2.1.1

Note that these answers are not the only possible correct ones.

1. $\forall x N x \to 0 < x$
2. $\forall x(Nx \wedge Ix \to I0)$ (another possibility is $\forall x N x \to (Ix \to I0))$
3. $\forall x \neg x < 0$
4. $\forall x((\neg Ix \wedge \forall y y < x \to Iy) \to Ix)$
5. $\neg \exists x N x \wedge (\forall y N y \to y < x)$
6. $\neg \exists x N x \wedge (\neg \exists y N y \wedge y < x)$

2.2.3

By definition, we are trying to show that whenever $\mathfrak{A}$ is a structure and s a substitution such that

$$\models_{\mathfrak{A}} \forall x(\alpha \to \beta)[s]$$

and

$$\models_{\mathfrak{A}} \forall x \alpha[s],$$

then also $\models_{\mathfrak{A}} \forall x \beta[s]$.

By the first assumption, for every $a \in |\mathfrak{A}|$,

$$\models_{\mathfrak{A}} \alpha \to \beta[s(x \mapsto a)],$$

which is equivalent to saying that for every $a \in |\mathfrak{A}|$,

$$\text{either } \not\models_{\mathfrak{A}} \alpha[s(x \mapsto a)] \text{ or } \models_{\mathfrak{A}} \beta[s(x \mapsto a)].$$

Also, by the second assumption, for every $a \in |\mathfrak{A}|$

$$\models_{\mathfrak{A}} \alpha[s(x \mapsto a)].$$

Putting these together, for every $a \in |\mathfrak{A}|$,

$$\models_{\mathfrak{A}} \beta[s(x \mapsto a)].$$

But that is equivalent to

$$\models_{\mathfrak{A}} \forall x \beta[s],$$

which is what we wanted to show.

2.2.6

First, suppose θ is valid. That is, for every $\mathfrak{A}$ and every s, $\vDash_{\mathfrak{A}} \theta[s]$. Now we want to show that $\forall x\theta$ is valid, so let $\mathfrak{A}$ be any structure and s and substitution. We are trying to show that $\vDash_{\mathfrak{A}} \forall x\theta[s]$, which is equivalent to showing that for every $a \in |\mathfrak{A}|$, $\vDash_{\mathfrak{A}} \theta[s(x \mapsto a)]$. But for every $a \in |\mathfrak{A}|$, $\mathfrak{A}$ is a structure and $s(x \mapsto a)$ is a substitution, so by assumption we do have $\vDash_{\mathfrak{A}} \theta[s(x \mapsto a)]$.

Second, suppose $\forall x\theta$ is valid, so for every $\mathfrak{A}$ and every s, $\vDash_{\mathfrak{A}} \forall x\theta[s]$. This means that for every $\mathfrak{A}$, every $a \in |\mathfrak{A}|$, and every s, $\vDash_{\mathfrak{A}} \theta[s(x \mapsto a)]$. Now take any $\mathfrak{A}$ and any s; we want to show that $\vDash_{\mathfrak{A}} \theta[s]$. Observe that $s = s(x \mapsto s(x))$, so indeed we have $\vDash_{\mathfrak{A}} \theta[s(x \mapsto s(x))]$, which is the same as $\vDash_{\mathfrak{A}} \theta[s]$.

2.2.2

There are many possible answers. Note that (a) says P is transitive and (b) says P is antisymmetric (the propery $\leq$ has). So it is natural to think about structures that look like orderings. First, we want to satisfy (a) and (b), but not (c); in order to make (c) false, we need to satisfy $\forall x\exists yPxy$ but not $\exists y\forall xPxy$. If we take $|\mathfrak{N}| = \mathbb{N}$ and $P^{\mathfrak{N}} = \{\langle n, m\rangle \mid n \leq m\}$, (a) and (b) are the transitivity and antisymmetry of $\leq$, which are standard properties. $\vDash_{\mathfrak{N}} \forall x\exists yPxy$ since, for example, $\vDash_{\mathfrak{N}} \forall xPxx$. On the other hand, $\nvDash_{\mathfrak{N}} \exists y\forall xPxy$ since there is no largest element in $\mathbb{N}$.

To satisfy (a) and (c) but not (b), we want a structure which violates antisymmetry. One approach is to go all the way, and impose outright symmetry: take $|\mathfrak{A}| = \{u, v, w\}$ and let $P^{\mathfrak{A}} = \{\langle u, v\rangle, \langle v, u\rangle, \langle u, u\rangle, \langle u, v\rangle\}$. (a) is satisfied in this structure because if $\vDash_{\mathfrak{A}} Pxy[s]$ and $\vDash_{\mathfrak{A}} Pyz[s]$ then $s(x), s(y), s(z)$ must all be in $\{u, v\}$, so $\vDash_{\mathfrak{A}} Pxz[s]$ as well. (c) is satisfied because $\nvDash_{\mathfrak{A}} \forall x\exists yPxy$—consider the case where we interpret x as w. (b) is not satisfied because $\vDash_{\mathfrak{A}} Pxy[[u, v]]$, $\vDash_{\mathfrak{A}} Pyx[[u, v]]$, but $\nvDash_{\mathfrak{A}} x = y[[u, v]]$.

To satisfy (b) and (c) but not (a), we can take $|\mathfrak{B}| = \{u, v, w\}$ and let $P^{\mathfrak{A}} = \{\langle u, v\rangle, \langle v, w\rangle\}$. $\mathfrak{B}$ satisfies (b) because for any s, $\nvDash_{\mathfrak{B}} Pxy \land Pyx$. $\mathfrak{B}$ satisfies (c) because $\nvDash_{\mathfrak{B}} \forall x\exists yPxy$—again, consider the case where we interpret x as w. And $\mathfrak{B}$ does not satisfy (a) because $\vDash_{\mathfrak{B}} Pxy[[u, v, w]]$, $\vDash_{\mathfrak{B}} Pyz[[u, v, w]]$, but $\nvDash_{\mathfrak{B}} Pxz[[u, v, w]]$.

2.2.11

a

One possibility is $\forall x x + v_1 = x$.

b

One possibility is $\forall x x \cdot v_1 = x$.

c

One possibility is $\forall y (\forall x x \cdot y = x) \to y + v_1 = v_2$. Another is $\exists y (\forall x x \cdot v_1 = x) \land y + v_1 = v_2$.

d

One possibility is $\neg v_1 = v_2 \land \exists x x + v_1 = v_2$.

2.2.12

a

One possibility is $\exists x x \cdot x = v_1$.

b

One possibility is $\exists x (\forall y y \cdot x = y) \land v_1 = x + x$.

c

Note the hints above that we can put together definitions to make more complicated definitions. Let's build this up in pieces. Let's write $\phi_1(v_1)$ for the formula $\forall z z \cdot v_1 = z$ (that is, $\phi_1(x)$ means "x is 1"). If n is a positive integer, we can define n by $\phi_n(v_1)$, which is

$$\exists x \phi_1(x) \land v_1 = x + x + \cdots + x.$$

We can define 0 by $\phi_0(v_1)$: $\forall z z + v_1 = z$, we can define -1 by $\phi_{-1}(v_1)$:

$$\exists y \phi_1(y) \land v_1 \neq y \land v_1 \cdot v_1 = y$$

and any negative number $-n$ by $\phi_{-n}(v_1)$, which is

$$\exists x \phi_{-1}(x) \wedge v_1 = x + x + \cdots + x.$$

Now if n/m is any rational number, we have $\phi_{n/m}(v_1)$: $\exists y \exists z \phi_n(y) \wedge \phi_m(z) \wedge v_1 \cdot m = n$.

Suppose that $r < s$ are values we can define—that is, that we have formulas $\phi_r(v_1)$ and $\phi_s(v_1)$ so

$$\vDash_{\mathfrak{R}} \phi_r(v_1)[[a]]$$

iff $a = r$, and

$$\vDash_{\mathfrak{R}} \phi_s(v_1)[[a]]$$

iff $a = s$. We can define a formula $\phi_{[r,\infty)}(v_1)$ defining the interval $[r, \infty)$ by:

$$\exists x \exists z \exists w \phi_r(x) \wedge (w \cdot w = z) \wedge x + z = v_1.$$

Similarly, we can define a formula $\phi_{(-\infty,s]}(v_1)$ by:

$$\exists x \exists z \exists w \phi_r(x) \wedge (w \cdot w = z) \wedge v_1 + z = x.$$

Then the interval $[r, s]$ is defined by $\phi_{[r,s]}(v_1)$, which is $\phi_{[r,\infty)}(v_1) \wedge \phi_{(-\infty,s]}(v_1)$.

Suppose r is an algebraic number, so there are integers $n_0, \ldots, n_k$ so that

$$\sum_{i \leq k} n_i r^i = 0.$$

We can write down a formula which holds exactly of the roots of this equation $\phi_{\sum_{i \leq k} n_i r^i = 0}(v_1)$:

$$\exists x_0 \exists x_1 \cdots \exists x_k \phi_{n_0}(x_0) \wedge \phi_{n_1}(x_1) \wedge \cdots \phi_{n_k}(x_k) \wedge (n_0 + n_1 \cdot v_1 + n_2 \cdot v_1 \cdot v_1 + \cdots = 0).$$

To pick out a particular root, we can find some rational interval $[p, q]$ containing the root we want and no other roots,

$$\phi_{\sum_{i \leq k} n_i r^i = 0}(v_1) \wedge \phi_{[p,q]}(v_1).$$

This formula defines any algebraic number, so we also have all closed intervals. To get open intervals, we can always combine with $\neg \phi_r(v_1)$ to rule out an endpoint.