# MIDTERM 2

Math 340
11/15/2012

**Name:** ______________________

**ID:** ______________________

"I have adhered to the Penn Code of Academic Integrity in completing this exam."

**Signature:** ______________________

**Read all of the following information before starting the exam:**

- Check your exam to make sure all pages are present.
- You do not need to simplify answers. You may include factorials, $P(n,k)$, $\binom{n}{k}$, etc. in your answers. Do not use $\sum$ in your answers.
- You may use writing implements and a single 3"$x$5" notecard.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Good luck!

| | | |
|---|---|---|
| 1 | 15 | |
| 2 | 10 | |
| 3 | 20 | |
| 4 | 15 | |
| 5 | 20 | |
| 6 | 10 | |
| 7 | 10 | |
| Total | 100 | |

**1.** (*10 points*) **(a)** Find the general solution to the recurrence relation given by

$$a_{n+2} = 4a_{n+1} - 8a_n.$$

The characteristic equation is $r^2 = 4r - 8$, so $r^2 - 4r + 8 = 0$. The solutions are $2 \pm 2i$, which has magnitude $\sqrt{8}$ and angle $\pi/4$, so

$$a_n = \sqrt{8}^n \left( k_1 \cos \frac{n\pi}{4} + k_2 \sin \frac{n\pi}{4} \right).$$

**(b)** Find the particular solution to $a_{n+2} = 4a_{n+1} - 8a_n$ where $a_0 = 1$ and $a_1 = 8$.

We have $1 = a_0 = k_1$ while $8 = \sqrt{8}(k_1\sqrt{2}/2 + k_2\sqrt{2}/2) = 2(k_1 + k_2)$, so $k_1 = 1$ and $k_2 = 3$, so the general solution is

$$a_n = \sqrt{8}^n \left( \cos \frac{n\pi}{4} + 3 \sin \frac{n\pi}{4} \right).$$

**(c)** Find one particular solution to the recurrence relation given by

$$a_{n+2} = 4a_{n+1} - 8a_n + n.$$

We guess $a_n = bn + c$, so

$$b(n+2) + c = 4(b(n+1) + c) - 8(bn + c) + n.$$

Expanding,

$$bn + 2b + c = 4bn + 4b + 4c - 8bn - 8c + n.$$

This gives two equations; the first is $b = 4b - 8b + 1$, which implies $b = 1/5$, and the second is then $2/5 + c = 4/5 + 4c - 8c$, which implies $c = 2/25$. So a particular solution is $a_n = n/5 + 2/25$.

**2.** (*10 points*) Find the solution to the recurrence relation given by

$$a_{n+4} = 7a_{n+2} + 8a_n$$

with $a_0 = 1, a_1 = 1, a_2 = 17, a_3 = 8$. (Hint: it will be easier to give this as two separate equations for two different cases.)

We observe that $a_{n+4}$ depends only on $a_{n+2}$ and $a_n$. Let us define $b_n = a_{2n}$ and $c_n = a_{2n+1}$. Then

$$b_{n+2} = 7b_{n+1} + 8b_n, b_0 = 1, b_1 = 17.$$

The characteristic equation is $r^2 - 7r - 8 = 0$, which factors as $(r-8)(r+1) = 0$, so $r = 8$ or $r = -1$. The general solution is $b_n = k_0 8^n + k_1(-1)^n$. If $1 = k_0 + k_1$ and $17 = 8k_0 - k_1$, we add the equations to find $18 = 9k_0$, so $k_0 = 2$, and therefore $k_1 = -1$. So $a_{2n} = b_n = 2 \cdot 8^n + (-1)^{n+1}$.

The $c_n$ satisfy the same general equation, but with $c_0 = 1$ and $c_1 = 8$, so we have $1 = k_0 + k_1$ and $8 = 8k_0 - k_1$, so $9 = 9k_0$, so $k_0 = 1$ and $k_1 = 0$, so $a_{2n+1} = c_n = 8^n$.

**3.** (*10 points*) You have a bag with a very large number of marbles in the colors green, red, and yellow. Marbles are indistinguishable except for color. Let $a_n$ be the number of ways to draw $n$ marbles such that:

- There are an even number of yellow marbles,
- There are at least three red marbles.

(**a**) Write an expression which evaluates to the generating function for the sequence $a_0, a_1, \ldots$. (Your expression may involve infinite sums, $\cdots$, and so on..)

$$(1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + \cdots)(x^3 + x^4 + \cdots)$$

(**b**) Find an explict form for the generating function as a ratio of polynomials.

$$\frac{1}{1-x}\frac{1}{1-x^2}\frac{x^3}{1-x}.$$

**4.** (*10 points*) Let $a_0, a_1, a_2, \ldots$ be a sequence with generating function $G(x) = \sum_{n=0}^{\infty} a_n x^n$ and let $b_0, \ldots$ be the sequence whose generating function is $F(x) = G(3x^2)$. Give an explicit formula for calculating $b_n$ in terms of the $a_k$.

$F(x) = \sum_{n=0}^{\infty} a_n 3^n x^{2n}$, so $b_{2n} = a_n 3^n$ while $b_{2n+1} = 0$.
If we want to combine these into a single formula, we could write

$$b_n = \frac{a_{\lceil n/2 \rceil} 3^{\lceil n/2 \rceil} + (-1)^n a_{\lceil n/2 \rceil} 3^{\lceil n/2 \rceil}}{2}.$$

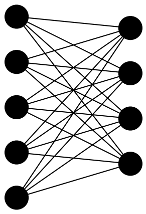
**5.** (*10 points*) **(a)** Prove that every connected graph has a connected spanning (containing every vertex) subgraph with no cycles.

By induction on the number of vertices. If there is one vertex, this vertex by itself is a spanning subgraph. For the inductive case, suppose that every graph with fewer vertices has a spanning subgraph with no cycles. Pick a vertex $v$, and consider the graph $G - v$; this graph has fewer vertices, so it has a spanning subgraph $H$ with no cycles. Since $G$ is connected, there is at least one edge between $v$ and some vertex in $H$; add exactly one of these edges. The result is obviously connected and spanning. To see that it contains no cycles, observe that any cycle would have to contain $v$ (since otherwise it would be a cycle in $H$), but $deg(v) = 1$, and a cycle contains no vertices of degree 1.

**(b)** Prove that if a graph has no cycles then the graph is planar.

If $G$ has no cycles, every subgraph of $G$ also has no cycles. Also, if $G$ has no cycles, $G/e$ has no cycles. In particular, any minor of $G$ has no cycles. If $G$ were non-planar, it would have either $K_5$ or $K_{3,3}$ as a minor, and both of these have cycles, contradicting the fact that $G$ has no cycles.

**6.** (*10 points*) Consider a complete bipartite graph $K_{m,n}$: that is there are two blocks of vertices $V_1$ and $V_2$ with $|V_1| = m$ and $|V_2| = n$, and the edges present are exactly those which connect one vertex of $V_1$ with one vertex of $V_2$. For instance, the graph below is $K_{5,4}$:



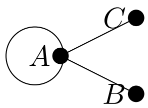
By a *cycle of length* 6, we mean a sequence of 6 distinct vertices $v_1, v_2, \ldots, v_6$ such that each pair $v_i, v_{i+1}$ is adjacent, and also $v_6, v_1$ is adjacent. Since the vertices of a cycle are symmetric, we do not consider changes in rotation or starting point significant: if $a, b, c, x, y, z$ is a cycle of length 6, so are $b, c, x, y, z, a$, $x, c, b, a, z, y$ and $a, z, y, x, c, b, a$. However $a, x, c, b, y, z$ (if it is a cycle) is different.
If $m \geq 3$ and $n \geq 3$, how many distinct cycles of length 6 are there in the graph $K_{m,n}$.

Clearly a cycle consists of 3 vertices from each side, and there are $\binom{m}{3}\binom{n}{3}$ ways of choosing these vertices. However each choice of 6 vertices could determine several cycles, depending on the order. We reason as follows: suppose $a, b, c$ are the vertices chosen from the left side and $x, y, z$ the vertices from the right. In any cycle, $a$ will have two of $x, y, z$ as neighbors, and there are $\binom{3}{2}$ ways to choose these; once we've chosen the neighbors, say $x$ and $y$, we have to decide which of $b, c$ will be next to $x$, and there are two ways to do this. Once we have made both these decisions, we have completely specified the cycle. There are

$$\binom{m}{3}\binom{n}{3}\binom{3}{2}2$$

total cycles.

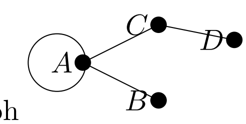
**7.** (*10 points*) Consider the graph below:




We would like to count how many closed walks of length $n$ there are starting from the vertex $A$. Recall that a closed walk is a sequence of $n+1$ vertices $v_0, \ldots, v_n$ so that each pair is adjacent. Since the walk is closed, we should have $v_0 = v_n = A$. In a walk, we are allowed to repeat both edges and vertices freely. For instance, there is one walk of length 1 (the walk which goes around the loop) and three walks of length 2 (the walk which goes around the loop, the one which goes to $B$ and then back, and the one which goes to $C$ and then back). Find a recurrence relation for $a_n$, the number of walks of length $n$. (You do not need to solve this recurrence relation.)

We consider what the first step of some walk of length $n$ must be: either it loops around $A$, and is then followed by some walk of length $n-1$, or it goes up to $B$ and back and is followed by some walk of length $n-2$, or up to $C$ and back and is followed by some walk of length $n-2$. So

$$a_n = a_{n-1} + 2a_{n-2}.$$

For the original version of the problem, with the graph 

we have to consider the possibility that the walk goes from $A$ to $C$, and then back and forth between $C$ and $D$ any number of times before coming back to $A$. Therefore

$$a_n = a_{n-1} + a_{n-2} + \sum_{k<n/2} a_{n-2k}.$$