Inference

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A Model with Sticky Price and Sticky Wage

- Household $j \in [0, 1]$ maximizes utility function:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[ \frac{G_t (C_t^j)^{1-\frac{1}{\sigma}}}{1 - \frac{1}{\sigma}} - \frac{(N_t^j)^{1+\gamma}}{1 + \gamma} + \frac{\eta}{1 - \xi} \left( \frac{M_t^j}{P_t} \right)^{1-\xi} \right]$$

$0 < \beta < 1$ is the discount factor, $\sigma > 0$ the elasticity of intertemporal substitution, $\xi > 1$ the elasticity of money holdings, and $\gamma > 0$ the inverse of the elasticity of labor supply with respect to real wages.
Subject the budget constrain given by:

\[ P_t C_t^j + M_t^j - M_{t-1}^j + \sum_{h_{t+\tau}} Q_t(h_{t+\tau}) D_t^j(h_{t+\tau}) + \frac{B_t^{j+1}}{R_t} \]

\[ = W_t^j N_t^j + \Pi_t^j + T_t^j + D_t^j + B_t^j, \]

where \( \Pi_t^j \) are firms’ profits, \( T_t^j \) are nominal transfers, \( D_t^j(h_{t+\tau}) \) denotes holdings of contingent bonds, \( B_{t+1}^j \) denotes holdings of an un-contingent bonds, and \( W_t^j \) is the hourly nominal wage.
Technology

- Intermediate Goods producer \( i \in [0, 1] \) use the following production function:

\[
Y^i_t = A_t K^\delta_{sr} \left\{ \left[ \int_0^1 \left( N^{ij}_t \right)^{\phi-1} dj \right]^{\phi} \right\}^{1-\delta}
\]

\( A_t \) is a technology factor, which is common to the whole economy. \( N^{ij}_t \) is the amount of hours of type \( j \) labor used by intermediate good producer \( i \). \( \phi > 1 \) is the elasticity of substitution between different types of labor, and \( 0 > \delta > 1 \) is the capital share of output. The production function is concave in the labor aggregate, and we assume that capital is fixed in the short run at a level \( K_{sr} \).
• Final good:

\[ Y_t = \left[ \int_0^1 \left( Y_t^i \right) \frac{\varepsilon_t - 1}{\varepsilon_t} \, di \right]^{\frac{\varepsilon_t}{\varepsilon_t - 1}} \]

where \( \varepsilon_t > 1 \) the elasticity of substitution between intermediate goods, \( \Lambda_t = \varepsilon_t / (\varepsilon_t - 1) \) price markup. There is a shock to the elasticity of substitution, \( \varepsilon_t \).
Final Goods Price Setting

- Final good producers are competitive and maximize profits.

- The input demand functions associated with this problem are

\[
Y^i_t = \left[ \frac{P^i_t}{P_t} \right]^{-\varepsilon_t} Y_t \quad \forall i,
\]

- The zero profit condition \( \Rightarrow \) the price of the final good

\[
P_t = \left[ \int_0^1 P_t^{i1-\varepsilon_t} di \right]^{1\over 1-\varepsilon_t}
\]
Intermediate Goods Producers Problem

- Operate in a monopolistic competition environment. They maximize profits taking as given all prices and wages but their own price.

- The profit maximization problem of the intermediate good producers is divided into two stages: In the first stage, given all wages, firms choose \( \{ N_{ij}^t \}_{j \in [0,1]} \) to obtain the optimal labor mix. Hence, the demand of producer \( i \) for type of labor \( j \) is

\[
N_{ij}^t = \left[ \frac{W_t^j}{W_t} \right]^{-\phi} \left[ \frac{Y_t^i}{A_t} \right]^{\frac{1}{1-\delta}} \quad \forall j,
\]
Where the aggregate wage $W_t$ can be expressed as

$$W_t = \left[ \int_0^1 (W_t^j)^{1-\phi} \, dj \right]^{\frac{1}{1-\phi}}.$$
• In the second stage, they set prices. They can reset their price only when they receive a stochastic signal to do so. This signal is received with probability $1 - \theta_p$.

• If they can change the price, they choose the price that maximizes:

$$E_t \sum_{\tau=0}^{\infty} \theta_p^\tau Q_t^{t+\tau} \left[ P_t^i Y_{t+\tau}^i - W_{t+\tau} \left( \frac{Y_{t+\tau}^i}{A_{t+\tau}} \right)^{\frac{1}{1-\delta}} \right]$$

subject to

$$Y_{t+\tau}^i = \left[ \frac{P_t^i}{P_{t+\tau}} \right]^{-\varepsilon_{t+\tau}} Y_{t+\tau} \quad \forall i, \tau$$
• The solution is:

\[
E_t \sum_{\tau=0}^{\infty} \theta_p^\tau Q_t^{t+\tau} \left\{ \left[ \frac{P_t^i,*}{P_t^{t+\tau}} - \Lambda_t M C_t^i \right] Y_t^i \right\} = 0,
\]

• The evolution of the aggregate price level is:

\[
P_t = \left[ \theta_p (P_{t-1})^{1-\varepsilon_t} + (1 - \theta_p) (P_t^*)^{1-\varepsilon_t} \right]^{\frac{1}{1-\varepsilon_t}}
\]
Consumers problem

• Intertemporal substitution Equation

\[ G_t C_t^{-\frac{1}{\sigma}} = \beta E_t \{ G_{t+1} C_{t+1}^{-\frac{1}{\sigma}} R_t \frac{P_t}{P_{t+1}} \} \]

• Demand for money, at a given interest rate, always satisfied.
Wage Setting Problem

- Consumers operate in a monopolistic competition environment. They maximize utility given all wages, but their own. They reset wages if signal to do so. They receive the signal with probability \((1 - \theta_w)\). As before, the signal is independent across intermediate good producers and past history of signals.

- If they can change their wage, they choose the wage, \(W_{t,*}^j\), that maximizes:

\[
E_t \sum_{\tau=0}^{\infty} (\beta \theta_w)^\tau \left\{ \left[ G_{t+\tau} C_{t+\tau}^{-\frac{1}{\sigma}} \frac{W_{t,*}^j}{P_{t+\tau}} - \theta \left( N_{t+\tau}^* j \right)^\gamma \right] N_{t+\tau}^* j \right\} = 0
\]
subject to

\[ N_{t+\tau}^{*j} = \left( \frac{W_{t}^{j,*}}{W_{t+\tau}} \right)^{-\phi} \int_{0}^{1} \left( \frac{Y_{t+\tau}^{i}}{A_{t+\tau}} \right)^{\frac{1}{1-\delta}} \, d\iota \quad \forall j, \tau \]

- The evolution of the aggregate wage level is:

\[ W_{t} = \left[ \theta_{w} W_{t-1}^{1-\phi} + (1 - \theta_{w}) (W_{t}^{*})^{1-\phi} \right]^{\frac{1}{1-\phi}}. \]
Fiscal and Monetary Policy

- On the fiscal side, the government cannot run deficits or surpluses, so its budget constraint is

\[
\int_0^1 T(h_t, j) dj = M(h_t) - M(h_{t-1}),
\]

- On the Monetary side, as suggested by Taylor (1993), we assume that the monetary authority conducts monetary policy using the nominal interest rate, through a Taylor rule.

\[
\frac{r_t}{r} = \left( \frac{r_{t-1}}{r} \right)^{\rho_r} \left( \frac{\pi_t}{\pi} \right)^{\gamma_\pi} \left( \frac{y_t}{y} \right)^{\gamma_y} e^{m_{st}}
\]
Dynamics

\begin{align*}
    a_t + (1 - \delta)n_t - y_t &= 0 \\
    mc_t - (w_t - p_t) + y_t - n_t &= 0 \\
    \frac{1}{\sigma}ct + \gamma n_t - gt - mrs_t &= 0 \\
    \rho_r r_{t-1} + (1 - \rho_r) \left[ \gamma_{\pi} \pi_t + \gamma_y y_t \right] + mst - r_t &= 0 \\
    -y_t + c_t &= 0 \\
    w_t - p_t - (w_{t-1} - p_{t-1} - \Delta w_t + \pi_t) &= 0
\end{align*}
\[ E_t [-\sigma r_t + \sigma \pi_{t+1} - \sigma g_{t+1} + \sigma g_t - c_t + c_{t+1}] = 0 \]

\[ E_t [\kappa_p mc_t + \kappa_p \mu_t - \pi_t + \beta \pi_{t+1}] = 0 \]

\[ E_t [\kappa_w mrs_t - \kappa_w (w_t - p_t) - \Delta w_t + \beta \Delta w_{t+1}] = 0 \]
where

\[ a_t = \rho_a a_{t-1} + \varepsilon_{at} \]

\[ \mu_t = \varepsilon_{\mu t} \]

\[ m_{st} = \varepsilon_{mst} \]

\[ g_t = \rho_g g_{t-1} + \varepsilon_{gt} \]

and

\[ \kappa_p = (1 - \delta)(1 - \theta_p \beta)(1 - \theta_p)/(\theta_p(1 + \delta(\bar{\varepsilon} - 1))) \]

\[ \kappa_w = (1 - \theta_w)(1 - \beta \theta_w)/[\theta_w(1 + \phi \gamma)] \]
Solve the Model (Uhlig Algorithm)

- Derive the system:

\[
0 = A_s t + B_{s_{t-1}} + C e_t + D z_t
\]

\[
0 = E_t [F_{s_{t+1}} + G_{s_t} + H_{s_{t-1}} + J_{e_{t+1}} + K e_t + L_z z_{t+1} + M_z z_t]
\]

\[
z_{t+1} = N z_t + \epsilon_{t+1} \quad \epsilon_{t+1} \sim N (0, \Sigma)
\]

- \( s_t = (w_t - p_t, r_t, \pi_t, \Delta w_t, y_t)' \) is the endogenous state, \( e_t = (n_t, m c_t, m r s_t, c_t) \) are endogenous variables, and \( z_t = (a_t, m s_t, \mu_t, g_t)' \) is the exogenous state.
\[ N = \begin{bmatrix}
\rho_a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \rho_g 
\end{bmatrix} \]

Solution

\[ s_t = PPs_{t-1} + QQz_t \]

\[ e_t = RRs_{t-1} + SSz_t \]
Writing the Solution in State Space Form

- **Transition equation**

\[
\begin{pmatrix}
    s_t \\
    z_t
\end{pmatrix} = \begin{pmatrix}
    PP & QQ \ast N \\
    0 & N
\end{pmatrix} \begin{pmatrix}
    s_{t-1} \\
    z_{t-1}
\end{pmatrix} + \begin{pmatrix}
    Q \\
    I
\end{pmatrix} \varepsilon_t = \\
\begin{pmatrix}
    s_t \\
    z_t
\end{pmatrix} = F \begin{pmatrix}
    s_{t-1} \\
    z_{t-1}
\end{pmatrix} + G \varepsilon_t
\]

- **Measurement equation**

\[
s_t = \begin{pmatrix}
    I & 0 \\
    0 & 0
\end{pmatrix} \begin{pmatrix}
    s_t \\
    z_t
\end{pmatrix} = H \begin{pmatrix}
    s_t \\
    z_t
\end{pmatrix}
\]

- Evaluate the Likelihood function using the Kalman Filter.
• Apply Metropolis-Hastings Algorithm to get a draw from the posterior.

• Compute moments and marginal likelihood.
<table>
<thead>
<tr>
<th>Prior Distribution</th>
<th>Mean/std</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{1-\theta_p}$</td>
<td>gamma$(2, 1) + 1$</td>
</tr>
<tr>
<td>$\frac{1}{1-\theta_w}$</td>
<td>gamma$(3, 1) + 1$</td>
</tr>
<tr>
<td>$\gamma_{\pi}$</td>
<td>normal$(1.5, 0.25)$</td>
</tr>
<tr>
<td>$\gamma_{y}$</td>
<td>normal$(0.125, 0.125)$</td>
</tr>
<tr>
<td>$\rho_r$</td>
<td>uniform$[0, 1]$</td>
</tr>
<tr>
<td>$\sigma^{-1}$</td>
<td>gamma$(2, 1.25)$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>normal$(1, 0.5)$</td>
</tr>
<tr>
<td>$\rho_{a}$</td>
<td>uniform$[0, 1]$</td>
</tr>
<tr>
<td>$\rho_{g}$</td>
<td>uniform$[0, 1]$</td>
</tr>
<tr>
<td>$\sigma_{a}(%)$</td>
<td>uniform$[0, 1]$</td>
</tr>
<tr>
<td>$\sigma_{m}(%)$</td>
<td>uniform$[0, 1]$</td>
</tr>
<tr>
<td>$\sigma_{\lambda}(%)$</td>
<td>uniform$[0, 1]$</td>
</tr>
<tr>
<td>$\sigma_{g}(%)$</td>
<td>uniform$[0, 1]$</td>
</tr>
</tbody>
</table>
Algorithm (gencoefsehl.m)

Step 0 Read data (usadefl1d.txt)

Step 1 Initial value for $\theta_0$, $N$ and set $j = 1$.

Step 2 Evaluate $f(Y^T|\theta_0)$ and $\pi(\theta_0)$ and make sure $f(Y^T|\theta_0), \pi(\theta_0) > 0$

(a) Given $\theta_0$ evaluate prior $\pi(\theta_0)$ (priorehl.m)

(b) Given $\theta_0$: Ulish algorithm to solve the model (modelehl.m and solve2.m)

(c) Kalman Filter to evaluate $f(Y^T|\theta_0)$ (likeliehl.m)
Step 3 \( \theta_j^* = \theta_{j-1} + \varepsilon \sim N(0, \Sigma_{\varepsilon}) \) and \( u \) from \( \text{Uniform}[0, 1] \)

(a) Given \( \theta_j^* \) evaluate prior \( \pi(\theta_j^*) \) (priorehl.m)

(b) Given \( \theta_j^* \): Uligh algorithm to solve the model (modelehl.m and solve2.m)

(c) Kalman Filter to evaluate \( f(Y^T|\theta_j^*) \) (likeliehl.m)

Step 4 If \( u \leq \alpha(\theta_{j-1}, \theta_j^*) = \min \left\{ \frac{f(Y^T|\theta_j^*)\pi(\theta_j^*)}{f(Y^T|\theta_{j-1})\pi(\theta_{j-1})}, 1 \right\} \) then \( \theta_j = \theta_j^* \), \( \theta_j = \theta_{j-1} \) otherwise.

Step 5 If \( j \leq N \) then \( j \sim j + 1 \) and got to 3.
<table>
<thead>
<tr>
<th>Prior</th>
<th>Mean (Std)</th>
<th>Mean of Posterior (Std)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{1-\theta_p}$</td>
<td>gamma$(2, 1) + 1$</td>
<td>3.00 (1.42)</td>
</tr>
<tr>
<td>$\frac{1}{1-\theta_w}$</td>
<td>gamma$(3, 1) + 1$</td>
<td>4.00 (1.71)</td>
</tr>
<tr>
<td>$\gamma_\pi$</td>
<td>normal$(1.5, 0.25)$</td>
<td>1.5 (0.25)</td>
</tr>
<tr>
<td>$\gamma_y$</td>
<td>normal$(0.125, 0.125)$</td>
<td>0.125 (0.125)</td>
</tr>
<tr>
<td>$\rho_r$</td>
<td>uniform$[0, 1)$</td>
<td>0.5 (0.28)</td>
</tr>
<tr>
<td>$\sigma^{-1}$</td>
<td>gamma$(2, 1.25)$</td>
<td>2.5 (1.76)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>normal$(1, 0.5)$</td>
<td>1.0 (0.5)</td>
</tr>
<tr>
<td>$\rho_a$</td>
<td>uniform$[0, 1)$</td>
<td>0.5 (0.28)</td>
</tr>
<tr>
<td>$\rho_g$</td>
<td>uniform$[0, 1)$</td>
<td>0.5 (0.28)</td>
</tr>
<tr>
<td>$\sigma_a(%)$</td>
<td>uniform$[0, 1)$</td>
<td>50.0 (28.0)</td>
</tr>
<tr>
<td>$\sigma_m(%)$</td>
<td>uniform$[0, 1)$</td>
<td>50.0 (28.0)</td>
</tr>
<tr>
<td>$\sigma_\lambda(%)$</td>
<td>uniform$[0, 1)$</td>
<td>50.0 (28.0)</td>
</tr>
<tr>
<td>$\sigma_g(%)$</td>
<td>uniform$[0, 1)$</td>
<td>50.0 (28.0)</td>
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