MIDTERM 2

Math 340
11/15/2012

Name: ______________________________

ID: ______________________________

"I have adhered to the Penn Code of Academic Integrity in completing this exam."

Signature: ______________________________

Read all of the following information before starting the exam:

- Check your exam to make sure all pages are present.
- You do not need to simplify answers. You may include factorials, $P(n, k)$, $\binom{n}{k}$, etc. in your answers. Do not use $\sum$ in your answers.
- You may use writing implements and a single 3"x5" notecard.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Good luck!

1	15	
2	15	
3	15	
4	10	
5	20	
6	15	
7	10	
Total	100	

1. (*15 points*) (**a**) Find the general solution to the recurrence relation given by

$$a_{n+2} = 4a_{n+1} - 8a_n.$$

(**b**) Find the particular solution to $a_{n+2} = 4a_{n+1} - 8a_n$ where $a_0 = 1$ and $a_1 = 8$.

(**c**) Find one particular solution to the recurrence relation given by

$$a_{n+2} = 4a_{n+1} - 8a_n + n.$$

2. (*15 points*) Find the solution to the recurrence relation given by

$$a_{n+4} = 7a_{n+2} + 8a_n$$

with $a_0 = 1, a_1 = 1, a_2 = 17, a_3 = 8$. (Hint: it will be easier to give this as two separate equations for two different cases.)

3. (*15 points*) You have a bag with a very large number of marbles in the colors green, red, and yellow. Marbles are indistinguishable except for color. Let a_n be the number of ways to draw n marbles such that:

- There are an even number of yellow marbles,
- There are at least three red marbles.

(**a**) Write an expression which evaluates to the generating function for the sequence $a_0, a_1, \ldots$. (Your expression may involve infinite sums, $\cdots$, and so on..)

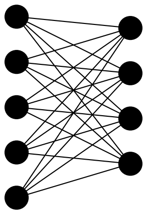
(**b**) Find an explict form for the generating function as a ratio of polynomials.

4. (*10 points*) Let $a_0, a_1, a_2, \ldots$ be a sequence with generating function $G(x) = \sum_{n=0}^{\infty} a_n x^n$ and let $b_0, \ldots$ be the sequence whose generating function is $F(x) = G(3x^2)$. Give an explicit formula for calculating b_n in terms of the sequence $a_0, \ldots$.

5. (*20 points*) (**a**) Prove that every connected graph has a connected spanning (containing every vertex) subgraph with no cycles.

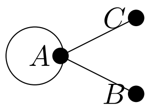
(**b**) Prove that if a graph has no cycles then the graph is planar.

6. (*15 points*) Consider a complete bipartite graph $K_{m,n}$: that is there are two blocks of vertices V_1 and V_2 with $|V_1| = m$ and $|V_2| = n$, and the edges present are exactly those which connect one vertex of V_1 with one vertex of V_2. For instance, the graph below is $K_{5,4}$:



By a *cycle of length* 6, we mean a sequence of 6 distinct vertices $v_1, v_2, \ldots, v_6$ such that each pair v_i, v_{i+1} is adjacent, and also v_6, v_1 is adjacent. Since the vertices of a cycle are symmetric, we do not consider changes in rotation or starting point significant: if a, b, c, x, y, z is a cycle of length 6, so are $b, c, x, y, z, a,$ x, c, b, a, z, y and a, z, y, x, c, b, a. However a, x, c, b, y, z (if it is a cycle) is different.
If $m \geq 3$ and $n \geq 3$, how many distinct cycles of length 6 are there in the graph $K_{m,n}$.

7. (*10 points*) Consider the graph below:



We would like to count how many closed walks of length n there are starting from the vertex A. Recall that a closed walk is a sequence of $n+1$ vertices $v_0, \ldots, v_n$ so that each pair is adjacent. Since the walk is closed, we should have $v_0 = v_n = A$. In a walk, we are allowed to repeat both edges and vertices freely. For instance, there is one walk of length 1 (the walk which goes around the loop) and three walks of length 2 (the walk which goes around the loop, the one which goes to B and then back, and the one which goes to C and then back). Find a recurrence relation for a_n, the number of walks of length n. (You do not need to solve this recurrence relation.)