Inference

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A Model with Sticky Price and Sticky Wage

- Household $j \in [0, 1]$ maximizes utility function:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{G_t \left(C_t^j \right)^{1-\frac{1}{\sigma}}}{1 - \frac{1}{\sigma}} - \frac{\left(N_t^j \right)^{1+\gamma}}{1+\gamma} + \frac{\eta}{1-\xi} \left(\frac{M_t^j}{P_t} \right)^{1-\xi} \right]$$

$0 < \beta < 1$ is the discount factor, $\sigma > 0$ the elasticity of intertemporal substitution, $\xi > 1$ the elasticity of money holdings, and $\gamma > 0$ the inverse of the elasticity of labor supply with respect to real wages.

- Subject the budget constrain given by:

$$P_t C_t^j + M_t^j - M_{t-1}^j + \sum_{h_{t+\tau}} Q_t(h_{t+\tau}) D_t^j(h_{t+\tau}) + \frac{B_{t+1}^j}{R_t}$$
$$= W_t^j N_t^j + \Pi_t^j + T_t^j + D_t^j + B_t^j,$$

where Π_t^j are firms' profits, T_t^j are nominal transfers, $D_t^j(h_{t+\tau})$ denotes holdings of contingent bonds, B_{t+1}^j denotes holdings of an uncontingent bonds, and W_t^j is the hourly nominal wage.

Technology

- Intermediate Goods producer $i \in [0,1]$ use the following production function:

$$Y_t^i = A_t K_{sr}^{\delta} \left\{ \left[\int_0^1 \left(N_t^{ij} \right)^{\frac{\phi-1}{\phi}} dj \right]^{\frac{\phi}{\phi-1}} \right\}^{1-\delta}$$

  $A_t$is a technology factor, which is common to the whole economy. N_t^{ij} is the amount of hours of type j labor used by intermediate good producer i. $\phi > 1$ is the elasticity of substitution between different types of labor, and $0 > \delta > 1$ is the capital share of output. The production function is concave in the labor aggregate, and we assume that capital is fixed in the short run at a level K_{sr}.

- Final good:

$$Y_t = \left[\int_0^1 \left(Y_t^i \right)^{\frac{\varepsilon_t - 1}{\varepsilon_t}} di \right]^{\frac{\varepsilon_t}{\varepsilon_t - 1}}$$

$\varepsilon_t > 1$ the elasticity of substitution between intermediate goods, $\Lambda_t = \varepsilon_t / (\varepsilon_t - 1)$ price markup. There is a shock to the elasticity of substitution, ε_t.

Final Goods Price Setting

- Final good producers are competitive and maximize profits.

- The input demand functions associated with this problem are

$$Y_t^i = \left[\frac{P_t^i}{P_t}\right]^{-\varepsilon_t} Y_t \qquad \forall i,$$

- The zero profit condition $\Rightarrow$ the price of the final good

$$P_t = \left[\int_0^1 P_t^{i\,1-\varepsilon_t} di\right]^{\frac{1}{1-\varepsilon_t}}$$

Intermediate Goods Producers Problem

- Operate in a monopolistic competition environment. They maximize profits taking as given all prices and wages but their own price.

- The profit maximization problem of the intermediate good producers is divided into two stages: In the first stage, given all wages, firms choose $\left\{N_t^{ij}\right\}_{j\in[0,1]}$ to obtain the optimal labor mix. Hence, the demand of producer i for type of labor j is

$$N_t^{ij} = \left[\frac{W_t^j}{W_t}\right]^{-\phi} \left[\frac{Y_t^i}{A_t}\right]^{\frac{1}{1-\delta}} \qquad \forall j,$$

- Where the aggregate wage W_t can be expressed as

$$W_t = \left[\int_0^1 \left(W_t^j \right)^{1-\phi} dj \right]^{\frac{1}{1-\phi}}.$$

- In the second stage, they set prices. They can reset their price only when they receive a stochastic signal to do so. This signal is received with probability $1 - \theta_p$.

- If they can change the price, they choose the price that maximizes:

$$E_t \sum_{\tau=0}^{\infty} \theta_p^{\tau} Q_t^{t+\tau} \left[P_t^i Y_{t+\tau}^i - W_{t+\tau} \left(\frac{Y_{t+\tau}^i}{A_{t+\tau}} \right)^{\frac{1}{1-\delta}} \right]$$

subject to

$$Y_{t+\tau}^i = \left[\frac{P_t^i}{P_{t+\tau}} \right]^{-\varepsilon_{t+\tau}} Y_{t+\tau} \qquad \forall i, \tau$$

- The solution is:

$$E_t \sum_{\tau=0}^{\infty} \theta_p^{\tau} Q_t^{t+\tau} \left\{ \left[\frac{P_t^{i,*}}{P_{t+\tau}} - \Lambda_t MC_t^i \right] Y_t^i \right\} = 0,$$

- The evolution of the aggregate price level is:

$$P_t = \left[\theta_p \left(P_{t-1} \right)^{1-\varepsilon_t} + (1-\theta_p) \left(P_t^* \right)^{1-\varepsilon_t} \right]^{\frac{1}{1-\varepsilon_t}}$$

Consumers problem

- Intertemporal susbstitution Equation

$$G_t C_t^{-\frac{1}{\sigma}} = \beta E_t \{ G_{t+1} C_{t+1}^{-\frac{1}{\sigma}} R_t \frac{P_t}{P_{t+1}} \}$$

- Demand for money, at a given interest rate, always satisfied.

Wage Setting Problem

- Consumers operate in a monopolistic competition environment. They maximize utility given all wages, but their own. They reset wages if signal to do so. They receive the signal with probability $(1-\theta_w)$. As before, the signal is independent across intermediate good producers and past history of signals.

- If they can change their wage, they choose the wage, $W_t^{j,*}$, that maximizes:

$$E_t \sum_{\tau=0}^{\infty} (\beta\theta_w)^{\tau} \left\{ \left[G_{t+\tau} C_{t+\tau}^{-\frac{1}{\sigma}} \frac{W_t^{j,*}}{P_{t+\tau}} - \vartheta \left(N_{t+\tau}^{*j} \right)^{\gamma} \right] N_{t+\tau}^{*j} \right\} = 0$$

subject to

$$N_{t+\tau}^{*j} = \left(\frac{W_t^{j,*}}{W_{t+\tau}}\right)^{-\phi} \int_0^1 \left(\frac{Y_{t+\tau}^i}{A_{t+\tau}}\right)^{\frac{1}{1-\delta}} di \qquad \forall j, \tau$$

- The evolution of the aggregate wage level is:

$$W_t = \left[\theta_w W_{t-1}^{1-\phi} + (1-\theta_w)\left(W_t^*\right)^{1-\phi}\right]^{\frac{1}{1-\phi}}.$$

Fiscal and Monetary Policy

- On the fiscal side, the government cannot run deficits or surpluses, so its budget constraint is

$$\int_0^1 T(h_t, j)dj = M(h_t) - M(h_{t-1}),$$

- On the Monetary side, as suggested by Taylor (1993), we assume that the monetary authority conducts monetary policy using the nominal interest rate, through a Taylor rule.

$$\frac{r_t}{r} = \left(\frac{r_{t-1}}{r}\right)^{\rho_r} \left(\frac{\pi_t}{\pi}\right)^{\gamma_\pi} \left(\frac{y_t}{y}\right)^{\gamma_y} e^{ms_t}$$

Dynamics

$$a_t + (1 - \delta) n_t - y_t = 0$$

$$mc_t - (w_t - p_t) + y_t - n_t = 0$$

$$\frac{1}{\sigma} c_t + \gamma n_t - g_t - mrs_t = 0$$

$$\rho_r r_{t-1} + (1 - \rho_r) \left[\gamma_\pi \pi_t + \gamma_y y_t \right] + ms_t - r_t = 0$$

$$-y_t + c_t = 0$$

$$w_t - p_t - (w_{t-1} - p_{t-1} - \Delta w_t + \pi_t) = 0$$

$$E_t\left[-\sigma r_t + \sigma \pi_{t+1} - \sigma g_{t+1} + \sigma g_t - c_t + c_{t+1}\right] = 0$$

$$E_t\left[\kappa_p mc_t + \kappa_p \mu_t - \pi_t + \beta \pi_{t+1}\right] = 0$$

$$E_t\left[\kappa_w mrs_t - \kappa_w (w_t - p_t) - \Delta w_t + \beta \Delta w_{t+1}\right] = 0$$

where

$$a_t = \rho_a a_{t-1} + \varepsilon_{at}$$

$$\mu_t = \varepsilon_{\mu t}$$

$$ms_t = \varepsilon_{mst}$$

$$g_t = \rho_g g_{t-1} + \varepsilon_{gt}$$

and

$$\kappa_p = (1-\delta)(1-\theta_p\beta)(1-\theta_p)/(\theta_p(1+\delta(\bar{\varepsilon}-1)))$$

$$\kappa_w = (1-\theta_w)(1-\beta\theta_w)/\left[\theta_w\left(1+\phi\gamma\right)\right]$$

Solve the Model (Uhlig Algorithm)

- Derive the system:

$$0 = As_t + Bs_{t-1} + Ce_t + Dz_t$$

$$0 = E_t[Fs_{t+1} + Gs_t + Hs_{t-1} + Je_{t+1} + Ke_t + Lz_{t+1} + Mz_t]$$

$$z_{t+1} = Nz_t + \varepsilon_{t+1} \qquad \varepsilon_{t+1} \sim N(0, \Sigma)$$

- $s_t = (w_t - p_t, r_t, \pi_t, \Delta w_t, y_t)'$ is the endogenous state, $e_t = (n_t, mc_t, mrs_t, c_t)$ are endogenous variables, and $z_t = (a_t, ms_t, \mu_t, g_t)'$ is the exogenous state.

- $N = \begin{bmatrix} \rho_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_g \end{bmatrix}$

- Solution

$$s_t = PPs_{t-1} + QQz_t$$

$$e_t = RRs_{t-1} + SSz_t$$

Writing the Solution in State Space Form

- Transition equation

$$\begin{pmatrix} s_t \\ z_t \end{pmatrix} = \begin{pmatrix} PP & QQ * N \\ 0 & N \end{pmatrix} \begin{pmatrix} s_{t-1} \\ z_{t-1} \end{pmatrix} + \begin{pmatrix} Q \\ I \end{pmatrix} \varepsilon_t =$$
$$\begin{pmatrix} s_t \\ z_t \end{pmatrix} = F \begin{pmatrix} s_{t-1} \\ z_{t-1} \end{pmatrix} + G\varepsilon_t$$

- Measurement equation

$$s_t = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} s_t \\ z_t \end{pmatrix} = H \begin{pmatrix} s_t \\ z_t \end{pmatrix}$$

- Evaluate the Likelihood function using the Kalman Filter.

- Apply Metropolis-Hastings Algorigthm to get a draw from the posterior.

- Compute moments and marginal likelihood.

	Prior Distribution	Mean/std
$\frac{1}{1-\theta_p}$	gamma$(2,1)+1$	3.00 (1.42)
$\frac{1}{1-\theta_w}$	gamma$(3,1)+1$	4.00 (1.71)
γ_π	normal$(1.5, 0.25)$	1.5 (0.25)
γ_y	normal$(0.125, 0.125)$	0.125 (0.125)
ρ_r	uniform$[0,1)$	0.5 (0.28)
σ^{-1}	gamma$(2, 1.25)$	2.5 (1.76)
γ	normal$(1, 0.5)$	1.0 (0.5)
ρ_a	uniform$[0,1)$	0.5 (0.28)
ρ_g	uniform$[0,1)$	0.5 (0.28)
$\sigma_a(\%)$	uniform$[0,1)$	50.0 (28.0)
$\sigma_m(\%)$	uniform$[0,1)$	50.0 (28.0)
$\sigma_\lambda(\%)$	uniform$[0,1)$	50.0 (28.0)
$\sigma_g(\%)$	uniform$[0,1)$	50.0 (28.0)

Algorithm (`gencoeffsehl.m`)

Step 0 Read data (usadefl1d.txt)

Step 1 Intial value for θ_0, N and set $j = 1$.

Step 2 Evaluate $f(Y^T|\theta_0)$ and $\pi(\theta_0)$ and make sure $f(Y^T|\theta_0), \pi(\theta_0) > 0$

(a) Given θ_0 evaluate prior $\pi(\theta_0)$ (priorehl.m)

(b) Given θ_0: Uligh algorithm to solve the model (modelehl.m and solve2.m)

(c) Kalman Filter to evaluate $f(Y^T|\theta_0)$ (likeliehl.m)

Step 3 $\theta_j^* = \theta_{j-1} + \varepsilon \sim N\left(0, \Sigma_\varepsilon\right)$ and u from $Uniform[0,1]$

(a) Given θ_j^* evaluate prior $\pi(\theta_j^*)$ (priorehl.m)

(b) Given θ_j^*: Uligh algorithm to solve the model (modelehl.m and solve2.m)

(c) Kalman Filter to evaluate $f(Y^T|\theta_j^*)$ (likeliehl.m)

Step 4 If $u \leq \alpha\left(\theta_{j-1}, \theta_j^*\right) = \min\left\{\frac{f(Y^T|\theta_j^*)\pi\left(\theta_j^*\right)}{f(Y^T|\theta_{j-1})\pi\left(\theta_{j-1}\right)}, 1\right\}$ then $\theta_j = \theta_j^*$, $\theta_j = \theta_{j-1}$ otherwise.

Step 5 If $j \leq N$ then $j \rightsquigarrow j+1$ and got to 3.

	Prior	Mean (Std)	Mean of Posterior (Std)
$\frac{1}{1-\theta_p}$	gamma$(2,1)+1$	3.00 (1.42)	4.37 (0.35)
$\frac{1}{1-\theta_w}$	gamma$(3,1)+1$	4.00 (1.71)	2.72 (0.27)
γ_π	normal$(1.5, 0.25)$	1.5 (0.25)	1.08 (0.09)
γ_y	normal$(0.125, 0.125)$	0.125 (0.125)	0.26 (0.06)
ρ_r	uniform$[0,1)$	0.5 (0.28)	0.74 (0.02)
σ^{-1}	gamma$(2, 1.25)$	2.5 (1.76)	8.33 (2.50)
γ	normal$(1, 0.5)$	1.0 (0.5)	1.74 (0.29)
ρ_a	uniform$[0,1)$	0.5 (0.28)	0.74 (0.05)
ρ_g	uniform$[0,1)$	0.5 (0.28)	0.82 (0.03)
$\sigma_a(\%)$	uniform$[0,1)$	50.0 (28.0)	3.88 (1.09)
$\sigma_m(\%)$	uniform$[0,1)$	50.0 (28.0)	0.33 (0.02)
$\sigma_\lambda(\%)$	uniform$[0,1)$	50.0 (28.0)	31.67 (5.32)
$\sigma_g(\%)$	uniform$[0,1)$	50.0 (28.0)	11.88 (3.28)