Question 1. *We have four balls, colored Emerald, Teal, Blue, and Red. How many possible orderings of these four balls are there?*

There are 4 objects, and we are ordering all 4, so the answer should be $P(4,4) = 4! = 24$. We list the possible arrangements, abbreviating each ball by its first letter:

ETBR	TEBR	ETRB	TERB
BETR	BTER	RETB	RTEB
BRET	BRTE	RBET	RBTE
EBTR	TBER	ERTB	EBTR
BERT	BTRE	REBT	RTBE
EBRT	TBRE	ERBT	TRBE

Question 2. *A philistine considers the Emerald and Teal balls to be indistinguishably Green. How many possible orderings does this person recognize?*

First approach: Start with the $4! = 24$ possible orderings where all the balls are distinct. Group these orderings into collections that all look the same to the philistine. Each of these collections contains exactly $2! = 2$ orderings: for example, one collection is $ETBR$ and $TEBR$, which both look like $GGBR$ to this person. When we group 24 orders into groups of 2 each, there are $24/2 = 12$ possible groups, so this person sees 12 orderings.

GGBR	GGRB
BGGR	RGGB
BRGG	RBGG
GBGR	GRGB
BGRG	RGBG
GBRG	GRBG

Second approach: We use the multiplication principle and a two-step process. In the first step, we assign two of the four slots to be green. There are $\binom{4}{2} = 6$ ways of doing this:

GG??
?GG?
??GG
G?G?
?G?G
G??G

As a second step, we take this setup and assign B to one of the blank spots and R to the other. There are $P(2,2) = 2$ ways of doing this. For instance, GG?? can be turned into either $GGBR$ or $GGRB$. This again gives us $\binom{4}{2} \cdot 2 = 12$ possible orderings.

Question 3. *Now suppose someone is red-green colorblind, and the Emerald, Teal, and Red balls all look indistinguishably Gray. How many possible orderings are there?*

First approach: We start with the 24 original orderings, and put them in groups that the colorblind person can't tell apart. For instance,

$$ETBR, TEBR, ERBT, TRBE, REBT, RTBE$$

are all indistinguishable to this person. There are $3! = 6$ ways of permuting the E, T, R balls, so each of these groups contains 6 of the orginal orderings. Therefore there are $24/6 = 4$ possible orderings:

GGGB GGBG GBGG BGGG

Second approach: We use a two step process. In the first step, we choose 3 of the 4 positions to contain Gray balls. In the final step, we assign the Blue ball to the remaining spot. There are $\binom{4}{3} = 4$ ways of doing the first step and only 1 way of doing the second, so there are $\binom{4}{3} \cdot 1 = 4$ possible orderings.

Question 4. *We now have* 10 *balls.* 3 *are Green,* 2 *are Red, and the remaining* 5 *are Yellow, Blue, Purple, Mauve, and Chartreuse (and all* 5 *can be distinguished from each other). How many possible orderings are there?*

First approach: If the Green and Red were distinguishable, there would be 10! possible orderings. We can then group these into indistinguishable collections. For instance one ordering is

$$G_1G_2YBR_1MR_2PG_3C.$$

Then all rearrangements of the Green and Red balls are indistinguishable:

$$\begin{array}{lll} G_1G_2YBR_1MR_2PG_3C & G_2G_3YBR_1MR_2PG_1C & G_3G_1YBR_1MR_2PG_2C \\ G_3G_2YBR_1MR_2PG_1C & G_2G_1YBR_1MR_2PG_3C & G_1G_3YBR_1MR_2PG_2C \\ G_1G_2YBR_2MR_1PG_3C & G_2G_3YBR_2MR_1PG_1C & G_3G_1YBR_2MR_1PG_2C \\ G_3G_2YBR_2MR_1PG_1C & G_2G_1YBR_2MR_1PG_3C & G_1G_3YBR_2MR_1PG_2C \end{array}$$

Given one of the 10! original colorings, there are $3! = 6$ ways of arranging the Green balls and $2! = 2$ ways of arranging the Red balls, for a total of 12 possible orderings. So there are $\frac{10!}{3!2!}$ orderings where we can't distinguish the green balls from each other or the blue balls from each other.

Second approach: A multistep process. In the first step, we choose 3 out of the 10 positions to be green. In the second step, we choose 2 of the remaining 7 positions to be red. In the final step, we pick an arrangement of the 5 remaining balls in the 5 remaining positions. There are

$$\binom{10}{3}\binom{7}{2}5! = \frac{10!}{7!3!}\frac{7!}{5!2!}5! = \frac{10!}{3!2!}$$

possible orderings.

Third approach: A different multistep process. We first place the 5 miscellaneous balls into 5 of the 10 positions in some order ($P(10,5)$ options). In the second step, we choose 3 of the 5 remaining positions to receive green balls ($\binom{5}{3}$ options). There are 2 positions left, and they will both receive red balls ($\binom{2}{2}$ options). There are

$$P(10,5)\binom{5}{3}\binom{2}{2} = \frac{10!}{5!}\frac{5!}{3!2!}\cdot 1 = \frac{10!}{3!2!}$$

possible orderings.

A question to see if you have the idea:

Question 5. *A track and field team is attending a competition. There are 8 team members (all distinguishable!) and 14 events. The two team captains will participate in two events each, the other six will only participate in one each. (The team will skip four events.) How many possible arrangements are there?*