rightarrow \phi_0, \Sigma \qquad \Gamma' \Rightarrow \phi_1, \Sigma'}{\Gamma\Gamma' \Rightarrow \phi_0 \otimes \phi_1, \Sigma\Sigma'}$$

$$\frac{\Gamma, \phi_i \Rightarrow \Sigma}{\Gamma, \phi_0 \& \phi_1 \Rightarrow \Sigma} \qquad \frac{\Gamma \Rightarrow \phi_0, \Sigma \qquad \Gamma \Rightarrow \phi_1, \Sigma}{\Gamma \Rightarrow \phi_0 \& \phi_1, \Sigma}$$

1.5. **Completeness.** We recall the usual semantics for the classical propositional calculus, in which formulas are assigned the values T and F, corresponding to the intended interpration of formulas as either *true* or *false*, respectively.

Definition 1.13. A *truth assignment* for ϕ is a function ν mapping the propositional variables which appear in ϕ to $\{T, F\}$. Given such a ν, we define $\overline{\nu}$ by:

- $\overline{\nu}(p) = \nu(p)$,
- $\overline{\nu}(\perp) = F$,
- $\overline{\nu}(\phi_0 \wedge \phi_1) = 1$ if $\overline{\nu}(\phi_0) = \overline{\nu}(\phi_1) = 1$ and 0 otherwise,
- $\overline{\nu}(\phi_0 \wedge \phi_1) = 0$ if $\overline{\nu}(\phi_0) = \overline{\nu}(\phi_1) = 0$ and 1 otherwise,
- $\overline{\nu}(\phi_0 \to \phi_1) = 0$ if $\overline{\nu}(\phi_0) = 1$ and $\overline{\nu}(\phi_1) = 0$, and 1 otherwise.

We write $\vDash \phi$ if for every truth assignment ν, $\overline{\nu}(\phi) = T$.

A straightforward induction on deductions gives:

Theorem 1.14 (Soundness)**.** *If* $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma$ *with* Γ *finite then* $\vDash \bigwedge \Gamma \to \bigvee \Sigma$.

An alternate way of stating this is:

Theorem 1.15 (Soundness)**.** *If* $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma$ *and* $\overline{\nu}(\gamma) = T$ *for every* $\gamma \in \Gamma$ *then there is some* $\sigma \in \Sigma$ *such that* $\overline{\nu}(\sigma) = T$.

Theorem 1.16 (Completeness)**.** *Suppose there is no deduction of* $\Rightarrow \phi$ *in* $\mathbf{P}_c$. *Then there is an assignment* ν *of truth values* T *and* F *to the propositional variables of* ϕ *so that* $\overline{\nu}(\phi) = F$.

Proof. We prove the corresponding statement about sequents: given a finite sequent $\Gamma \Rightarrow \Sigma$, if there is no deduction of this sequent then there is such an assignment of truth values for the formula $\bigwedge \Gamma \to \bigvee \Sigma$. We proceed by induction on the number of connectives $\wedge, \vee, \to$ appearing in $\Gamma \Rightarrow \Sigma$.

Suppose there are no connectives in $\Gamma \Rightarrow \Sigma$. If $\perp$ appeared in Γ or any propositional variable appeared in both Σ and Γ then there would be a one-step deduction of this sequent, so neither of these can happen. Therefore neither of these occur, and we define a truth assignment ν by making every propositional variable in Γ false and every propositional variable in Σ true.

Suppose $\Sigma = \Sigma', \phi_0 \wedge \phi_1$ where $\phi_0 \wedge \phi_1$ is not in Σ'. If there were deductions of both $\Gamma \Rightarrow \Sigma', \phi_0$ and $\Gamma \Rightarrow \Sigma', \phi_1$ then there would be a deduction of $\Gamma \Rightarrow \Sigma$. Since this is not the case, there is some i such that there is not a deduction of $\Gamma \Rightarrow \Sigma', \phi_i$, and therefore by IH a truth assignment demonstrating the falsehood of $\bigwedge \Gamma \Rightarrow \bigvee \Sigma' \vee \phi_i$, and therefore the falsehood we desire.

Suppose $\Gamma = \Gamma', \phi_0 \wedge \phi_1$. If there is no deduction of $\Gamma', \phi_0 \wedge \phi_1 \Rightarrow \Sigma$ then there can also be no deduction of $\Gamma', \phi_0, \phi_1 \Rightarrow \Sigma$, and so by IH there is a truth assignment making $\bigwedge \Gamma' \wedge \phi_0 \wedge \phi_1 \Rightarrow \Sigma$ false, which suffices.

The cases for $\vee$ and $\rightarrow$ are similar. □

Lemma 1.17 (Compactness)**.** *If* $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma$ *then there is a finite* $\Gamma_0 \subseteq \Gamma$ *such that* $\mathbf{P}_c \vdash \Gamma_0 \Rightarrow \Sigma$.

Proof. The idea is that since $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma$, there is a deduction in the sequent calculus witnessing this, and we may therefore restrict Γ to those formulas which actually get used in the deduction. Let d be a deduction of $\Gamma \Rightarrow \Sigma$, and let Γ^d be the set of all formulas appearing as the main formula of one of the inference rules $\mathrm{L}\wedge, \mathrm{L}\vee, \mathrm{L}\rightarrow, \mathrm{L}\bot, \mathrm{Ax}$ anywhere in the deduction. (These are essentially the left-side rules, viewing Ax as both a left-side and a right-side rule.) Since d is finite, there are only finitely many inference rules, each of which has only one main formula, so Γ^d is finite.

We now show by induction that if d' is any subdeduction of d deducing $\Delta \Rightarrow \Lambda$ then there is a deduction of $\Delta \cap \Gamma^d \Rightarrow \Lambda$. We will do one of the base cases, for Ax, and four of the inductive cases, W, $\mathrm{L}\wedge$, $\mathrm{L}\rightarrow$, and Cut. The right side rules are quite simple, and the $\mathrm{L}\vee$ case is similar to $\mathrm{L}\rightarrow$.

- Suppose d is simply an instance of Ax,

  $$\frac{}{p \Rightarrow p}$$

  for p atomic. Then $p \in \Gamma^d$, so the deduction is unchanged.
- Suppose the final inference rule of d' is an application W to a deduction d'',

  $$\frac{d'' \; \overline{\overline{\Delta \Rightarrow \Lambda}}}{\Delta\Delta' \Rightarrow \Lambda\Lambda'}$$

  Then

  $$\frac{\mathrm{IH} \; \overline{\overline{\Delta \cap \Gamma^d \Rightarrow \Lambda}}}{(\Delta\Delta') \cap \Gamma^d \Rightarrow \Lambda\Lambda'}$$

  is a valid deduction.
- Suppose the final inference rule of d' is an application of $L\wedge$ to a deduction d'',

  $$\frac{d'' \; \overline{\overline{\Delta, \phi_i \Rightarrow \Lambda}}}{\Delta, \phi_0 \wedge \phi_1 \Rightarrow \Lambda}$$

  If $\phi_i \notin \Gamma^d$ then $(\Delta, \phi) \cap \Gamma^d = \Delta \cap \Gamma^d$, so we have

$$\mathrm{W}\ \cfrac{\mathrm{IH}\ \cfrac{}{\Delta \cap \Gamma^d \Rightarrow \Lambda}}{\cfrac{\Delta \cap \Gamma^d, \phi_0 \wedge \phi_1 \Rightarrow \Lambda}{(\Delta, \phi_0 \wedge \phi_1) \cap \Gamma^d \Rightarrow \Lambda}}$$

If $\phi_i \in \Gamma^d$ then $(\Delta, \phi_i) \cap \Gamma^d = \Delta \cap \Gamma^d, \phi_i$, so

$$\cfrac{\mathrm{IH}\ \cfrac{}{\Delta \cap \Gamma^d, \phi_i \Rightarrow \Lambda}}{\Delta \cap \Gamma^d, \phi_0 \wedge \phi_1 \Rightarrow \Lambda}$$

is the desired deduction.

- Suppose the final inference rule of d' is an application of L$\rightarrow$ to deductions d'_0, d'_1,

$$\cfrac{d'_0\ \cfrac{}{\Delta \Rightarrow \Lambda, \phi} \qquad d'_1\ \cfrac{}{\Delta', \psi \Rightarrow \Lambda'}}{\Delta\Delta', \phi \rightarrow \psi \Rightarrow \Lambda\Lambda'}$$

First, suppose $\psi \notin \Gamma^d$; then $(\Delta', \psi) \cap \Gamma^d = \Delta' \cap \Gamma^d$, so we have

$$\mathrm{W}\ \cfrac{\mathrm{IH}\ \cfrac{}{\Delta' \cap \Gamma^d \Rightarrow \Lambda'}}{\cfrac{\Delta\Delta' \cap \Gamma^d, \phi \rightarrow \psi \Rightarrow \Lambda\Lambda'}{(\Delta\Delta', \phi \rightarrow \psi) \cap \Gamma^d \Rightarrow \Lambda\Lambda'}}$$

Otherwise $\psi \in \Gamma^d$, and we have

$$\cfrac{\mathrm{IH}\ \cfrac{}{\Delta \cap \Gamma^d \Rightarrow \Lambda, \phi} \qquad \mathrm{IH}\ \cfrac{}{\Delta' \cap \Gamma^d, \psi \Rightarrow \Lambda'}}{\Delta\Delta' \cap \Gamma^d, \phi \rightarrow \psi \Rightarrow \Lambda\Lambda'}$$

- Suppose the final inference rule of d' is an application of *Cut* to deductions d'_0, d'_1,

$$\cfrac{d'_0\ \cfrac{}{\Delta \Rightarrow \Lambda, \phi} \qquad d'_1\ \cfrac{}{\Delta', \phi \Rightarrow \Lambda'}}{\Delta\Delta' \Rightarrow \Lambda\Lambda'}$$

If $\phi \notin \Gamma^d$ then $(\Delta', \phi) \cap \Gamma^d = \Delta' \cap \Gamma^d$, and so we have

$$\cfrac{\mathrm{IH}\ \cfrac{}{\Delta' \cap \Gamma^d \Rightarrow \Lambda'}}{\Delta\Delta' \cap \Gamma^d \Rightarrow \Lambda\Lambda'}$$

If $\phi \in \Gamma^d$ we have

$$\cfrac{\mathrm{IH}\ \cfrac{}{\Delta \Rightarrow \Lambda, \phi} \qquad \mathrm{IH}\ \cfrac{}{\Delta' \cap \Gamma^d, \phi \Rightarrow \Lambda'}}{\Delta\Delta' \Rightarrow \Lambda\Lambda'}$$

$\square$

1.6. **Cut-Free Proofs.** Closer examination of the completeness theorem reveals that we proved something stronger: we never used the rule **Cut** in the proof, and therefore we can weaken the assumption.

Definition 1.18. $\mathbf{P}_c^{cf}$ consists of the rules of $\mathbf{P}_c$ other than Cut.

Therefore our proof of completness actually gave us the following:

Theorem 1.19 (Completeness)**.** *Suppose there is no deduction of* $\Gamma \Rightarrow \Sigma$ *in* $\mathbf{P}_c^{cf}$*. Then there is an assignment* ν *of truth values* T *and* F *to the propositional variables of* Γ *and* Σ *so that for every* $\phi \in \Gamma$, $\overline{\nu}(\phi) = T$ *while for every* $\psi \in \Sigma$, $\overline{\nu}(\psi) = F$.

An immediate consequence is that the cut rule is admissible:

Theorem 1.20. *If* $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma$ *then* $\mathbf{P}_c^{cf} \vdash \Gamma \Rightarrow \Sigma$.

Proof. If $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma$ then, by soundness, every ν satisfies $\overline{\nu}(\bigwedge \Gamma \rightarrow \bigvee \Sigma) = T$. But then completeness implies that there must be a deduction of $\Gamma \Rightarrow \Sigma$ in $\mathbf{P}_c^{cf}$. □

Ultimately, we will want a constructive proof of this theorem: we would like an explicit set of steps for taking a proof involving cut and transforming it into a proof without cut; we give such a proof below.

For now, we note an important property of cut-free proofs which hints at why we are interested in obtaining them:

Definition 1.21. We define the *subformulas* of ϕ recursively by:

- If ϕ is atomic then ϕ is the only subformula of ϕ,
- The subformulas of $\phi \circledast \psi$ are the subformulas of ϕ, the subformulas of ψ, and $\phi \circledast \psi$.

A proof system has the *subformula property* if every formula in every sequent in any proof of $\Gamma \Rightarrow \Sigma$ is a subformula of some formula in $\Gamma \cup \Sigma$.

It is easy to see by inspection of the proof rules that:

Lemma 1.22. $\mathcal{P}_c^{cf}$ *has the subformula property.*

1.7. **Intuitionistic Logic.** We will want to study an important fragment of classical logic: intuitionistic logic. Intuitionistic logic was introduced to satisfy the philosophical concerns introduced by Brouwer, but our concerns will be purely practical: on the one hand, intuitionistic logic has useful properties that classical logic lacks, and on the other, intuitionistic logic is very close to classical logic—in fact, we will be able to translate results in classical logic into intuitionistic logic in a formal way.

We will also (largely for technical reasons) introduce minimal logic, which restricts intuitionistic logic even further by dropping the rule L$\bot$. In other words, minimal logic treats $\bot$ as just another propositional variable, rather than having the distinctive property of implying all other statements.

Definition 1.23. $\mathbf{P}_i$ is the fragment of $\mathbf{P}_c$ in which we require that the right-hand part of each sequent consist of 1 or 0 formulas.

$\mathbf{P}_m$ is the fragment of $\mathbf{P}_i$ omitting L$\bot$.

$\mathbf{P}_i^{cf}$ and $\mathbf{P}_m^{cf}$ are the rules of $\mathbf{P}_i$ and $\mathbf{P}_m$, respectively, other than **Cut**.

We call these intuitionistic and minimal logic, respectively. We will later show that the cut-rule is admissible over both $\mathbf{P}_i^{cf}$ and $\mathbf{P}_m^{cf}$. (As we will see, $\mathbf{P}_i$ and $\mathbf{P}_m$ are *not* complete, so the proof of admissibility over $\mathbf{P}_c^{cf}$ given above does not help.)

Many, though not all, of the properties we have shown for classical logic still hold for intuitionistic and minimal logic. In particular, it is easy to check that the proof of the following lemma is identical to the proof for classical logic:

Lemma 1.24. *For any ϕ, $\mathbf{P}_m \vdash \phi \Rightarrow \phi$,*

The main property of intuitionistic logic we will find useful is the following:

Theorem 1.25 (Disjunction Property)**.** *If $\mathbf{P}_i^{cf} \vdash \phi \vee \psi$ then either $\mathbf{P}_i^{cf} \vdash \phi$ or $\mathbf{P}_i^{cf} \vdash \psi$.*

Proof. Consider the last step of such a proof. The only thing the last step can be is $R\vee$. Since the sequence can only have one element, the previous line is either $\Rightarrow \phi$ or $\Rightarrow \psi$. $\square$

Corollary 1.26. $\mathbf{P}_i^{cf} \nvdash p \vee \neg p$.

1.8. **Double Negation.** We now show that minimal and intuitionistic logic are not so different, either from each other or from classical logic, by creating various translations between these systems.

Theorem 1.27. *If $\perp$ is not a subformula of ϕ and $\mathbf{P}_i^{cf} \vdash \phi$ then also $\mathbf{P}_m^{cf} \vdash \phi$.*

We say $\mathbf{P}_i^{cf}$ is *conservative* over $\mathbf{P}_m^{cf}$ for formulas not involving $\perp$.

Proof. Follows immediately from the subformula property: the inference rule $L\perp$ cannot appear in the deduction of ϕ, and therefore the same deduction is a valid deduction in $\mathbf{P}_m^{cf}$. $\square$

Definition 1.28. If ϕ is a formula, we define a formula ϕ^* recursively by:

- p^* is $p \vee \perp$,
- $\perp^*$ is $\perp$,
- $(\phi \circledast \psi)^*$ is $\phi^* \circledast \psi^*$.

When Γ is a set of formulas, we write $\Gamma^* = \{\gamma^* \mid \gamma \in \Gamma\}$.

Theorem 1.29. *For any formula ϕ,*

1. $\mathbf{P}_i \vdash \phi \leftrightarrow \phi^*$*,*
2. $\mathbf{P}_m \vdash \perp \to \phi^*$*,*
3. *If $\mathbf{P}_i \vdash \phi$ then $\mathbf{P}_m \vdash \phi^*$.*

Proof. We will only prove the third part here. Naturally, this is done by induction on deductions of sequents $\Gamma \Rightarrow \phi$. We need to be a bit careful, because $\mathbf{P}_i$ allows the right hand side of a sequent to be empty, while $\mathbf{P}_m$ does not ($\mathbf{P}_m$ does not explicitly prohibit it, but an easy induction shows that it cannot occur). So more precisely, what we want is:

Suppose $\mathbf{P}_i \vdash \Gamma \Rightarrow \Sigma$. If $\Sigma = \{\phi\}$ then $\mathbf{P}_m \vdash \Gamma^* \Rightarrow \phi^*$, and if $\Sigma = \emptyset$ then $\mathbf{P}_m \vdash \Gamma^* \Rightarrow \bot$.

We prove this by induction on the the deduction of $\Gamma \Rightarrow \Sigma$. If this is an application of Ax, so $\Gamma = \Sigma = p$, there is a deduction of $p \vee \bot \Rightarrow p \vee \bot$.

If the deduction is a single application of L$\bot$, $\bot^* \Rightarrow \emptyset^*$ is $\bot^* \Rightarrow \bot^*$, which is an instance of Ax.

If the last inference is an application of W, we have

$$\dfrac{\overline{\overline{\Gamma \Rightarrow \Sigma}}}{\Gamma\Gamma' \Rightarrow \Sigma\Sigma'}$$

If $\Sigma\Sigma' = \Sigma$, this follows immediately from IH. Otherwise $\Sigma' = \{\phi\}$ while $\Sigma = \emptyset$. By the previous part, there must be a deduction of $\bot \to \phi^*$, and so there is a deduction

$$\dfrac{\dfrac{\dfrac{\overline{\overline{\Gamma^* \Rightarrow \Sigma^*}}\ \text{IH}}{\Gamma^* \Rightarrow \bot}}{(\Gamma\Gamma)^* \Rightarrow \bot} \qquad \dfrac{\overline{\overline{\bot \to \phi^*}}}{\bot \Rightarrow \phi^*}}{(\Gamma\Gamma)^* \Rightarrow \phi^*}$$

The other cases follow easily from IH. $\square$

Definition 1.30. We define the double negation interpretation of ϕ, ϕ^N, inductively by:

- $\bot^N$ is $\bot$,
- p^N is $\neg\neg p$,
- $(\phi_0 \wedge \phi_1)^N$ is $\phi_0^N \wedge \phi_1^N$,
- $(\phi_0 \vee \phi_1)^N$ is $\neg(\neg\phi_0^N \wedge \neg\phi_1^N)$,
- $(\phi_0 \to \phi_1)^N$ is $\phi_0^N \to \phi_1^N$.

Again $\Gamma^N = \{\gamma^N \mid \gamma \in \Gamma\}$.

It will save us some effort to observe that the following rule is admissible, even in minimal logic:

$$\dfrac{\Gamma \Rightarrow \phi}{\overline{\Gamma, \neg\phi \Rightarrow \bot}}$$

This is because it abbreviates the deduction

$$\dfrac{\Gamma \Rightarrow \phi \qquad \overline{\bot \Rightarrow \bot}}{\Gamma, \neg\phi \Rightarrow \bot}$$

The key point is that the formula ϕ^N is in a fairly special form—for example, ϕ^N does not contain $\vee$. In particular, while $\neg\neg\phi \to \phi$ is not, in general, provable in $\mathbf{P}_m$, or even $\mathbf{P}_i$, this is provable for formulas in the special form ϕ^N. (It is not hard to see that formulas like $\neg\neg p \to p$ are not provable without use of the cut-rule; non-provability in general will follow from the fact that the cut-rule is admissible.)

Lemma 1.31. $\mathbf{P}_m \vdash \neg\neg\phi^N \Rightarrow \phi^N$.

Proof. By induction on ϕ. When $\phi = \bot$, we have

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\bot \Rightarrow \bot$}
\UnaryInfC{$\Rightarrow \neg\bot$}
\doubleLine
\UnaryInfC{$\neg\neg\bot \Rightarrow \bot$}
\end{prooftree}
$$

When $\phi = p$, we have

$$
\begin{prooftree}
\AxiomC{}
\doubleLine
\UnaryInfC{$\neg p \Rightarrow \neg p$}
\doubleLine
\UnaryInfC{$\neg p, \neg\neg p \Rightarrow \bot$}
\UnaryInfC{$\neg p \Rightarrow \neg\neg\neg p$}
\doubleLine
\UnaryInfC{$\neg\neg\neg\neg p, \neg p \Rightarrow \bot$}
\UnaryInfC{$\neg\neg\neg\neg p \Rightarrow \neg\neg p$}
\end{prooftree}
$$

For $\phi = \phi_0 \wedge \phi_1$, we have

$$
\begin{prooftree}
\AxiomC{}
\doubleLine
\UnaryInfC{$\phi_0^N \Rightarrow \phi_0^N$}
\UnaryInfC{$\phi_0^N \wedge \phi_1^N \Rightarrow \phi_0^N$}
\doubleLine
\UnaryInfC{$\neg\phi_0^N, \phi_0^N \wedge \phi_1^N \Rightarrow \bot$}
\UnaryInfC{$\neg\phi_0^N \Rightarrow \neg(\phi_0^N \wedge \phi_1^N)$}
\doubleLine
\UnaryInfC{$\neg\neg(\phi_0^N \wedge \phi_1^N), \neg\phi_0^N \Rightarrow \bot$}
\UnaryInfC{$\neg\neg(\phi_0^N \wedge \phi_1^N) \Rightarrow \neg\neg\phi_0^N$}
\AxiomC{}
\doubleLine
\LeftLabel{IH}
\UnaryInfC{$\neg\neg\phi_0^N \Rightarrow \phi_0^N$}
\BinaryInfC{$\neg\neg(\phi_0^N \wedge \phi_1^N) \Rightarrow \phi_0^N$}
\AxiomC{}
\doubleLine
\UnaryInfC{$\phi_1^N \Rightarrow \phi_1^N$}
\UnaryInfC{$\phi_0^N \wedge \phi_1^N \Rightarrow \phi_1^N$}
\doubleLine
\UnaryInfC{$\neg\phi_1^N, \phi_0^N \wedge \phi_1^N \Rightarrow \bot$}
\UnaryInfC{$\neg\phi_1^N \Rightarrow \neg(\phi_0^N \wedge \phi_1^N)$}
\doubleLine
\UnaryInfC{$\neg\neg(\phi_0^N \wedge \phi_1^N), \neg\phi_1^N \Rightarrow \bot$}
\UnaryInfC{$\neg\neg(\phi_0^N \wedge \phi_1^N) \Rightarrow \neg\neg\phi_1^N$}
\AxiomC{}
\doubleLine
\LeftLabel{IH}
\UnaryInfC{$\neg\neg\phi_1^N \Rightarrow \phi_1^N$}
\BinaryInfC{$\neg\neg(\phi_0^N \wedge \phi_1^N) \Rightarrow \phi_1^N$}
\BinaryInfC{$\neg\neg(\phi_0^N \wedge \phi_1^N) \Rightarrow \phi_0^N \wedge \phi_1^N$}
\end{prooftree}
$$

For $\phi = \phi_0 \vee \phi_1$, we have

$$
\begin{prooftree}
\AxiomC{}
\doubleLine
\UnaryInfC{$\neg\phi_0^N \Rightarrow \neg\phi_0^N$}
\UnaryInfC{$\neg\phi_0^N \wedge \neg\phi_1^N \Rightarrow \neg\phi_0^N$}
\AxiomC{}
\doubleLine
\UnaryInfC{$\neg\phi_1^N \Rightarrow \neg\phi_1^N$}
\UnaryInfC{$\neg\phi_0^N \wedge \neg\phi_1^N \Rightarrow \neg\phi_1^N$}
\BinaryInfC{$\neg\phi_0^N \wedge \neg\phi_1^N \Rightarrow \neg\phi_0^N \wedge \neg\phi_1^N$}
\doubleLine
\UnaryInfC{$\neg\phi_0^N \wedge \neg\phi_1^N, \neg(\neg\phi_0^N \wedge \neg\phi_1^N) \Rightarrow \bot$}
\UnaryInfC{$\neg\phi_0^N \wedge \neg\phi_1^N \Rightarrow \neg\neg(\neg\phi_0^N \wedge \neg\phi_1^N)$}
\doubleLine
\LeftLabel{IH}
\UnaryInfC{$\neg\neg\neg(\neg\phi_0^N \wedge \neg\phi_1^N), \neg\phi_0^N \wedge \neg\phi_1^N \Rightarrow \bot$}
\UnaryInfC{$\neg\neg\neg(\neg\phi_0^N \wedge \neg\phi_1^N) \Rightarrow \neg(\neg\phi_0^N \wedge \neg\phi_1^N)$}
\end{prooftree}
$$

For $\phi = \phi_0 \rightarrow \phi_1$, we have

$$
\begin{prooftree}
\AxiomC{}
\doubleLine
\UnaryInfC{$\phi_0^N \Rightarrow \phi_0^N$}
\AxiomC{}
\doubleLine
\UnaryInfC{$\phi_1^N \Rightarrow \phi_1^N$}
\doubleLine
\UnaryInfC{$\neg\phi_1^N, \phi_1^N \Rightarrow \bot$}
\BinaryInfC{$\phi_0^N, \neg\phi_1^N, \phi_0^N \to \phi_1^N \Rightarrow \bot$}
\UnaryInfC{$\phi_0^N, \neg\phi_1^N \Rightarrow \neg(\phi_0^N \to \phi_1^N)$}
\doubleLine
\UnaryInfC{$\neg\neg(\phi_0^N \to \phi_1^N), \phi_0^N, \neg\phi_1^N \Rightarrow \bot$}
\UnaryInfC{$\neg\neg(\phi_0^N \to \phi_1^N), \phi_0^N \Rightarrow \neg\neg\phi_1^N$}
\AxiomC{}
\doubleLine
\UnaryInfC{$\neg\neg\phi_1^N \Rightarrow \phi_1^N$}
\BinaryInfC{$\neg\neg(\phi_0^N \to \phi_1^N), \phi_0^N \Rightarrow \phi_1^N$}
\UnaryInfC{$\neg\neg(\phi_0^N \to \phi_1^N) \Rightarrow \phi_0^N \to \phi_1^N$}
\end{prooftree}
$$

□

Corollary 1.32. *If* $\mathbf{P}_m \vdash \Gamma, \neg\phi^N \Rightarrow \bot$ *then* $\mathbf{P}_m \vdash \Gamma \Rightarrow \phi^N$.

Proof.

$$
\begin{prooftree}
\AxiomC{}
\doubleLine
\UnaryInfC{$\neg\phi^N \Rightarrow \bot$}
\UnaryInfC{$\Rightarrow \neg\neg\phi^N$}
\AxiomC{}
\doubleLine
\LeftLabel{1.31}
\UnaryInfC{$\neg\neg\phi^N \Rightarrow \phi^N$}
\BinaryInfC{$\Rightarrow \phi^N$}
\end{prooftree}
$$

□

Theorem 1.33. *Suppose* $\mathbf{P}_c \vdash \phi$. *Then* $\mathbf{P}_m \vdash \phi^N$.

Proof. We have to chose our inductive hypothesis a bit carefully. We will show by induction on deductions

$$\text{If } \mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma \text{ then } \mathbf{P}_m \vdash \Gamma^N, \neg(\Sigma^N) \Rightarrow \bot.$$

If our deduction consists only of Ax, we have either $\bot$ in Γ^N or $\neg\neg p$ in both Γ^N and Σ^N. In the former case we may take Ax followed by W. In the latter case we have

$$
\begin{prooftree}
\AxiomC{}
\doubleLine
\UnaryInfC{$\neg\neg p \Rightarrow \neg\neg p$}
\doubleLine
\UnaryInfC{$\neg\neg p, \neg\neg\neg p \Rightarrow \bot$}
\end{prooftree}
$$

If the deduction is an application of L$\bot$ then $\bot$ is in Γ^N, so the the claim is given by an instance of Ax.

If the last inference rule is L$\wedge$, we have $\phi_0 \wedge \phi_1$ in Γ, and therefore $\phi_0^N \wedge \phi_1^N$ in Γ^N, so the claim follows from L$\wedge$ applied to IH.

If the last inference rule is R$\wedge$, we have $\phi_0 \wedge \phi_1$ in Σ, and therefore $\neg(\phi_0^N \wedge \phi_1^N)$ in $\neg\Sigma^N$, so we have

$$
\begin{prooftree}
\AxiomC{}
\doubleLine
\LeftLabel{IH}
\UnaryInfC{$\Gamma^N, \neg\Sigma^N, \neg\phi_0^N \Rightarrow \bot$}
\doubleLine
\LeftLabel{1.32}
\UnaryInfC{$\Gamma^N, \neg\Sigma^N \Rightarrow \phi_0^N$}
\AxiomC{}
\doubleLine
\LeftLabel{IH}
\UnaryInfC{$\Gamma^N, \neg\Sigma^N, \neg\phi_1^N \Rightarrow \bot$}
\doubleLine
\LeftLabel{1.32}
\UnaryInfC{$\Gamma^N, \neg\Sigma^N \Rightarrow \phi_1^N$}
\BinaryInfC{$\Gamma^N, \neg\Sigma^N \Rightarrow \phi_0^N \wedge \phi_1^N$}
\doubleLine
\UnaryInfC{$\Gamma^N, \neg\Sigma^N, \neg(\phi_0^N \wedge \phi_1^N) \Rightarrow \bot$}
\end{prooftree}
$$

If the last inference rule is L$\vee$, we have $\phi_0 \vee \phi_1$ in Γ, and therefore $\neg(\neg\phi_0^N \wedge \neg\phi_1^N)$ in Γ^N. We have

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{IH}
\doubleLine
\UnaryInfC{$\Gamma^N, \phi_0^N, \neg\Sigma^N \Rightarrow \bot$}
\UnaryInfC{$\Gamma^N, \neg\Sigma^N \Rightarrow \neg\phi_0^N$}
\AxiomC{}
\LeftLabel{IH}
\doubleLine
\UnaryInfC{$\Gamma^N, \phi_1^N, \neg\Sigma^N \Rightarrow \bot$}
\UnaryInfC{$\Gamma^N, \neg\Sigma^N \Rightarrow \neg\phi_1^N$}
\BinaryInfC{$\Gamma^N, \neg\Sigma^N \Rightarrow \neg\phi_0^N \wedge \neg\phi_1^N$}
\doubleLine
\UnaryInfC{$\Gamma^N, \neg(\neg\phi_0^N \wedge \neg\phi_1^N), \neg\Sigma^N \Rightarrow \bot$}
\end{prooftree}
$$

If the last inference is R$\vee$ then we have $\neg\neg(\neg\phi_0^N \wedge \neg\phi_1^N)$ in $\neg\Sigma^N$, and so

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{IH}
\doubleLine
\UnaryInfC{$\Gamma^N, \neg\Sigma^N, \neg\phi_i^N \Rightarrow \bot$}
\UnaryInfC{$\Gamma^N, \neg\Sigma^N, \neg\phi_0^0 \wedge \neg\phi_1^N \Rightarrow \bot$}
\AxiomC{}
\LeftLabel{1.31}
\doubleLine
\UnaryInfC{$\neg\neg(\neg\phi_0^N \wedge \neg\phi_1^N) \Rightarrow \neg\phi_0^N \wedge \neg\phi_1^N$}
\BinaryInfC{$\Gamma^N, \neg\Sigma^N, \neg\neg(\neg\phi_0^N \wedge \neg\phi_1^N) \Rightarrow \bot$}
\end{prooftree}
$$

If the last inference is L$\to$ then we have $\phi_0^N \to \phi_1^N$ in Γ^N, and so

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{IH}
\doubleLine
\UnaryInfC{$\Gamma^N, \neg\Sigma^N, \neg\phi_0^N \Rightarrow \bot$}
\LeftLabel{1.32}
\doubleLine
\UnaryInfC{$\Gamma^N, \neg\Sigma^N \Rightarrow \phi_0^N$}
\AxiomC{}
\LeftLabel{IH}
\doubleLine
\UnaryInfC{$\Gamma^N, \phi_1^N, \neg\Sigma^N \Rightarrow \bot$}
\BinaryInfC{$\Gamma^N, \phi_0^N \to \phi_1^N, \neg\Sigma^N \Rightarrow \bot$}
\end{prooftree}
$$

If the last inference is R$\to$ then we have $\neg(\phi_0^N \to \phi_1^N)$ in $\neg\Sigma^N$, and so

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{IH}
\doubleLine
\UnaryInfC{$\Gamma^N, \phi_0^N, \neg\Sigma^N, \neg\phi_1^N \Rightarrow \bot$}
\LeftLabel{1.32}
\doubleLine
\UnaryInfC{$\Gamma^N, \phi_0^N, \neg\Sigma^N \Rightarrow \phi_1^N$}
\UnaryInfC{$\Gamma^N, \neg\Sigma^N \Rightarrow \phi_0^N \to \phi_1^N$}
\doubleLine
\UnaryInfC{$\Gamma^N, \neg\Sigma^N, \neg(\phi_0^N \to \phi_1^N) \Rightarrow \bot$}
\end{prooftree}
$$

If the last inference is Cut then we have

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{IH}
\doubleLine
\UnaryInfC{$\Gamma^N, \phi^N, \neg\Sigma^N \Rightarrow \bot$}
\UnaryInfC{$\Gamma^N, \neg\Sigma^N \Rightarrow \neg\phi^N$}
\AxiomC{}
\LeftLabel{IH}
\doubleLine
\UnaryInfC{$\Gamma^N, \neg\Sigma^N, \neg\phi^N \Rightarrow \bot$}
\BinaryInfC{$\Gamma^N, \neg\Sigma^N \Rightarrow \bot$}
\end{prooftree}
$$

This completes the induction. In particular, if $\mathbf{P} \vdash \phi$ then $\mathbf{P}_m \vdash \neg\phi^N \Rightarrow \bot$, and so by the previous corollary, $\mathbf{P}_m \vdash \phi^N$. $\square$

The following can also be shown by methods similar to (or simpler than) the proof of Lemma 1.31:

Theorem 1.34.

(1) $\mathbf{P}_m \vdash \phi \to \phi^N$,
(2) $\mathbf{P}_c \vdash \phi^N \leftrightarrow \phi$.
(3) $\mathbf{P}_i \vdash \phi^N \leftrightarrow \neg\neg\phi$

The last part of this theorem, together with the theorem above, gives:

Theorem 1.35 (Glivenko's Theorem)**.** *If* $\mathbf{P}_c \vdash \phi$ *then* $\mathbf{P}_i \vdash \neg\neg\phi$*.*

1.9. **Cut Elimination.** Finally we come to our main structural theorem about propositional logic:

Theorem 1.36. Cut *is admissible for* $\mathbf{P}_c^{cf}$*,* $\mathbf{P}_i^{cf}$*, and* $\mathbf{P}_m^{cf}$*.*

We have already given an indirect proof of this for $\mathbf{P}_c$, but the proof we give now will be *effective*: we will give an explicit method for transforming a deduction in $\mathbf{P}_c$ into a deduction in $\mathbf{P}_c^{cf}$.

To avoid repetition, we will write $\mathbf{P}_\epsilon$ to indicate any of the three systems $\mathbf{P}_c, \mathbf{P}_i, \mathbf{P}_m$.

In the cut-rule

$$\frac{\Gamma \Rightarrow \Sigma, \phi \qquad \Gamma', \phi \Rightarrow \Sigma'}{\Gamma\Gamma' \Rightarrow \Sigma\Sigma'}$$

we call ϕ the *cut formula*.

Definition 1.37. We define the *rank* of a formula inductively by:

- $rk(p) = rk(\bot) = 0$,
- $rk(\phi \circledast \psi) = \max\{rk(\phi), rk(\psi)\} + 1$.

We write $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma$ if there is a deduction of $\Gamma \Rightarrow \Sigma$ in $\mathbf{P}_\epsilon$ such that all cut-formulas have rank $< r$.

Clearly $\mathbf{P}_\epsilon \vdash_0 \Gamma \Rightarrow \Sigma$ iff $\mathbf{P}_\epsilon^{cf} \vdash \Gamma \Rightarrow \Sigma$.

Lemma 1.38 (Inversion)**.**

- *If* $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \phi_0 \wedge \phi_1$ *then for each* i*,* $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \phi_i$*.*
- *If* $\mathbf{P}_\epsilon \vdash_r \Gamma, \phi_0 \vee \phi_1 \Rightarrow \Sigma$ *then for each* i*,* $\mathbf{P}_\epsilon \vdash_r \Gamma, \phi_i \Rightarrow \Sigma$*.*
- *If* $\mathbf{P}_c \vdash_r \Gamma, \phi_0 \to \phi_1 \Rightarrow \Sigma$ *then* $\mathbf{P}_c \vdash_r \Gamma \Rightarrow \Sigma, \phi_0$ *and* $\mathbf{P}_c \vdash_r \Gamma, \phi_1 \Rightarrow \Sigma$*.*
- *If* $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \bot$ *with* $\epsilon \in \{c, i\}$ *then* $\mathbf{P}_\epsilon \vdash \Gamma \Rightarrow \Sigma$*.*

Proof. All parts are proven by induction on deductions. We will only prove the first part, since the others are similar.

We have a deduction of $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \phi_0 \wedge \phi_1$. If $\phi_0 \wedge \phi_1$ is not the main formula of the last inference rule, the claim immediately follows from IH. For example, suppose the last inference is

$$\frac{\overline{\overline{\Gamma \Rightarrow \Sigma, \psi_0, \phi_0 \wedge \phi_1}}}{\Gamma \Rightarrow \Sigma, \psi_0 \vee \psi_1, \phi_0 \wedge \phi_1}$$

Then $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \psi_0, \phi_0 \wedge \phi_1$, and so by IH, $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \psi_0, \phi_i$, and therefore

$$\frac{\text{IH}\ \overline{\overline{\Gamma \Rightarrow \Sigma, \psi_0, \phi_i}}}{\Gamma \Rightarrow \Sigma, \psi_0 \vee \psi_1, \phi_i}$$

demonstrates that $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \psi_0 \vee \psi_1, \phi_i$.

The interesting case is when $\phi_0 \wedge \phi_1$ is the main formula of the last inference. Then we must have

$$\dfrac{\overline{\overline{\Gamma \Rightarrow \Sigma, \phi_0 \wedge \phi_1, \phi_0}} \qquad \overline{\overline{\Gamma \Rightarrow \Sigma, \phi_0 \wedge \phi_1, \phi_1}}}{\Gamma \Rightarrow \Sigma, \phi_0 \wedge \phi_1}$$

In particular, $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \phi_0 \wedge \phi_1, \phi_i$, and therefore by IH, $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \phi_i$. (Of course, if $\epsilon \neq c$, we do not need to worry about the case where $\phi_0 \wedge \phi_1$ appears again in the premises, but we need to worry about this in all three systems in the $\vee$ and $\rightarrow$ cases.) □

Lemma 1.39.

(1) *Suppose* $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \phi_0 \wedge \phi_1$ *and* $\mathbf{P}_\epsilon \vdash_r \Gamma', \phi_0 \wedge \phi_1 \Rightarrow \Sigma'$, $rk(\phi_0 \wedge \phi_1) \leq r$. *Then* $\mathbf{P}_\epsilon \vdash_r \Gamma\Gamma' \Rightarrow \Sigma\Sigma'$.

(2) *Suppose* $\mathbf{P}_\epsilon \vdash_r \Gamma, \phi_0 \vee \phi_1 \Rightarrow \Sigma$ *and* $\mathbf{P}_\epsilon \vdash_r \Gamma' \Rightarrow \Sigma', \phi_0 \vee \phi_1$, $rk(\phi_0 \vee \phi_1) \leq r$. *Then* $\mathbf{P}_\epsilon \vdash_r \Gamma\Gamma' \Rightarrow \Sigma\Sigma'$.

(3) *Suppose* $\mathbf{P}_c \vdash_r \Gamma, \phi_0 \rightarrow \phi_1 \Rightarrow \Sigma$ *and* $\mathbf{P}_c \vdash_r \Gamma' \Rightarrow \Sigma', \phi_0 \rightarrow \phi_1$ *and* $rk(\phi_0 \rightarrow \phi_1) \leq r$. *Then* $\mathbf{P}_c \vdash_r \Gamma\Gamma' \Rightarrow \Sigma\Sigma'$.

Proof. Again, we only prove the first case, since the others are similar. We proceed by induction on the deduction of $\Gamma', \phi_0 \wedge \phi_1 \Rightarrow \Sigma'$.

If $\phi_0 \wedge \phi_1$ is not the main formula of the last inference rule of this deduction, the claim follows immediately from IH. If $\phi_0 \wedge \phi_1$ is the main formula of the last inference rule of this deduction, we must have

$$\dfrac{\overline{\overline{\Gamma', \phi_0 \wedge \phi_1, \phi_i \Rightarrow \Sigma'}}}{\Gamma', \phi_0 \wedge \phi_1 \Rightarrow \Sigma'}$$

Since $\mathbf{P}_\epsilon \vdash_r \Gamma', \phi_0 \wedge \phi_1, \phi_i \Rightarrow \Sigma'$, IH shows that $\mathbf{P}_\epsilon \vdash_r \Gamma\Gamma', \phi_i \Rightarrow \Sigma\Sigma'$. By Inversion, we know that $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \phi_i$, so we may apply a cut:

$$\dfrac{\overline{\overline{\Gamma\Gamma', \phi_i \Rightarrow \Sigma\Sigma'}} \qquad \overline{\overline{\Gamma \Rightarrow \Sigma, \phi_i}}}{\Gamma\Gamma' \Rightarrow \Sigma\Sigma'}$$

Since $rk(\phi_i) < rk(\phi_0 \wedge \phi_1) \leq r$, the cut formula has rank $< r$, so we have shown that $\mathbf{P}_\epsilon \vdash_r \Gamma\Gamma' \Rightarrow \Sigma\Sigma'$. □

While a similar argument works for $\vee$, the corresponding argument for $\rightarrow$ only works in $\mathbf{P}_c$. The reader might correctly intuit that in intuitionistic and minimal logic, the movement of formulas between the right and left sides of a sequent creates a conflict with the requirement that the right side of a sequent have at most one element. (Of course, the reader should check the classical case carefully, and note the step that fails in intuitionistic and minimal logic.)

The argument below covers the intuitionisitc and minimal cases (and, indeed, works fine in classical logic as well).

Lemma 1.40. *Suppose* $\mathbf{P}_\epsilon \vdash_r \Gamma, \phi_0 \to \phi_1 \Rightarrow \Sigma$ *and* $\mathbf{P}_\epsilon \vdash_r \Gamma' \Rightarrow \phi_0 \to \phi_1$ *and* $rk(\phi_0 \to \phi_1) \leq r$*. Then* $\mathbf{P}_\epsilon \vdash_r \Gamma\Gamma' \Rightarrow \Sigma$*.*

Proof. We handle this case by simultaneous induction on both deductions. (If this seems strange, we may think of this as being by induction on the sum of the sizes of the two deductions.)

If $\phi_0 \to \phi_1$ is not the main formula of the last inference of both deductions, the claim follows immediately from IH. So we may assume that $\phi_0 \to \phi_1$ is the last inference of both deductions. We therefore have deductions of

- $\Gamma, \phi_0 \to \phi_1, \phi_1 \Rightarrow \Sigma$,
- $\Gamma, \phi_0 \to \phi_1 \Rightarrow \phi_0$,
- $\Gamma', \phi_0 \Rightarrow \phi_1$.

Applying IH gives deductions of

- $\Gamma, \phi_1 \Rightarrow \Sigma$,
- $\Gamma \Rightarrow \phi_0$,
- $\Gamma', \phi_0 \Rightarrow \phi_1$.

Then we have

$$\dfrac{\dfrac{\overline{\overline{\Gamma', \phi_0 \Rightarrow \phi_1}} \qquad \overline{\overline{\Gamma \Rightarrow \phi_0}}}{\Gamma\Gamma' \Rightarrow \phi_1} \qquad \overline{\overline{\Gamma, \phi_1 \Rightarrow \Sigma}}}{\Gamma\Gamma' \Rightarrow \Sigma}$$

Since both ϕ_0 and ϕ_1 have rank $< rk(\phi_0 \to \phi_1) \leq r$, the resulting deduction demonstrates $\mathbf{P}_\epsilon \vdash_r \Gamma\Gamma' \Rightarrow \Sigma$. $\square$

Lemma 1.41. *Suppose ϕ is atomic,* $\mathbf{P}_\epsilon \vdash_0 \Gamma \Rightarrow \Sigma, \phi$*, and* $\mathbf{P}_\epsilon \vdash_0 \Gamma', \phi \Rightarrow \Sigma'$*. Then* $\mathbf{P}_\epsilon \vdash_0 \Gamma\Gamma' \Rightarrow \Sigma\Sigma'$*.*

Proof. By induction on the deduction of $\Gamma \Rightarrow \Sigma, \phi$. If the deduction is anything other than an axiom, the claim follows by IH. The deduction cannot be an instance of L$\bot$. If the deduction is an instance of Ax introducing ϕ then $\Gamma = \{\phi\}$, and therefore the deduction of $\Gamma', \phi \Rightarrow \Sigma'$ is also a deduction of $\Gamma\Gamma' \Rightarrow \Sigma'$, so the result follows by an application of W. $\square$

Theorem 1.42. *Suppose* $\mathbf{P}_\epsilon \vdash_{r+1} \Gamma \Rightarrow \Sigma$*. Then* $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma$*.*

Proof. By induction on deductions. If the last inference is anything other than a cut over a formula of rank r, the claim follows immediately from IH. If the last inference is a cut over a formula of rank r, we have

$$\dfrac{\overline{\overline{\Gamma \Rightarrow \Sigma, \phi}} \qquad \overline{\overline{\Gamma', \phi \Rightarrow \Sigma'}}}{\Gamma\Gamma' \Rightarrow \Sigma\Sigma'}$$

Therefore $\mathbf{P}_\epsilon \vdash_{r+1} \Gamma \Rightarrow \Sigma, \phi$ and $\mathbf{P}_\epsilon \vdash_{r+1} \Gamma', \phi \Rightarrow \Sigma'$, and by IH, $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma, \phi$ and $\mathbf{P}_\epsilon \vdash_r \Gamma', \phi \Rightarrow \Sigma'$.

If ϕ is $\phi_0 \circledast \phi_1$, Lemma 1.39 (or Lemma 1.40 when $\circledast = \to$ and $\epsilon \in \{i, m\}$) shows that $\mathbf{P}_\epsilon \vdash_r \Gamma\Gamma' \Rightarrow \Sigma\Sigma'$. If ϕ is atomic, we obtain the same result by Lemma 1.41. $\square$

Theorem 1.43. *Suppose* $\mathbf{P}_\epsilon \vdash_r \Gamma \Rightarrow \Sigma$. *Then* $\mathbf{P}_\epsilon \vdash_0 \Gamma \Rightarrow \Sigma$.

Proof. By induction on r, applying the previous theorem. $\square$

We have already mentioned the subformula property. We observe an explicit consequence of this, which will be particularly interesting when we move to first-order logic:

Theorem 1.44. *Suppose* $\mathbf{P}_\epsilon \vdash \Gamma \Rightarrow \Sigma$. *Then there is a formula* ψ *such that:*

- $\mathbf{P}_\epsilon \vdash \Gamma \Rightarrow \psi$,
- $\mathbf{P}_\epsilon \vdash \psi \Rightarrow \Sigma$,
- *Every propositional variable appearing in* ψ *appears in both* Γ *and* Σ.

Proof. We know that there is a cut-free deduction of $\Gamma \Rightarrow \Sigma$, and we proceed by induction on this deduction. We need a stronger inductive hypothesis:

> Suppose $\mathbf{P}_\epsilon \vdash \Gamma\Gamma' \Rightarrow \Sigma\Sigma'$ (where if $\epsilon \in \{i, m\}$ then $\Sigma = \emptyset$).
> Then there is a formula ψ such that:
> - $\mathbf{P}_\epsilon \vdash \Gamma \Rightarrow \Sigma, \psi$,
> - $\mathbf{P}_\epsilon \vdash \Gamma', \psi \Rightarrow \Sigma'$,
> - Every propositional variable appearing in ψ appears in both $\Gamma\Sigma$ and $\Gamma'\Sigma'$.

If the deduction is Ax (we assume for a propositional variable; the $\bot$ case is identical), there are four possibilities. If $p \in \Gamma$ and $p \in \Sigma'$ then $\psi = p$ suffices. If $p \in \Gamma'$ and $p \in \Sigma$ then $\psi = \neg p$ suffices (note that this case only happens in classical logic). If $p \in \Gamma$ and $p \in \Sigma$ then $\psi = \bot$ (again, this case only happens in classical logic). Finally if $p \in \Gamma'$ and $p \in \Sigma'$ then $\psi = \bot \to \bot$.

If the deduction is L$\bot$, if $\bot \in \Gamma$ then $\psi = \bot$, and if $\bot \in \Gamma'$ then $\psi = \bot \to \bot$.

If the final inference is L$\wedge$, we are deducing $\Gamma\Gamma', \phi_0 \wedge \phi_1 \Rightarrow \Sigma$ from $\Gamma\Gamma', \phi_i \Rightarrow \Sigma$. We apply IH with ϕ_i belonging to Γ if $\phi_0 \wedge \phi_1$ does, and otherwise belonging to Γ', and the ψ given by IH suffices.

If the final inference is R$\wedge$, suppose $\phi_0 \wedge \phi_1 \in \Sigma'$. Then we apply IH to $\Gamma\Gamma' \Rightarrow \Sigma\Sigma', \phi_0$ and $\Gamma \Rightarrow \Sigma\Sigma', \phi_1$, in both cases taking ϕ_i to belong to the Σ' component. This gives ψ_0, ψ_1 and deductions of $\Gamma \Rightarrow \Sigma, \psi_i$ and $\Gamma', \psi_i \Rightarrow \Sigma', \phi_i$. We take $\psi = \psi_0 \wedge \psi_1$, and obtain deductions of $\Gamma \Rightarrow \Sigma, \psi_0 \wedge \psi_1$ and $\Gamma', \psi_0 \wedge \psi_1 \Rightarrow \Sigma', \phi_0 \wedge \phi_1$.

If $\phi_0 \wedge \phi_1 \in \Sigma$ then we apply IH to $\Gamma\Gamma' \Rightarrow \Sigma\Sigma', \phi_0$ and $\Gamma \Rightarrow \Sigma\Sigma', \phi_1$, in both cases taking ϕ_i to belong to the Σ component. We obtain ψ_0, ψ_1 and deductions of $\Gamma \Rightarrow \Sigma, \phi_i, \psi_i$ and $\Gamma', \psi_i \Rightarrow \Sigma'$. We take $\psi = \psi_0 \vee \psi_1$ and obtain deductions of $\Gamma \Rightarrow \Sigma, \phi_0 \wedge \phi_1, \psi_0 \wedge \psi_1$ and $\Gamma, \psi_0 \vee \psi_1 \Rightarrow \Sigma'$.

The cases for L$\vee$, R$\vee$, R$\to$ are similar. If the final inference is L$\to$, $\phi_0 \to \phi_1$ is in Γ, and $\epsilon = c$, we apply IH to $\Gamma\Gamma' \Rightarrow \Sigma\Sigma', \phi_0$ and $\Gamma\Gamma', \phi_1 \Rightarrow \Sigma\Sigma'$ taking ϕ_0 to belong to the Σ component and ϕ_1 to belong to the Γ component, obtaining deductions of $\Gamma \Rightarrow \Sigma, \phi_0, \psi_0$, $\Gamma', \psi_0 \Rightarrow \Sigma'$, $\Gamma, \phi_1 \Rightarrow \Sigma, \psi_1$, and $\Gamma', \psi_1 \Rightarrow \Sigma'$. We take $\psi = \psi_0 \vee \psi_1$ and obtain deductions of $\Gamma, \phi_0 \to \phi_1 \Rightarrow \Sigma, \psi_0 \vee \psi_1$ and $\Gamma', \psi_0 \vee \psi_1 \Rightarrow \Sigma'$.

If the final inference is $\mathrm{L}\!\to$, $\phi_0 \to \phi_1$ is in Γ, and $\epsilon \in \{i, m\}$ then this argument does not quite work (Σ is required to be empty). But we may apply IH to $\Gamma\Gamma' \Rightarrow \phi_0$ and $\Gamma\Gamma', \phi_1 \Rightarrow \Sigma'$; the key step is that we *swap* the roles of Γ and Γ' in the first of these, and take ϕ_1 to belong to Γ in the second, so IH gives ψ_0, ψ_1, and deductions of $\Gamma' \Rightarrow \psi_0$, $\Gamma, \psi_0 \Rightarrow \phi_0$, $\Gamma, \phi_1 \Rightarrow \psi_1$, and $\Gamma', \psi_1 \Rightarrow \Sigma'$. Then we take $\psi = \psi_0 \to \psi_1$, which gives us deductions of $\Gamma', \psi_0 \to \psi_1 \Rightarrow \Sigma'$ and $\Gamma, \phi_0 \to \phi_1 \Rightarrow \psi_0 \to \psi_1$.

If $\phi_0 \to \phi_1 \in \Gamma'$, we apply IH to $\Gamma\Gamma' \Rightarrow \Sigma\Sigma', \phi_0$ and $\Gamma\Gamma', \phi_1 \Rightarrow \Sigma\Sigma'$ taking ϕ_0 to belong to the Σ' component and ϕ_1 to belong to the Γ' component, obtaining deductions of $\Gamma \Rightarrow \Sigma, \psi_0$, $\Gamma', \psi_0 \Rightarrow \Sigma', \phi_0$, $\Gamma \Rightarrow \Sigma, \psi_1$, and $\Gamma', \phi_1, \psi_1 \Rightarrow \Sigma'$. We take $\psi = \psi_0 \wedge \psi_1$ and obtain deductions of $\Gamma \Rightarrow \Sigma, \psi_0 \wedge \psi_1$ and $\Gamma', \phi_0 \to \phi_1, \psi_0 \wedge \psi_1 \Rightarrow \Sigma'$.

(Note that the $\to$ cases are what force us to use the more general inductive hypothesis: even if we begin with $\Gamma' = \Sigma = \emptyset$, the inductive step at $\to$ inference rules bring in the general situation.) $\square$

The results about cut-free proofs can be sharpened: we can distinguish between positive and negative occurrences of propositional variables in the interpolation theorem, we can improve the disjunction property of intuitionistic logic to include deductions from premises which have restrictions on disjunctions, and so on. In the next section we will need the following strengthened form of the subformula property.

Definition 1.45. If ϕ is a formula we define the *positive* and *negative* subformulas of ϕ, $pos(\phi)$ and $neg(\phi)$, recursively by:

- If ϕ is atomic then $pos(\phi) = \{\phi\}$ and $neg(\phi) = \emptyset$,
- For $\circledast \in \{\wedge, \vee\}$, $pos(\phi \circledast \psi) = pos(\phi) \cup pos(\psi) \cup \{\phi \circledast \psi\}$ and $neg(\phi \circledast \psi) = neg(\phi) \cup neg(\psi)$,
- $pos(\phi \to \psi) = pos(\psi) \cup neg(\phi) \cup \{\phi \to \psi\}$ while $neg(\phi \to \psi) = pos(\phi) \cup neg(\psi)$.

We say ϕ *occurs positively* in a sequent $\Gamma \Rightarrow \Sigma$ if $\phi \in \bigcup_{\gamma \in \Gamma} neg(\gamma) \cup \bigcup_{\sigma \in \Sigma} pos(\sigma)$ and ϕ *occurs negatively* in $\Gamma \Rightarrow \Sigma$ if $\phi \in \bigcup_{\gamma \in \Gamma} pos(\gamma) \cup \bigcup_{\sigma \in \Sigma} neg(\Sigma)$.

Theorem 1.46. *If d is a cut-free deduction of $\Gamma \Rightarrow \Sigma$ then any formula which occurs positively (resp. negatively) in any sequent anywhere in d also occurs positively (resp. negatively) in $\Gamma \Rightarrow \Sigma$.*

Proof. By induction on deductions. For axioms this is trivial. For rules this follows easily from IH; we give one example.

Suppose d is an application of $\mathrm{L}\!\to$ to d_0, d_1:

$$\dfrac{\begin{matrix} d_0 & & d_1 \\ \Gamma \Rightarrow \Sigma, \phi & & \Gamma', \psi \Rightarrow \Sigma' \end{matrix}}{\Gamma\Gamma', \phi \to \psi \Rightarrow \Sigma\Sigma'}$$

Observe that by IH, if θ occurs positively (resp. negatively) in d_0 then θ occurs positively (resp. negatively) in $\Gamma \Rightarrow \Sigma, \phi$, and similarly for d_1, so it suffices to show that any formula which occurs positively (resp. negatively) in $\Gamma \Rightarrow \Sigma, \phi$ or $\Gamma', \psi \Rightarrow \Sigma'$ also occurs positively (resp. negatively) in $\Gamma\Gamma', \phi \to \psi \Rightarrow \Sigma\Sigma'$.

For any formula in $\Gamma\Gamma'\Sigma\Sigma'$ this is immediate. The formula ϕ appears positively in $\Gamma \Rightarrow \Sigma, \phi$, and also in $\Gamma\Gamma', \phi \to \psi \Rightarrow \Sigma\Sigma'$, while ψ appears negatively in $\Gamma', \psi \Rightarrow \Sigma'$ and also $\Gamma\Gamma', \phi \to \psi \Rightarrow \Sigma\Sigma'$. □

Theorem 1.47. *Suppose $\mathbf{P}^{cf}_{\epsilon} \vdash \Gamma \Rightarrow \Sigma, p$ where p does not appear negatively in $\Gamma \Rightarrow \Sigma$. Then $\mathbf{P}^{cf}_{\epsilon} \vdash \Gamma \Rightarrow \Sigma$.*

Proof. By induction on deductions. If the deduction is an axiom, it must be some axiom other than $p \Rightarrow p$, since that would involve p negatively. For an inference rule, p, as an atomic formula, can never be the main formula of an inference, so the claim always follows immediately from the inductive hypothesis. □

1.10. **Optimality of Cut-Elimination.** While cut-elimination is admissible, we can show that it provides a substantial *speed-up* relative to cut-free proofs. More precisely, we will show that there are sequents which have short proofs using cut but only very long cut-free proofs.

Definition 1.48. The *size* of a deduction is the number of inference rules appearing in the deduction.

Size is a fairly coarse measure, and we usually prefer *height*—the length of the longest path from the conclusion to an axiom. But size will be easier to work with for our purposes.

A crucial observation is that the inversion lemma is actually size preserving: inspection of the proof reveals that we only remove inference rules from the deduction we start with, never add any, so the resulting deduction is no larger than the starting one. The only cases we need are:

Lemma 1.49.
1. *Suppose there is a deduction showing $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma, \phi_0 \wedge \phi_1$ and that this deduction has size m. Then for each $i \in \{0, 1\}$, there is a deduction showing $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma, \phi_i$ with size $\leq m$.*
2. *Suppose there is a deduction showing $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma, \phi_0 \vee \phi_1$ and that this deduction has size m. Then there is a deduction showing $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma, \phi_0, \phi_1$ with size $\leq m$.*
3. *Suppose there is a deduction showing $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma, \bot$ and that this deduction has size m. Then there is a deduction showing $\mathbf{P}_c \vdash \Gamma \Rightarrow \Sigma$ with size $\leq m$.*

Lower case letters will all represent propositional variables, while upper case letters will represent various formulas. We define the following families of formulas:

- $F_0 = \neg\bot$,
- $F_{n+1} = F_n \wedge (p_{n+1} \vee q_{n+1})$,
- $A_{n+1} = F_n \wedge \neg p_{n+1}$,
- $B_{n+1} = F_n \wedge \neg q_{n+1}$.

We then let Λ_n be the sequent

$$A_1 \wedge B_1, \dots, A_n \wedge B_n$$

and Γ_n be the sequent

$$\Lambda_n, p_n, q_n.$$

Consider why the sequent Γ_n must be true. F_0 is always true, and so $A_1 \wedge B_1$ is equivalent to $\neg p_1 \wedge \neg q_1$. F_1 is equivalent $p_1 \vee q_1$, so if $A_1 \wedge B_1$ fails, F_1 must be true. When F_1 is true, $A_2 \wedge B_2$ is equivalent to $\neg p_2 \wedge \neg q_2$, and so on.

The following lemma is easy to see by inspection of our proof that there is a deduction of $\phi \Rightarrow \phi$:

Lemma 1.50. *If ϕ is any formula, there is a deduction of $\phi \Rightarrow \phi$ with size linear in the number of symbols in ϕ (that is, the deduction is of size at most cn where c is a constant and n is the number of symbols in ϕ).*

Lemma 1.51. *Γ_n has a deduction of size quadratic in n. (That is, the size of the deduction is of size at most $cn^2 + dn + e$ for some constants c, d, e.)*

Proof. We construct several useful side deductions. First, for every n there is a deduction of $F_n \Rightarrow A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}$, and the size of this deduction is linear in n:

$$\dfrac{\dfrac{\dfrac{\overline{\overline{F_n \Rightarrow F_n}} \qquad \overline{\overline{\Rightarrow \neg p_{n+1}, p_{n+1}}}}{F_n \Rightarrow F_n \wedge \neg p_{n+1}, p_{n+1}}}{F_n \Rightarrow A_{n+1}, p_{n+1}} \qquad \dfrac{\dfrac{\overline{\overline{F_n \Rightarrow F_n}} \qquad \overline{\overline{\Rightarrow \neg q_{n+1}, q_{n+1}}}}{F_n \Rightarrow F_n \wedge \neg q_{n+1}, q_{n+1}}}{F_n \Rightarrow B_{n+1}, q_{n+1}}}{F_n \Rightarrow A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}}$$

(Indeed, this is almost constant, except for the deduction of $F_n \Rightarrow F_n$.)

Next, by induction on n we construct a deduction of Λ_n, F_n with size linear in n. For $n = 0$, this is just F_0, which is easily deduced with size 2. For the inductive case, we need to construct a deduction of

$$\Lambda_{n+1}, F_{n+1} = \Lambda_n, A_{n+1} \wedge B_{n+1}, F_{n+1}.$$

This is easily done using the deduction built above:

$$\dfrac{\dfrac{}{\Lambda_n, F_n} \qquad \dfrac{\dfrac{F_n \Rightarrow F_n \qquad \dfrac{\dfrac{}{F_n \Rightarrow A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}}}{F_n \Rightarrow A_{n+1} \wedge B_{n+1}, p_{n+1} \vee q_{n+1}}}{F_n \Rightarrow A_{n+1} \wedge B_{n+1}, F_n \wedge (p_{n+1} \vee q_{n+1})}}{F_n \Rightarrow A_{n+1} \wedge B_{n+1}, \mathbf{F_{n+1}}}}{\Lambda_n, A_{n+1} \wedge B_{n+1}, F_{n+1}}$$

From our deduction Λ_n, F_n, we wish to find a deduction of $\Gamma_{n+1} = \Lambda_n, A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}$. This is given by applying cut to the deduction of Λ_n, F_n and the deduction of $F_n \Rightarrow A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}$ given above.

□

We need the following lemma:

Lemma 1.52. *Suppose there is a cut-free deduction of*

$$\Gamma_n \setminus \{A_i \wedge B_i\}, \neg p_i, A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}$$

for some $i \leq n$. Then there is a cut-free deduction of Γ_n of the same size.

Proof. Since the deduction is cut-free, only the listed formulas can appear. We define an ad hoc transformation $\cdot^*$ on formulas appearing in this deduction:

- p_i^* is $\bot$,
- $\neg p_i^*$ is $\bot$,
- q_i^* is $\bot$,
- For any other atomic formula or negation of an atomic formula p, p^* is p,
- $(p_i \vee q_i)^*$ is $\bot$,
- For any other disjunction, ϕ^* is ϕ,
- For $k < i$, F_k^* is F_k,
- F_i^* is F_{i-1},
- F_{k+1}^* is $F_k^* \wedge (p_k \vee q_k)$ for $k > i$,
- A_k^* is $F_{k-1}^* \wedge \neg p_k$ for $k \neq i$,
- B_k^* is $F_{k-1}^* \wedge \neg q_k$ for $k \neq i$,
- $(A_k \wedge B_k)^*$ is $A_k^* \wedge B_k^*$.

Let d be a deduction of a sequent $\Delta \Rightarrow \Sigma$ consisting of subformulas of $\Gamma_n \setminus \{A_i \wedge B_i\}, \neg p_i, A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}$ and with the property that $p_i \vee q_i, p_i$, and q_i do not appear except as subformulas of F_i. (In particular, do not appear themselves.) Let m be the size of d. We show by induction on d that there is a cut-free deduction of $\Delta^* \Rightarrow \Sigma^*$ of size $\leq m$.

Any axiom must be introducing some atomic formula which is unchanged, and so the axiom remains valid.

Suppose d consists of an introduction rule with main formula F_i; then one of the subbranches contains a deduction of $F_i^* = F_{i-1}$, so we simply take the result of IH applied to this subdeduction. (Note that $p_i \vee q_i, p_i$, and q_i

may appear above the *other* branch, which we have just discarded, and so do not need to apply IH to it.)

The only other inference rules which can appear are R $\to$, R$\wedge$, R$\vee$ applied to formulas with the property that $(\phi \circledast \psi)^* = \phi^* \circledast \psi^*$, and therefore remain valid, so the claim in those cases followed immediately from IH.

In particular, we began with a deduction of

$$A_1 \wedge B_1, \ldots, A_{i-1} \wedge B_{i-1}, A_{i+1} \wedge B_{i+1}, \ldots, A_{n+1} \wedge B_{n+1}, \neg p_i, p_{n+1}, q_{n+1}.$$

After the transformation, this becomes a deduction of

$$A_1 \wedge B_1, \ldots, A_{i-1} \wedge B_{i-1}, A^*_{i+1} \wedge B^*_{i+1}, \ldots, A^*_{n+1} \wedge B^*_{n+1}, \bot, p_{n+1}, q_{n+1}.$$

Applying inversion to eliminate $\bot$ and renaming the propositional variables p_{k+1} to p_k and q_{k+1} to q_k for $k \geq i$, we obtain a deduction of Γ_n. □

Theorem 1.53. *A cut-free deduction of* Γ_n *has size* $\geq 2^n$ *for all* $n \geq 1$.

Proof. By induction on n.

Γ_1 is the sequent $A_1 \wedge B_1, p_1, q_1$. This is not an axiom, and so has size ≥ 2.

Suppose the claim holds for Γ_n. We show that it holds for Γ_{n+1}. The only possible final rule in a cut-free deduction is R$\wedge$. We consider two cases.

In the first case, the final rule introduces $A_{n+1} \wedge B_{n+1}$. We thus have a deduction of $\Lambda_n, A_{n+1}, A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}$ of size m and a deduction of $\Lambda_n, B_{n+1}, A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}$ of size m', with $m + m' + 1$ the size of the whole deduction.

We work with the first of these deductions. Applying inversion twice, we have a deduction of

$$\Lambda_n, F_n, p_{n+1}, q_{n+1}$$

of at most size m. Note that p_{n+1}, q_{n+1} do not appear elsewhere in Λ_n, F_n. In particular, since p_{n+1}, q_{n+1} do not appear negatively, there can be no axiom rule for these variables anywhere in the deduction. Therefore we have a deduction of

$$\Lambda_n, F_n = \Lambda_n, F_{n-1} \wedge (p_n \vee q_n)$$

of size at most m. Another application of inversion tells us that there must be a deduction of

$$\Lambda_n, p_n, q_n$$

of size at most m. But this is actually Γ_n, and therefore $2^n \leq m$. The same argument applied to the second of these deductions shows that $2^n \leq m'$. Then the size of the deduction of Γ_{n+1} was $m + m' + 1 \geq 2^n + 2^n + 1 = 2^{n+1} + 1$.

In the second case, we must introduce $A_i \wedge B_i$ for some $i \leq n$. We thus have a deduction of $\Lambda_n, A_i, A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}$ of size m, a deduction of $\Lambda_n, B_i, A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}$ of size m', with $m + m' + 1$ the size of the whole deduction.

Applying inversion to the first of these gives a deduction of

$$\Lambda_n \setminus \{A_i \wedge B_i\}, \neg p_i, A_{n+1} \wedge B_{n+1}, p_{n+1}, q_{n+1}$$

of height at most m. By the previous lemma, it follows that there is a deduction of Γ_n of height at most m. By similar reasoning, there must be a deduction of Γ_n of height at most m', and so again $m+m'+1 \geq 2^{n+1}+1$. $\square$