Solving a Dynamic Equilibrium Model

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Basic RBC

- Social Planner's problem:

$$\max E \sum_{t=0}^{\infty} \beta^t \left\{ \log c_t + \psi \log (1 - l_t) \right\}$$

$$\begin{aligned} c_t + k_{t+1} &= k_t^{\alpha} \left(e^{z_t} l_t\right)^{1-\alpha} + (1 - \delta) k_t, \ \forall\ t > 0 \\ z_t &= \rho z_{t-1} + \varepsilon_t,\ \ \varepsilon_t \sim \mathcal{N}(0, \sigma) \end{aligned}$$

- This is a dynamic optimization problem.

Computing the RBC

- The previous problem does not have a known “paper and pencil” solution.

- We will work with an approximation: Perturbation Theory.

- We will undertake a first order perturbation of the model.

- How well will the approximation work?

Equilibrium Conditions

From the household problem+firms's problem+aggregate conditions:

$$\frac{1}{c_t} = \beta E_t \left\{ \frac{1}{c_{t+1}} \left(1 + \alpha k_t^{\alpha-1} \left(e^{z_t} l_t \right)^{1-\alpha} - \delta \right) \right\}$$

$$\psi \frac{c_t}{1 - l_t} = (1 - \alpha) k_t^{\alpha} \left(e^{z_t} l_t \right)^{1-\alpha} l_t^{-1}$$

$$c_t + k_{t+1} = k_t^{\alpha} \left(e^{z_t} l_t \right)^{1-\alpha} + (1 - \delta) k_t$$

$$z_t = \rho z_{t-1} + \varepsilon_t$$

Finding a Deterministic Solution

- We search for the first component of the solution.

- If $\sigma = 0$, the equilibrium conditions are:

$$\frac{1}{c_t} = \beta \frac{1}{c_{t+1}} \left(1 + \alpha k_t^{\alpha-1} l_t^{1-\alpha} - \delta\right)$$
$$\psi \frac{c_t}{1 - l_t} = (1 - \alpha) k_t^{\alpha} l_t^{-\alpha}$$
$$c_t + k_{t+1} = k_t^{\alpha} l_t^{1-\alpha} + (1 - \delta) k_t$$

Steady State

- The equilibrium conditions imply a steady state:

$$\frac{1}{c} = \beta \frac{1}{c} \left(1 + \alpha k^{\alpha-1} l^{1-\alpha} - \delta\right)$$
$$\psi \frac{c}{1-l} = (1-\alpha) k^{\alpha} l^{-\alpha}$$
$$c + \delta k = k^{\alpha} l^{1-\alpha}$$

- The first equation can be written as:

$$\frac{1}{\beta} = 1 + \alpha k^{\alpha-1} l^{1-\alpha} - \delta$$

Solving the Steady State

Solution:

$$\begin{aligned} k &= \frac{\mu}{\Omega + \varphi\mu} \\ l &= \varphi k \\ c &= \Omega k \\ y &= k^{\alpha} l^{1-\alpha} \end{aligned}$$

where $\varphi = \left(\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)\right)^{\frac{1}{1-\alpha}}$, $\Omega = \varphi^{1-\alpha} - \delta$ and $\mu = \frac{1}{\psi}(1-\alpha)\varphi^{-\alpha}$.

Linearization I

- Loglinearization or linearization?

- Advantages and disadvantages

- We can linearize and perform later a change of variables.

Linearization II

We linearize:

$$\frac{1}{c_t} = \beta E_t \left\{ \frac{1}{c_{t+1}} \left(1 + \alpha k_t^{\alpha-1} \left(e^{z_t} l_t\right)^{1-\alpha} - \delta \right) \right\}$$
$$\psi \frac{c_t}{1-l_t} = (1-\alpha)\, k_t^{\alpha} \left(e^{z_t} l_t\right)^{1-\alpha} l_t^{-1}$$
$$c_t + k_{t+1} = k_t^{\alpha} \left(e^{z_t} l_t\right)^{1-\alpha} + (1-\delta)\, k_t$$
$$z_t = \rho z_{t-1} + \varepsilon_t$$

around l, k, and c with a First-order Taylor Expansion.

Linearization III

We get:

$$-\frac{1}{c}(c_t - c) = E_t \left\{ \begin{array}{c} -\frac{1}{c}(c_{t+1} - c) + \alpha(1-\alpha)\beta\frac{y}{k}z_{t+1} + \\ \alpha(\alpha-1)\beta\frac{y}{k^2}(k_{t+1} - k) + \alpha(1-\alpha)\beta\frac{y}{kl}(l_{t+1} - l) \end{array} \right\}$$

$$\frac{1}{c}(c_t - c) + \frac{1}{(1-l)}(l_t - l) = (1-\alpha)z_t + \frac{\alpha}{k}(k_t - k) - \frac{\alpha}{l}(l_t - l)$$

$$(c_t - c) + (k_{t+1} - k) = \left\{ \begin{array}{c} y\left((1-\alpha)z_t + \frac{\alpha}{k}(k_t - k) + \frac{(1-\alpha)}{l}(l_t - l)\right) \\ +(1-\delta)(k_t - k) \end{array} \right\}$$

$$z_t = \rho z_{t-1} + \varepsilon_t$$

Rewriting the System I

Or:

$$\alpha_1 (c_t - c) = E_t \{\alpha_1 (c_{t+1} - c) + \alpha_2 z_{t+1} + \alpha_3 (k_{t+1} - k) + \alpha_4 (l_{t+1} - l)\}$$

$$(c_t - c) = \alpha_5 z_t + \frac{\alpha}{k} c (k_t - k) + \alpha_6 (l_t - l)$$

$$(c_t - c) + (k_{t+1} - k) = \alpha_7 z_t + \alpha_8 (k_t - k) + \alpha_9 (l_t - l)$$

$$z_t = \rho z_{t-1} + \varepsilon_t$$

Rewriting the System II

where

$$
\begin{aligned}
\alpha_1 &= -\frac{1}{c} & \alpha_2 &= \alpha\left(1-\alpha\right)\beta\frac{y}{k} \\
\alpha_3 &= \alpha\left(\alpha-1\right)\beta\frac{y}{k^2} & \alpha_4 &= \alpha\left(1-\alpha\right)\beta\frac{y}{kl} \\
\alpha_5 &= \left(1-\alpha\right)c & \alpha_6 &= -\left(\frac{\alpha}{l}+\frac{1}{(1-l)}\right)c \\
\alpha_7 &= \left(1-\alpha\right)y & \alpha_8 &= y\frac{\alpha}{k}+\left(1-\delta\right) \\
\alpha_9 &= y\frac{(1-\alpha)}{l} & y &= k^{\alpha}l^{1-\alpha}
\end{aligned}
$$

Rewriting the System III

After some algebra the system is reduced to:

$$A\left(k_{t+1}-k\right)+B\left(k_t-k\right)+C\left(l_t-l\right)+Dz_t=0$$

$$E_t\left(G\left(k_{t+1}-k\right)+H\left(k_t-k\right)+J\left(l_{t+1}-l\right)+K\left(l_t-l\right)+Lz_{t+1}+Mz_t\right)=0$$

$$E_t z_{t+1}=\rho z_t$$

Guess Policy Functions

We guess policy functions of the form $(k_{t+1} - k) = P(k_t - k) + Qz_t$ and $(l_t - l) = R(k_t - k) + Sz_t$, plug them in and get:

$$A(P(k_t - k) + Qz_t) + B(k_t - k) \\ +C(R(k_t - k) + Sz_t) + Dz_t = 0$$

$$G(P(k_t - k) + Qz_t) + H(k_t - k) + J(R(P(k_t - k) + Qz_t) + SNz_t) \\ +K(R(k_t - k) + Sz_t) + (LN + M)z_t = 0$$

Solving the System I

Since these equations need to hold for any value $(k_{t+1} - k)$ or z_t we need to equate each coefficient to zero, on $(k_t - k)$:

$$\begin{aligned} AP + B + CR &= 0 \\ GP + H + JRP + KR &= 0 \end{aligned}$$

and on z_t:

$$\begin{aligned} AQ + CS + D &= 0 \\ (G + JR)\, Q + JSN + KS + LN + M &= 0 \end{aligned}$$

Solving the System II

- We have a system of four equations on four unknowns.

- To solve it note that $R = -\frac{1}{C}(AP + B) = -\frac{1}{C}AP - \frac{1}{C}B$

- Then:

$$P^2 + \left(\frac{B}{A} + \frac{K}{J} - \frac{GC}{JA}\right)P + \frac{KB - HC}{JA} = 0$$

a quadratic equation on P.

Solving the System III

- We have two solutions:

$$P = -\frac{1}{2}\left(-\frac{B}{A} - \frac{K}{J} + \frac{GC}{JA} \pm \left(\left(\frac{B}{A} + \frac{K}{J} - \frac{GC}{JA}\right)^2 - 4\frac{KB - HC}{JA}\right)^{0.5}\right)$$

  one stable and another unstable.

- If we pick the stable root and find $R = -\frac{1}{C}(AP + B)$ we have to a system of two linear equations on two unknowns with solution:

$$Q = \frac{-D(JN + K) + CLN + CM}{AJN + AK - CG - CJR}$$
$$S = \frac{-ALN - AM + DG + DJR}{AJN + AK - CG - CJR}$$

Practical Implementation

- How do we do this in practice?

- Solving quadratic equations: “A Toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily” by Harald Uhlig.

- Using `dynare`.

General Structure of Linearized System

Given m states x_t, n controls y_t, and k exogenous stochastic processes z_{t+1}, we have:

$$Ax_t + Bx_{t-1} + Cy_t + Dz_t = 0$$
$$E_t\left(Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t\right) = 0$$
$$E_t z_{t+1} = Nz_t$$

where C is of size $l \times n$, $l \geq n$ and of rank n, that F is of size $(m + n - l) \times n$, and that N has only stable eigenvalues.

Policy Functions

We guess policy functions of the form:

$$x_t = Px_{t-1} + Qz_t$$
$$y_t = Rx_{t-1} + Sz_t$$

where P, Q, R, and S are matrices such that the computed equilibrium is stable.

Policy Functions

For simplicity, suppose $l = n$. See Uhlig for general case (I have never be in the situation where $l = n$ did not hold).

Then:

1. P satisfies the matrix quadratic equation:

   $$\left(F - JC^{-1}A\right)P^2 - \left(JC^{-1}B - G + KC^{-1}A\right)P - KC^{-1}B + H = 0$$

   The equilibrium is stable iff $\max\left(abs\left(eig\left(P\right)\right)\right) < 1$.

2. R is given by:

   $$R = -C^{-1}\left(AP + B\right)$$

3. Q satisfies:

$$N' \otimes \left(F - JC^{-1}A\right) + I_k \otimes \left(JR + FP + G - KC^{-1}A\right) vec\,(Q)$$
$$= vec\left(\left(JC^{-1}D - L\right) N + KC^{-1}D - M\right)$$

4. S satisfies:

$$S = -C^{-1}\,(AQ + D)$$

How to Solve Quadratic Equations

To solve

$$\Psi P^2 - \Gamma P - \Theta = 0$$

for the $m \times m$ matrix P:

1. Define the $2m \times 2m$ matrices:

$$\Xi = \begin{bmatrix} \Gamma & \Theta \\ I_m & 0_m \end{bmatrix}, \text{ and } \Delta = \begin{bmatrix} \Psi & 0_m \\ 0_m & I_m \end{bmatrix}$$

2. Let s be the generalized eigenvector and λ be the corresponding generalized eigenvalue of Ξ with respect to Δ. Then we can write $s' = [\lambda x', x']$ for some $x \in \Re^m$.

3. If there are m generalized eigenvalues $\lambda_1, \lambda_2, ..., \lambda_m$ together with generalized eigenvectors $s_1, ..., s_m$ of Ξ with respect to Δ, written as $s' = \left[\lambda x_i', x_i'\right]$ for some $x_i \in \Re^m$ and if $(x_1, ..., x_m)$ is linearly independent, then:

$$P = \Omega \Lambda \Omega^{-1}$$

is a solution to the matrix quadratic equation where $\Omega = [x_1, ..., x_m]$ and $\Lambda = [\lambda_1, ..., \lambda_m]$. The solution of P is stable if $\max |\lambda_i| < 1$. Conversely, any diagonalizable solution P can be written in this way.

How to Implement This Solver

Available Code:

1. My own code: `undeter1.m`.

2. Uhlig's web page: `http://www.wiwi.hu-berlin.de/wpol/html/toolkit.htm`

An Alternative `Dynare`

- What is `Dynare`? A platform for the solution, simulation, and estimation of DSGE models in economics.
- Developed by Michel Juilliard and collaborators.
- I am one of them:)
- `http://www.cepremap.cnrs.fr/dynare/`

- Dynare takes a more “blackbox approach”.

- However, you can access the files...

- ...and it is very easy to use.

- Short tutorial.

Our Benchmark Model

- We are now ready to compute our benchmark model.

- We begin finding the steady state.

- As before, a variable x with no time index represent the value of that variable in the steady state.

Steady State I

- From the first order conditions of the household:.

$$c^{-\sigma} = \beta c^{-\sigma} \left(r + 1 - \delta\right)\}$$
$$c^{-\sigma} = \beta c^{-\sigma} \frac{R}{\pi}$$
$$\psi l^{\gamma} = c^{-\sigma} w$$

- We forget the money condition because the central bank, through open market operations, will supply all the needed money to support the chosen interest rate.

- Also, we normalize the price level to one.

Steady State II

- From the problem of the intermediate good producer:

$$k = \frac{\alpha}{1-\alpha}\frac{w}{r}l$$

- Also:

$$mc = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha} w^{1-\alpha} r^{\alpha}$$

$$\frac{p^*}{p} = \frac{\varepsilon}{\varepsilon - 1} mc$$

where $A = 1$.

Steady State III

- Now, since $p^* = p$:

$$\left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha} w^{1-\alpha} r^{\alpha} = \frac{\varepsilon - 1}{\varepsilon}$$

- By markets clearing:

$$c + \delta k = y = k^{\alpha} l^{1-\alpha}$$

where we have used the fact that $x = \delta k$ and that:

$$\frac{A}{v} = 1$$

- The Taylor rule will be trivially satisfied and we can drop it from the computation.

Steady State IV

- Our steady state equations, cancelling redundant constants are:

$$r = \frac{1}{\beta} - 1 + \delta$$

$$R = \frac{1}{\beta}\pi$$

$$\psi l^{\gamma} = c^{-\sigma} w$$

$$k = \frac{\alpha}{1-\alpha}\frac{w}{r} l$$

$$\left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha} w^{1-\alpha} r^{\alpha} = \frac{\varepsilon - 1}{\varepsilon}$$

$$c + \delta k = k^{\alpha} l^{1-\alpha}$$

- A system of six equations on six unknowns.

Solving for the Steady State I

- Note first that:

$$w^{1-\alpha} = (1-\alpha)^{1-\alpha}\,\alpha^{\alpha}\frac{\varepsilon-1}{\varepsilon}r^{-\alpha} \Rightarrow$$
$$w = (1-\alpha)\,\alpha^{\frac{\alpha}{1-\alpha}}\left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{1}{1-\alpha}}\left(\frac{1}{\beta}-1+\delta\right)^{\frac{\alpha}{\alpha-1}}$$

- Then:

$$\frac{k}{l} = \Omega = \frac{\alpha}{1-\alpha}\frac{w}{r} \Rightarrow k = \Omega l$$

Solving for the Steady State II

- We are left with a system of two equations on two unknowns:

$$\begin{aligned} \psi l^\gamma c^\sigma &= w \\ c + \delta \Omega l &= \Omega^\alpha l \end{aligned}$$

- Substituting $c = (\Omega^\alpha - \delta\Omega)\, l$, we have

$$\psi \left(\frac{c}{\Omega^\alpha - \delta\Omega} \right)^\gamma c^\sigma = w \Rightarrow$$

$$c = \left((\Omega^\alpha - \delta\Omega)^\gamma \frac{w}{\psi} \right)^{\frac{1}{\gamma+\sigma}}$$

Steady State

$$r = \frac{1}{\beta} - 1 + \delta \qquad mc = \frac{\varepsilon - 1}{\varepsilon}$$

$$R = \frac{1}{\beta}\pi \qquad k = \Omega l$$

$$w = (1-\alpha)\,\alpha^{\frac{\alpha}{1-\alpha}} \left(\frac{\varepsilon-1}{\varepsilon}\right)^{\frac{1}{1-\alpha}} \left(\frac{1}{\beta} - 1 + \delta\right)^{\frac{\alpha}{\alpha-1}} \qquad x = \delta k$$

$$c = \left(\left(\Omega^{\alpha} - \delta\Omega\right)^{\gamma} \frac{w}{\psi}\right)^{\frac{1}{\gamma+\sigma}} \qquad y = k^{\alpha} l^{1-\alpha}$$

$$l = \frac{c}{\Omega^{\alpha} - \delta\Omega} \qquad \Omega = \frac{\alpha}{1-\alpha}\frac{w}{r}$$

Log-Linearizing Equilibrium Conditions

- Take variable x_t.

- Substitute by $xe^{\widehat{x}_t}$ where:

$$\widehat{x}_t = \log \frac{x_t}{x}$$

- Notation: a variable $\widehat{x}_t$ represents the log-deviation with respect to the steady state.

- Linearize with respect to $\widehat{x}_t$.

Households Conditions I

- $\psi l_t^\gamma = c_t^{-\sigma} w_t$ or $\psi l^\gamma e^{\gamma \widehat{l_t}} = c^{-\sigma} e^{-\sigma \widehat{c}_t} w e^{\widehat{w}_t}$ gets loglinearized to:

$$\gamma \widehat{l_t} = -\sigma \widehat{c}_t + \widehat{w}_t$$

- Then:

$$c_t^{-\sigma} = \beta E_t \{c_{t+1}^{-\sigma} (r_{t+1} + 1 - \delta)\}$$

or:

$$c^{-\sigma} e^{-\sigma \widehat{c}_t} = \beta E_t \{c^{-\sigma} e^{-\sigma \widehat{c}_{t+1}} \left(r e^{\widehat{r}_{t+1}} + 1 - \delta\right)\}$$

that gets loglinearized to:

$$-\sigma \widehat{c}_t = -\sigma E_t \widehat{c}_{t+1} + \beta r E_t \widehat{r}_{t+1}$$

Households Conditions II

- Also:

$$c_t^{-\sigma} = \beta E_t\{c_{t+1}^{-\sigma} \frac{R_{t+1}}{\pi_{t+1}}\}$$

or:

$$c^{-\sigma} e^{-\sigma \widehat{c}_t} = \beta E_t\{c^{-\sigma} e^{-\sigma \widehat{c}_{t+1}} \left(\frac{R}{\pi} e^{\widehat{R}_{t+1} - \widehat{\pi}_{t+1}} \right)\}$$

that gets loglinearized to:

$$-\sigma \widehat{c}_t = -\sigma E_t \widehat{c}_{t+1} + E_t \left(\widehat{R}_{t+1} - \widehat{\pi}_{t+1} \right)$$

- We do not loglinearize the money condition because the central bank, through open market operations, will supply all the needed money to support the chosen interest rate.

Marginal Cost

- We know:

$$mc_t = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\frac{1}{A_t}w_t^{1-\alpha}r_t^{\alpha}$$

  or:

$$mce^{\widehat{mc}_t} = \left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{1}{\alpha}\right)^{\alpha}\frac{w^{1-\alpha}r^{\alpha}}{A}e^{-\widehat{A}_t+(1-\alpha)\widehat{w}_t+\alpha\widehat{r}_t}$$

- Loglinearizes to:

$$\widehat{mc}_t = -\widehat{A}_t + (1-\alpha)\,\widehat{w}_t + \alpha\widehat{r}_t$$

Pricing Condition I

- We have:

$$E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau}\, v_{t+\tau} \left\{ \left(\frac{p_{it}^{*}}{p_{t+\tau}} - \frac{\varepsilon}{\varepsilon - 1} mc_{t+\tau} \right) y_{it+\tau}^{*} \right\} = 0,$$

where

$$y_{it+\tau}^{*} = \left(\frac{p_{ti}^{*}}{p_{t+\tau}} \right)^{-\varepsilon} y_{t+\tau},$$

Pricing Condition II

- Also:

$$E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} v_{t+\tau} \left\{ \left(\left(\frac{p_{it}^*}{p_{t+\tau}} \right)^{1-\varepsilon} - \frac{\varepsilon}{\varepsilon - 1} mc_{t+\tau} \left(\frac{p_{ti}^*}{p_{t+\tau}} \right)^{-\varepsilon} \right) y_{t+\tau} \right\} = 0 \Rightarrow$$

$$E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} v_{t+\tau} \left(\frac{p_{it}^*}{p_{t+\tau}} \right)^{1-\varepsilon} y_{t+\tau} =$$

$$E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} v_{t+\tau} \frac{\varepsilon}{\varepsilon - 1} mc_{t+\tau} \left(\frac{p_{ti}^*}{p_{t+\tau}} \right)^{-\varepsilon} y_{t+\tau}$$

Working on the Expression I

- If we prepare the expression for loglinearization (and eliminating the index i because of the symmetric equilibrium assumption):

$$E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} \, v \left(\frac{p^*}{p}\right)^{1-\varepsilon} y e^{\widehat{v}_{t+\tau}+(1-\varepsilon)\widehat{p}_t^*-(1-\varepsilon)\widehat{p}_{t+\tau}+\widehat{y}_{t+\tau}} =$$

$$E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} \, v \left(\frac{\varepsilon}{\varepsilon-1} mc\right) \left(\frac{p^*}{p}\right)^{-\varepsilon} y e^{\widehat{v}_{t+\tau}-\varepsilon\widehat{p}_t^*+\varepsilon\widehat{p}_{t+\tau}+\widehat{mc}_{t+\tau}+\widehat{y}_{t+\tau}}$$

Working on the Expression II

- Note that $\frac{\varepsilon}{\varepsilon-1}mc = 1, \frac{p^*}{p} = 1,$

- Dropping redundant constants, we get:

$$E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^\tau \, e^{\widehat{v}_{t+\tau}+(1-\varepsilon)\widehat{p}_t^*-(1-\varepsilon)\widehat{p}_{t+\tau}+\widehat{y}_{t+\tau}} =$$

$$E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^\tau \, e^{\widehat{v}_{t+\tau}-\varepsilon\widehat{p}_t^*+\varepsilon\widehat{p}_{t+\tau}+\widehat{mc}_{t+\tau}+\widehat{y}_{t+\tau}}$$

Working on the Expression III

Then

$$E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} \left(\widehat{v}_{t+\tau} + (1-\varepsilon)\,\widehat{p}_t^{*} - (1-\varepsilon)\,\widehat{p}_{t+\tau} + \widehat{y}_{t+\tau}\right) =$$

$$= E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} \left(\widehat{v}_{t+\tau} - \varepsilon\widehat{p}_t^{*} + \varepsilon\widehat{p}_{t+\tau} + \widehat{mc}_{t+\tau} + \widehat{y}_{t+\tau}\right)$$

Working on the Expression IV:

$$E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} (\widehat{p}_t^* - \widehat{p}_{t+\tau}) = E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} \widehat{mc}_{t+\tau} \Rightarrow$$

$$E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} \widehat{p}_t^* = E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} \widehat{p}_{t+\tau} + E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} \widehat{mc}_{t+\tau} \Rightarrow$$

$$\frac{1}{1-\beta\theta_p}\widehat{p}_t^* = E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} \widehat{p}_{t+\tau} + E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} \widehat{mc}_{t+\tau} \Rightarrow$$

$$\widehat{p}_t^* = (1-\beta\theta_p) E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} \widehat{p}_{t+\tau} + (1-\beta\theta_p) E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} \widehat{mc}_{t+\tau}$$

Working on the Expression V

- Note that:

$$
\begin{aligned}
(1-\beta\theta_p) E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} \widehat{p}_{t+\tau} &= (1-\beta\theta_p)\widehat{p}_t + (1-\beta\theta_p)\beta\theta_p E_t \widehat{p}_{t+1} + \dots \\
&= \widehat{p}_t + \beta\theta_p E_t (\widehat{p}_{t+1} - \widehat{p}_t) + \dots \\
&= \widehat{p}_{t-1} + \widehat{p}_t - \widehat{p}_{t-1} + \beta\theta_p E_t \widehat{\pi}_{t+1} + \dots \\
&= \widehat{p}_{t-1} + E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} \widehat{\pi}_{t+\tau}
\end{aligned}
$$

- Then:

$$
\widehat{p}_t^{*} = \widehat{p}_{t-1} + E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} \widehat{\pi}_{t+\tau} + (1-\beta\theta_p) E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau} \widehat{mc}_{t+\tau}
$$

Working on the Expression VI

- Now:

$$\widehat{p}_t^* = \widehat{p}_{t-1} + \widehat{\pi}_t + (1 - \beta\theta_p)\,\widehat{mc}_t + E_t \sum_{\tau=1}^{\infty} (\beta\theta_p)^{\tau}\,\widehat{\pi}_{t+\tau} + (1 - \beta\theta_p)\,E_t \sum_{\tau=1}^{\infty} (\beta\theta_p)^{\tau}\,\widehat{mc}_{t+\tau}$$

- If we forward the equation one term:

$$E_t\widehat{p}_{t+1}^* = \widehat{p}_t + E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau}\,\widehat{\pi}_{t+1+\tau} + (1 - \beta\theta_p)\,E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau}\,\widehat{mc}_{t+1+\tau}$$

Working on the Expression VII

- We multiply it by $\beta\theta_p$:

$$\begin{aligned}\beta\theta_p E_t \widehat{p}^*_{t+1} &= \beta\theta_p \widehat{p}_t + E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau+1} \widehat{\pi}_{t+1+\tau} \\ &+ (1-\beta\theta_p) E_t \sum_{\tau=0}^{\infty} (\beta\theta_p)^{\tau+1} \widehat{mc}_{t+1+\tau}\end{aligned}$$

- Then:

$$\widehat{p}^*_t - \widehat{p}_{t-1} = \beta\theta_p E_t \left(\widehat{p}^*_{t+1} - \widehat{p}_t\right) + \widehat{\pi}_t + (1-\beta\theta_p)\,\widehat{mc}_t$$

Price Index

- Since the price index is equal to $p_t = \left[\theta_p p_{t-1}^{1-\varepsilon} + (1-\theta_p)\, p_t^{*1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}$.

- we can write:

$$\begin{aligned} pe^{\widehat{p}_t} &= \left[\theta_p p^{1-\varepsilon} e^{(1-\varepsilon)\widehat{p}_{t-1}} + (1-\theta_p)\, p^{1-\varepsilon} e^{(1-\varepsilon)\widehat{p}_t^*}\right]^{\frac{1}{1-\varepsilon}} \Rightarrow \\ e^{\widehat{p}_t} &= \left[\theta_p e^{(1-\varepsilon)\widehat{p}_{t-1}} + (1-\theta_p)\, e^{(1-\varepsilon)\widehat{p}_t^*}\right]^{\frac{1}{1-\varepsilon}} \end{aligned}$$

- Loglinearizes to:

$$\widehat{p}_t = \theta_p \widehat{p}_{t-1} + (1-\theta_p)\, \widehat{p}_t^* \Rightarrow \widehat{\pi}_t = (1-\theta_p)\left(\widehat{p}_t^* - \widehat{p}_{t-1}\right)$$

Evolution of Inflation

- We can put together the price index and the pricing condition:

$$\frac{\widehat{\pi}_t}{1-\theta_p} = \beta\theta_p E_t \frac{\widehat{\pi}_{t+1}}{1-\theta_p} + \widehat{\pi}_t + (1-\beta\theta_p)\,\widehat{mc}_t$$

or:

$$\widehat{\pi}_t = \beta\theta_p E_t \widehat{\pi}_{t+1} + (1-\theta_p)\,\widehat{\pi}_t + (1-\theta_p)(1-\beta\theta_p)\,\widehat{mc}_t$$

- Simplifies to:

$$\widehat{\pi}_t = \beta E_t \widehat{\pi}_{t+1} + \lambda\left(-\widehat{A}_t + (1-\alpha)\,\widehat{w}_t + \alpha\widehat{r}_t\right)$$

where $\lambda = \frac{(1-\theta_p)(1-\beta\theta_p)}{\theta_p}$ and $\widehat{mc}_t = -\widehat{A}_t + (1-\alpha)\,\widehat{w}_t + \alpha\widehat{r}_t$

New Keynesian Phillips Curve

- The expression:

$$\widehat{\pi}_t = \beta E_t \widehat{\pi}_{t+1} + \lambda \widehat{mc}_t$$

  is known as the New Keynesian Phillips Curve

- Empirical performance?

- Large literature:

  1. Lagged inflation versus expected inflation.

  2. Measures of marginal cost.

Production Function I

- Now:

$$ye^{\widehat{y}_t} = \frac{Ae^{\widehat{A}_t}}{j^{-\varepsilon}e^{-\varepsilon\widehat{j}_t}p^{\varepsilon}e^{\varepsilon\widehat{p}_t}}k^{\alpha}l^{1-\alpha}e^{\alpha\widehat{k}_t+(1-\alpha)\widehat{l}_t}$$

- Cancelling constants:

$$e^{\widehat{y}_t} = \frac{e^{\widehat{A}_t}}{e^{-\varepsilon\widehat{j}_t}e^{\varepsilon\widehat{p}_t}}e^{\alpha\widehat{k}_t+(1-\alpha)\widehat{l}_t}$$

- Then:

$$\widehat{y}_t = \widehat{A}_t + \alpha\widehat{k}_t + (1-\alpha)\,\widehat{l}_t + \varepsilon\left(\widetilde{\widehat{j}}_t - \widetilde{\widehat{p}}_t\right)$$

Production Function II

- Now we find expressions for the loglinearized values of j_t and p_t:

$$\widehat{j}_t = \log j_t - \log j = -\frac{1}{\varepsilon}\log\left(\int_0^1 p_{it}^{-\varepsilon} di\right) - \log p$$

$$\widehat{p}_t = \log p_t - \log p = \frac{1}{1-\varepsilon}\log\left(\int_0^1 p_{it}^{1-\varepsilon} di\right) - \log p$$

- Then:

$$\widetilde{\widehat{j}}_t = -\frac{1}{p}\int_0^1 (p_{it} - p)\, di$$

$$\widetilde{\widehat{p}}_t = -\frac{1}{1-\varepsilon}\frac{1-\varepsilon}{p}\int_0^1 (p_{it} - p)\, di$$

Production Function II

- Clearly $\widetilde{\widehat{j}}_t = \widetilde{\widehat{p}}_t$.

- Then:

$$\widehat{y}_t = \widehat{A}_t + \alpha \widehat{k}_t + (1 - \alpha) \widehat{l}_t$$

- No first-order loss of efficiency!

Aggregate Conditions I

- We know $c_t + x_t = y_t$ or $ce^{\widehat{c}_t} + xe^{\widehat{x}_t} = ye^{\widehat{y}_t}$ that loglinearizes to:

$$c\widehat{c} + x\widehat{x}_t = y\widehat{y}_t$$

- Also $k_{t+1} = (1-\delta)\, k_t + x_t$ or $ke^{\widehat{k}_{t+1}} = (1-\delta)\, ke^{\widehat{k}_t} + xe^{\widehat{x}_t}$ that loglinearizes to:

$$k\widehat{k}_{t+1} = (1-\delta)\, k\widehat{k}_t + x\widehat{x}_t$$

Aggregate Conditions II

- Finally:

$$k_t = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t}l_t$$

or:

$$ke^{\widehat{k}_t} = \frac{\alpha}{1-\alpha}\frac{w}{r}le^{\widehat{w}_t+\widehat{l}_t-\widehat{r}_t}$$

- Loglinearizes to:

$$\widehat{k}_t = \widehat{w}_t + \widehat{l}_t - \widehat{r}_t$$

Government

- We have that:

$$\frac{R_{t+1}}{R} = \left(\frac{R_t}{R}\right)^{\gamma_R} \left(\frac{\pi_t}{\pi}\right)^{\gamma_\pi} \left(\frac{y_t}{y}\right)^{\gamma_y} e^{\varphi_t}$$

  or:

$$e^{\widehat{R}_{t+1}} = e^{\gamma_R \widehat{R}_t + \gamma_\pi \widehat{\pi}_t + \gamma_y \widehat{y}_t + \varphi_t}$$

- Loglinearizes to:

$$\widehat{R}_{t+1} = \gamma_R \widehat{R}_t + \gamma_\pi \widehat{\pi}_t + \gamma_y \widehat{y}_t + \varphi_t$$

Loglinear System

$$
\begin{aligned}
-\sigma\widehat{c}_t &= E_t\left(-\sigma\widehat{c}_{t+1} + \beta r\widehat{r}_{t+1}\right) \\
-\sigma\widehat{c}_t &= E_t\left(-\sigma\widehat{c}_{t+1} + \widehat{R}_{t+1} - \widehat{\pi}_{t+1}\right) \\
\gamma\widehat{l}_t &= -\sigma\widehat{c}_t + \widehat{w}_t \\
\widehat{y}_t &= \widehat{A}_t + \alpha\widehat{k}_t + (1-\alpha)\,\widehat{l}_t \\
y\widehat{y}_t &= c\widehat{c} + x\widehat{x}_t \\
k\widehat{k}_{t+1} &= (1-\delta)\,k\widehat{k}_t + x\widehat{x}_t \\
\widehat{k}_t &= \widehat{w}_t + \widehat{l}_t - \widehat{r}_t \\
\widehat{R}_{t+1} &= \gamma_R\widehat{R}_t + \gamma_\pi\widehat{\pi}_t + \gamma_y\widehat{y}_t + \varphi_t \\
\widehat{\pi}_t &= \beta E_t\widehat{\pi}_{t+1} + \lambda\left(-\widehat{A}_t + (1-\alpha)\,\widehat{w}_t + \alpha\widehat{r}_t\right) \\
\widehat{A}_t &= \rho\widehat{A}_{t-1} + z_t
\end{aligned}
$$

a system of 10 equations on 10 variables: $\left\{\widehat{c}_t, \widehat{l}_t, \widehat{x}_t, \widehat{y}_t, \widehat{k}_t, \widehat{w}_t, \widehat{r}_t, \widehat{R}_{t+1}, \widehat{\pi}_t, \widehat{A}_t\right\}$.

Solving the System

- We can put the system in Uhlig's form.

- To do so, we redefine $\widehat{R}_{t+1}$ and $\widehat{\pi}_t$ as (pseudo) state-variables in order to have at most as many control variables as deterministic equations.

- States, controls, and shocks:

$$\begin{aligned} X_t &= \begin{pmatrix} \widehat{k}_{t+1} & \widehat{R}_{t+1} & \widehat{\pi}_t \end{pmatrix}' \\ Y_t &= \begin{pmatrix} \widehat{c}_t & \widehat{l}_t & \widehat{x}_t & \widehat{y}_t & \widehat{w}_t & \widehat{r}_t \end{pmatrix}' \\ Z_t &= \begin{pmatrix} z_t & \varphi_t \end{pmatrix}' \end{aligned}$$

Deterministic Bloc

$$
\begin{aligned}
\gamma \widehat{l}_t + \sigma \widehat{c}_t - \widehat{w}_t &= 0 \\
\widehat{y}_t - \widehat{A}_t - \alpha \widehat{k}_t - (1-\alpha)\,\widehat{l}_t &= 0 \\
y \widehat{y}_t - c \widehat{c} - x \widehat{x}_t &= 0 \\
k \widehat{k}_{t+1} - (1-\delta)\, k \widehat{k}_t - x \widehat{x}_t &= 0 \\
\widehat{k}_t - \widehat{w}_t - \widehat{l}_t + \widehat{r}_t &= 0 \\
\widehat{R}_{t+1} - \gamma_R \widehat{R}_t - \gamma_\pi \widehat{\pi}_t - \gamma_y \widehat{y}_t - \varphi_t &= 0
\end{aligned}
$$

Expectational Bloc

$$\begin{aligned}
\sigma\widehat{c}_t + E_t\left(-\sigma\widehat{c}_{t+1} + \beta r\widehat{r}_{t+1}\right) &= 0 \\
\sigma\widehat{c}_t + \widehat{R}_{t+1} + E_t\left(-\sigma\widehat{c}_{t+1} - \widehat{\pi}_{t+1}\right) &= 0 \\
\widehat{\pi}_t - \beta E_t\widehat{\pi}_{t+1} - \lambda\left(-\widehat{A}_t + (1-\alpha)\,\widehat{w}_t + \alpha\widehat{r}_t\right) &= 0
\end{aligned}$$

Two stochastic processes

$$\begin{aligned}
\widehat{A}_t &= \rho\widehat{A}_{t-1} + z_t \\
&\varphi_t
\end{aligned}$$

Matrices of the Deterministic Bloc

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -\gamma_\pi \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \\ -(1-\delta)k & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -\gamma_R & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \sigma & \gamma & 0 & 0 & -1 & 0 \\ 0 & -(1-\alpha) & 0 & 1 & 0 & 0 \\ -c & 0 & -x & y & 0 & 0 \\ 0 & 0 & -x & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -\gamma_y & 0 & 0 \end{pmatrix}, \ D = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}$$

Matrices of the Expectational Bloc

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\beta \end{pmatrix}, G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$J = \begin{pmatrix} -\sigma & 0 & 0 & 0 & 0 & \beta r \\ -\sigma & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, K = \begin{pmatrix} \sigma & 0 & 0 & 0 & 0 & 0 \\ \sigma & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda(1-\alpha) & \lambda\alpha \end{pmatrix}$$

$$L = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \lambda & 0 \end{pmatrix}$$

Matrices of the Stochastic Process

$$N = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}$$

Solution of the Problem

$$\begin{aligned} X_t &= PX_{t-1} + QZ_t \\ Y_t &= RX_{t-1} + SZ_t \end{aligned}$$

Beyond Linearization

- We solved the model using one particular approach.

- How different are the computational answers provided by alternative solution methods for dynamic equilibrium economies?

- Why do we care?

- Stochastic neoclassical growth model is nearly linear for the benchmark calibration.

  - Linear methods may be good enough.

- Unsatisfactory answer for many economic questions: we want to use highly nonlinear models.

  - Linear methods not enough.

Solution Methods

1. Linearization: levels and logs.

2. Perturbation: levels and logs, different orders.

3. Projection methods: spectral and Finite Elements.

4. Value Function Iteration.

5. Other?

Evaluation Criteria

- Accuracy.
- Computational cost.
- Programming cost

What Do We Know about Other Methods?

- Perturbation methods deliver an interesting compromise between accuracy, speed and programming burden (Problem: Analytical derivatives).

- Second order perturbations much better than linear with trivial additional computational cost.

- Finite Elements method the best for estimation purposes.

- Linear methods can deliver misleading answers.

- Linearization in Levels can be better than in Logs.

A Quick Overview

- Numerous problems in macroeconomics involve functional equations of the form:

$$\mathcal{H}(d) = \mathbf{0}$$

- Examples: Value Function, Euler Equations.

- Regular equations are particular examples of functional equations.

- How do we solve functional equations?

Two Main Approaches

1. Projection Methods:

$$d^n(x,\theta) = \sum_{i=0}^{n} \theta_i \Psi_i(x)$$

We pick a basis $\{\Psi_i(x)\}_{i=0}^{\infty}$ and "project" $\mathcal{H}(\cdot)$ against that basis.

2. Perturbation Methods:

$$d^n(x,\theta) = \sum_{i=0}^{n} \theta_i (x - x_0)^i$$

We use implicit-function theorems to find coefficients θ_i.

Solution Methods I: Projection (Spectral)

- Standard Reference: Judd (1992).

- Choose a basis for the policy functions.

- Restrict the policy function to a be a linear combination of the elements of the basis.

- Plug the policy function in the Equilibrium Conditions and find the unknown coefficients.

- Use Chebyshev polynomial.

- Pseudospectral (collocation) weigthing.

Solution Methods II: Projection (Finite Elements)

- Standard Reference: McGrattan (1999)
- Bound the domain of the state variables.
- Partition this domain in nonintersecting elements.
- Choose a basis for the policy functions in each element.
- Plug the policy function in the Equilibrium Conditions and find the unknown coefficients.

- Use linear basis.
- Galerkin weighting.
- We can be smart picking our grid.

Solution Methods III: Perturbation Methods

- Most complicated problems have particular cases that are easy to solve.

- Often, we can use the solution to the particular case as a building block of the general solution.

- Very successful in physics.

- Judd and Guu (1993) showed how to apply it to economic problems.

A Simple Example

- Imagine we want to find the (possible more than one) roots of:

$$x^3 - 4.1x + 0.2 = 0$$

  such that $x < 0$.

- This a tricky, cubic equation.

- How do we do it?

Main Idea

- Transform the problem rewriting it in terms of a small perturbation parameter.

- Solve the new problem for a particular choice of the perturbation parameter.

- Use the previous solution to approximate the solution of original the problem.

Step 1: Transform the Problem

- Write the problem into a perturbation problem indexed by a small parameter ε.

- This step is usually ambiguous since there are different ways to do so.

- A natural, and convenient, choice for our case is to rewrite the equation as:

$$x^3 - (4 + \varepsilon)\, x + 2\varepsilon = 0$$

  where $\varepsilon \equiv 0.1$.

Step 2: Solve the New Problem

- Index the solutions as a function of the perturbation parameter $x = g(\varepsilon)$:

$$g(\varepsilon)^3 - (4+\varepsilon)\, g(\varepsilon) + 2\varepsilon = 0$$

and assume each of this solution is smooth (this can be shown to be the case for our particular example).

- Note that $\varepsilon = 0$ is easy to solve:

$$x^3 - 4x = 0$$

that has roots $g(0) = -2, 0, 2$. Since we require $x < 0$, we take $g(0) = -2$.

Step 3: Build the Approximated Solution

- By Taylor's Theorem:

$$x = g(\varepsilon)|_{\varepsilon=0} = g(0) + \sum_{n=1}^{\infty} \frac{g^n(0)}{n!}\varepsilon^n$$

- Substitute the solution into the problem and recover the coefficients $g(0)$ and $\frac{g^n(0)}{n!}$for $n = 1, ...$ in an iterative way.

- Let's do it!

Zeroth -Order Approximation

- We just take $\varepsilon = 0$.

- Before we found that $g(0) = -2$.

- Is this a good approximation?

$$x^3 - 4.1x + 0.2 = 0 \Rightarrow$$
$$-8 + 8.2 + 0.2 = 0.4$$

- It depends!

First -Order Approximation

- Take the derivative of $g(\varepsilon)^3 - (4+\varepsilon)\, g(\varepsilon) + 2\varepsilon = 0$ with respect to ε:

$$3g(\varepsilon)^2 g'(\varepsilon) - g(\varepsilon) - (4+\varepsilon)\, g'(\varepsilon) + 2 = 0$$

- Set $\varepsilon = 0$

$$3g(0)^2 g'(0) - g(0) - 4g'(0) + 2 = 0$$

- But we just found that $g(0) = -2$, so:

$$8g'(0) + 4 = 0$$

that implies $g'(0) = -\frac{1}{2}$.

First -Order Approximation

- By Taylor: $x = g(\varepsilon)|_{\varepsilon=0} \simeq g(0) + \frac{g^1(0)}{1!}\varepsilon^1$ or

$$x \simeq -2 - \frac{1}{2}\varepsilon$$

- For our case $\varepsilon \equiv 0.1$

$$x = -2 - \frac{1}{2} * 0.1 = -2.05$$

- Is this a good approximation?

$$x^3 - 4.1x + 0.2 = 0 \Rightarrow$$
$$-8.615125 + 8.405 + 0.2 = -0.010125$$

Second -Order Approximation

- Take the derivative of $3g(\varepsilon)^2 g'(\varepsilon) - g(\varepsilon) - (4+\varepsilon) g'(\varepsilon) + 2 = 0$ with respect to ε:

$$6g(\varepsilon)\left(g'(\varepsilon)\right)^2 + 3g(\varepsilon)^2 g''(\varepsilon) - g'(\varepsilon) - g'(\varepsilon) - (4+\varepsilon) g''(\varepsilon) = 0$$

- Set $\varepsilon = 0$

$$6g(0)\left(g'(0)\right)^2 + 3g(0)^2 g''(0) - 2g'(0) - 4g''(0) = 0$$

- Since $g(0) = -2$ and $g'(0) = -\frac{1}{2}$, we get:

$$8g''(0) - 2 = 0$$

that implies $g''(0) = \frac{1}{4}$.

Second -Order Approximation

- By Taylor: $x = g(\varepsilon)|_{\varepsilon=0} \simeq g(0) + \frac{g^1(0)}{1!}\varepsilon^1 + \frac{g^2(0)}{2!}\varepsilon^2$ or
$$x \simeq -2 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2$$

- For our case $\varepsilon \equiv 0.1$
$$x = -2 - \frac{1}{2} * 0.1 + \frac{1}{8} * 0.01 = -2.04875$$

- Is this a good approximation?
$$x^3 - 4.1x + 0.2 = 0 \Rightarrow$$
$$-8.59937523242188 + 8.399875 + 0.2 = 4.997675781240329e - 004$$

Some Remarks

- The exact solution (up to machine precession of 14 decimal places) is $x = -2.04880884817015$.

- A second-order approximation delivers: $x = -2.04875$

- Relative error: 0.00002872393906.

- Yes, this was a rigged, but suggestive, example.

A Couple of Points to Remember

1. We transformed the original problem into a perturbation problem in such a way that the zeroth-order approximation has an analytical solution.

2. Solving for the first iteration involves a nonlinear (although trivial in our case) equation. All further iterations only require to solve a linear equation in one unknown.

An Application in Macroeconomics: Basic RBC

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \{\log c_t\}$$

$$\begin{aligned} c_t + k_{t+1} &= e^{z_t} k_t^{\alpha} + (1-\delta) k_t, \ \forall \ t > 0 \\ z_t &= \rho z_{t-1} + \sigma \varepsilon_t, \ \ \varepsilon_t \sim \mathcal{N}(0,1) \end{aligned}$$

Equilibrium Conditions

$$\frac{1}{c_t} = \beta E_t \frac{1}{c_{t+1}} \left(1 + \alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} - \delta\right)$$

$$c_t + k_{t+1} = e^{z_t} k_t^{\alpha} + (1-\delta) k_t$$

$$z_t = \rho z_{t-1} + \sigma \varepsilon_t$$

Computing the RBC

- We already discuss that the previous problem does not have a known "paper and pencil" solution.

- One particular case the model has a closed form solution: $\delta = 1$.

- Why? Because, the income and the substitution effect from a productivity shock cancel each other.

- Not very realistic but we are trying to learn here.

Solution

- By "Guess and Verify"

$$c_t = (1 - \alpha\beta)\, e^{z_t} k_t^{\alpha}$$
$$k_{t+1} = \alpha\beta e^{z_t} k_t^{\alpha}$$

- How can you check? Plug the solution in the equilibrium conditions.

Another Way to Solve the Problem

- Now let us suppose that you missed the lecture where "Guess and Verify" was explained.

- You need to compute the RBC.

- What you are searching for? A policy functions for consumption:

$$c_t = c\left(k_t, z_t\right)$$

and another one for capital:

$$k_{t+1} = k\left(k_t, z_t\right)$$

Equilibrium Conditions

- We substitute in the equilibrium conditions the budget constraint and the law of motion for technology.

- Then, we have the equilibrium conditions:

$$\frac{1}{c(k_t, z_t)} = \beta E_t \frac{\alpha e^{\rho z_t + \sigma \varepsilon_{t+1}} k(k_t, z_t)^{\alpha-1}}{c(k(k_t, z_t), \rho z_t + \sigma \varepsilon_{t+1})}$$
$$c(k_t, z_t) + k(k_t, z_t) = e^{z_t} k_t^{\alpha}$$

- The Euler equation is the equivalent of $x^3 - 4.1x + 0.2 = 0$ in our simple example, and $c(k_t, z_t)$ and $k(k_t, z_t)$ are the equivalents of x.

A Perturbation Approach

- You want to transform the problem.

- Which perturbation parameter? standard deviation σ.

- Why σ?

- Set $\sigma = 0 \Rightarrow$deterministic model, $z_t = 0$ and $e^{z_t} = 1$.

Taylor's Theorem

- We search for policy function $c_t = c(k_t, z_t; \sigma)$ and $k_{t+1} = k(k_t, z_t; \sigma)$.

- Equilibrium conditions:

$$\begin{aligned} E_t \left(\frac{1}{c(k_t, z_t; \sigma)} - \beta \frac{\alpha e^{\rho z_t + \sigma \varepsilon_{t+1}} k(k_t, z_t; \sigma)^{\alpha-1}}{c(k(k_t, z_t; \sigma), \rho z_t + \sigma \varepsilon_{t+1}; \sigma)} \right) &= 0 \\ c(k_t, z_t; \sigma) + k(k_t, z_t; \sigma) - e^{z_t} k_t^{\alpha} &= 0 \end{aligned}$$

- We will take derivatives with respect to k_t, z_t, and σ.

Asymptotic Expansion $c_t = c(k_t, z_t; \sigma)|_{k,0,0}$

$$\begin{aligned} c_t &= c(k, 0; 0) \\ &+ c_k(k, 0; 0)(k_t - k) + c_z(k, 0; 0) z_t + c_\sigma(k, 0; 0)\sigma \\ &+ \frac{1}{2} c_{kk}(k, 0; 0)(k_t - k)^2 + \frac{1}{2} c_{kz}(k, 0; 0)(k_t - k) z_t \\ &+ \frac{1}{2} c_{k\sigma}(k, 0; 0)(k_t - k)\sigma + \frac{1}{2} c_{zk}(k, 0; 0) z_t (k_t - k) \\ &+ \frac{1}{2} c_{zz}(k, 0; 0) z_t^2 + \frac{1}{2} c_{z\sigma}(k, 0; 0) z_t \sigma \\ &+ \frac{1}{2} c_{\sigma k}(k, 0; 0)\sigma(k_t - k) + \frac{1}{2} c_{\sigma z}(k, 0; 0)\sigma z_t \\ &+ \frac{1}{2} c_{\sigma^2}(k, 0; 0)\sigma^2 + \ldots \end{aligned}$$

Asymptotic Expansion $k_{t+1} = k\left(k_t, z_t; \sigma\right)|_{k,0,0}$

$$
\begin{aligned}
k_{t+1} = \; & k\left(k, 0; 0\right) \\
& + k_k\left(k, 0; 0\right) k_t + k_z\left(k, 0; 0\right) z_t + k_\sigma\left(k, 0; 0\right) \sigma \\
& + \frac{1}{2} k_{kk}\left(k, 0; 0\right)\left(k_t - k\right)^2 + \frac{1}{2} k_{kz}\left(k, 0; 0\right)\left(k_t - k\right) z_t \\
& + \frac{1}{2} k_{k\sigma}\left(k, 0; 0\right)\left(k_t - k\right) \sigma + \frac{1}{2} k_{zk}\left(k, 0; 0\right) z_t \left(k_t - k\right) \\
& + \frac{1}{2} k_{zz}\left(k, 0; 0\right) z_t^2 + \frac{1}{2} k_{z\sigma}\left(k, 0; 0\right) z_t \sigma \\
& + \frac{1}{2} k_{\sigma k}\left(k, 0; 0\right) \sigma \left(k_t - k\right) + \frac{1}{2} k_{\sigma z}\left(k, 0; 0\right) \sigma z_t \\
& + \frac{1}{2} k_{\sigma^2}\left(k, 0; 0\right) \sigma^2 + \ldots
\end{aligned}
$$

Comment on Notation

- From now on, to save on notation, I will just write

$$F(k_t, z_t; \sigma) = E_t \left[\begin{array}{c} \frac{1}{c(k_t, z_t; \sigma)} - \beta \frac{\alpha e^{\rho z_t + \sigma \varepsilon_{t+1}} k(k_t, z_t; \sigma)^{\alpha-1}}{c(k(k_t, z_t; \sigma), \rho z_t + \sigma \varepsilon_{t+1}; \sigma)} \\ c(k_t, z_t; \sigma) + k(k_t, z_t; \sigma) - e^{z_t} k_t^{\alpha} \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \end{array} \right]$$

- Note that:

$$F(k_t, z_t; \sigma) = H(c(k_t, z_t; \sigma), c(k(k_t, z_t; \sigma), z_{t+1}; \sigma), k(k_t, z_t; \sigma), k_t, z_t; \sigma)$$

- I will use H_i to represent the partial derivative of H with respect to the i component and drop the evaluation at the steady state of the functions when we do not need it.

Zeroth -Order Approximation

- First, we evaluate $\sigma = 0$:

$$F(k_t, 0; 0) = 0$$

- Steady state:

$$\frac{1}{c} = \beta \frac{\alpha k^{\alpha-1}}{c}$$

or,

$$1 = \alpha \beta k^{\alpha-1}$$

Steady State

- Then:

$$c = c\,(k, 0; 0) = (\alpha\beta)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta)^{\frac{1}{1-\alpha}}$$
$$k = k\,(k, 0; 0) = (\alpha\beta)^{\frac{1}{1-\alpha}}$$

- How good is this approximation?

First -Order Approximation

- We take derivatives of $F(k_t, z_t; \sigma)$ around $k, 0$, and 0.

- With respect to k_t:

$$F_k(k, 0; 0) = 0$$

- With respect to z_t:

$$F_z(k, 0; 0) = 0$$

- With respect to σ:

$$F_\sigma(k, 0; 0) = 0$$

Solving the System I

Remember that:

$$F(k_t, z_t; \sigma) = H(c(k_t, z_t; \sigma), c(k(k_t, z_t; \sigma), z_{t+1}; \sigma), k(k_t, z_t; \sigma), k_t, z_t; \sigma)$$

Then:

$$F_k(k, 0; 0) = H_1 c_k + H_2 c_k k_k + H_3 k_k + H_4 = 0$$

$$F_z(k, 0; 0) = H_1 c_z + H_2 (c_k k_z + c_k \rho) + H_3 k_z + H_5 = 0$$

$$F_\sigma(k, 0; 0) = H_1 c_\sigma + H_2 (c_k k_\sigma + c_\sigma) + H_3 k_\sigma + H_6 = 0$$

Solving the System II

- Note that:

$$
\begin{aligned}
F_k(k, 0; 0) &= H_1 c_k + H_2 c_k k_k + H_3 k_k + H_4 = 0 \\
F_z(k, 0; 0) &= H_1 c_z + H_2 (c_k k_z + c_k \rho) + H_3 k_z + H_5 = 0
\end{aligned}
$$

  is a quadratic system of four equations on four unknowns: c_k, c_z, k_k, and k_z.

- Procedures to solve quadratic systems: Uhlig (1999).

- Why quadratic? Stable and unstable manifold.

Solving the System III

- Note that:

$$F_\sigma(k, 0; 0) = H_1 c_\sigma + H_2 (c_k k_\sigma + c_\sigma) + H_3 k_\sigma + H_6 = 0$$

  is a linear, and homogeneous system in c_σ and k_σ.

- Hence

$$c_\sigma = k_\sigma = 0$$

Comparison with Linearization

- After Kydland and Prescott (1982) a popular method to solve economic models has been the use of a LQ approximation.

- Close relative: linearization of equilibrium conditions.

- When properly implemented linearization, LQ, and first-order perturbation are equivalent.

- Advantages of linearization:

  1. Theorems.

  2. Higher order terms.

Second -Order Approximation

- We take second-order derivatives of $F(k_t, z_t; \sigma)$ around $k, 0$, and 0:

$$\begin{aligned}
F_{kk}(k, 0; 0) &= 0 \\
F_{kz}(k, 0; 0) &= 0 \\
F_{k\sigma}(k, 0; 0) &= 0 \\
F_{zz}(k, 0; 0) &= 0 \\
F_{z\sigma}(k, 0; 0) &= 0 \\
F_{\sigma\sigma}(k, 0; 0) &= 0
\end{aligned}$$

- Remember Young's theorem!

Solving the System

- We substitute the coefficients that we already know.

- A linear system of 12 equations on 12 unknowns. Why linear?

- Cross-terms $k\sigma$ and $z\sigma$ are zero.

- Conjecture on all the terms with odd powers of σ.

Correction for Risk

- We have a term in σ^2.
- Captures precautionary behavior.
- We do not have certainty equivalence any more!
- Important advantage of second order approximation.

Higher Order Terms

- We can continue the iteration for as long as we want.

- Often, a few iterations will be enough.

- The level of accuracy depends on the goal of the exercise: Fernández-Villaverde, Rubio-Ramírez, and Santos (2005).

A Computer

- In practice you do all this approximations with a computer.

- Burden: analytical derivatives.

- Why are numerical derivatives a bad idea?

- More theoretical point: do the derivatives exist? (Santos, 1992).

Code

- First and second order: `Matlab` and `Dynare`.

- Higher order: `Mathematica`, `Fortran` code by Jinn and Judd.

An Example

- Let me run a second order approximation.

- Our choices

Calibrated Parameters

Parameter	β	α	ρ	σ
Value	0.99	0.33	0.95	0.01

Computation

- Steady State:

$$c = (\alpha\beta)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta)^{\frac{1}{1-\alpha}} = 0.388069$$
$$k = (\alpha\beta)^{\frac{1}{1-\alpha}} = 0.1883$$

- First order components.

$$\begin{array}{ll} c_k(k,0;0) = 0.680101 & k_k(k,0;0) = 0.33 \\ c_z(k,0;0) = 0.388069 & k_z(k,0;0) = 0.1883 \\ c_\sigma(k,0;0) = 0 & k_\sigma(k,0;0) = 0 \end{array}$$

Comparison

$$c_t = 0.6733 e^{z_t} k_t^{0.33}$$
$$c_t \simeq 0.388069 + 0.680101 \left(k_t - k\right) + 0.388069 z_t$$

and:

$$k_{t+1} = 0.3267 e^{z_t} k_t^{0.33}$$
$$k_{t+1} \simeq 0.1883 + 0.1883 \left(k_t - k\right) + 0.33 z_t$$

Second-Order Terms

$$
\begin{array}{ll}
c_{kk}(k,0;0) = -2.41990 & k_{kk}(k,0;0) = -1.1742 \\
c_{kz}(k,0;0) = 0.680099 & k_{kz}(k,0;0) = 0.330003 \\
c_{k\sigma}(k,0;0) = 0. & k_{k\sigma}(k,0;0) = 0 \\
c_{zz}(k,0;0) = 0.388064 & k_{zz}(k,0;0) = 0.188304 \\
c_{z\sigma}(k,0;0) = 0 & k_{z\sigma}(k,0;0) = 0 \\
c_{\sigma^2}(k,0;0) = 0 & k_{\sigma^2}(k,0;0) = 0
\end{array}
$$

Non Local Accuracy test (Judd, 1992, and Judd and Guu, 1997)

Given the Euler equation:

$$\frac{1}{c^i\left(k_t\,,z_t\right)}=E_t\left(\frac{\alpha e^{z_{t+1}}k^i(k_t,z_t)^{\alpha-1}}{c^i\left(k^i(k_t,z_t),z_{t+1}\right)}\right)$$

we can define:

$$EE^i\left(k_t\,,z_t\right)\equiv 1-c^i\left(k_t\,,z_t\right)E_t\left(\frac{\alpha e^{z_{t+1}}k^i(k_t,z_t)^{\alpha-1}}{c^i\left(k^i(k_t,z_t),z_{t+1}\right)}\right)$$

Changes of Variables

- We approximated our solution in levels.

- We could have done it in logs.

- Why stop there? Why not in powers of the state variables?

- Judd (2002) has provided methods for changes of variables.

- We apply and extend ideas to the stochastic neoclassical growth model.

A General Transformation

- We look at solutions of the form:

$$\begin{aligned} c^{\mu} - c_0^{\mu} &= a\left(k^{\zeta} - k_0^{\zeta}\right) + cz \\ k'^{\gamma} - k_0^{\gamma} &= c\left(k^{\zeta} - k_0^{\zeta}\right) + dz \end{aligned}$$

- Note that:

1. If γ, ζ, μ and φ are 1 we get the linear representation.

2. As γ, ζ and μ tend to zero and φ is equal to 1 we get the loglinear approximation.

Theory

- The first order solution can be written as

$$f(x) \simeq f(a) + (x-a) f'(a)$$

- Expand $g(y) = h(f(X(y)))$ around $b = Y(a)$, where $X(y)$ is the inverse of $Y(x)$.

- Then:

$$g(y) = h(f(X(y))) = g(b) + g_{\alpha}(b)(Y^{\alpha}(x) - b^{\alpha})$$

where $g_{\alpha} = h_A f_i^A X_{\alpha}^i$ comes from the application of the chain rule.

- From this expression it is easy to see that if we have computed the values of f_i^A, then it is straightforward to find the value of g_{α}.

Coefficients Relation

- Remember that the linear solution is:

$$\begin{aligned} \left(k' - k_0\right) &= a_1 (k - k_0) + b_1 z \\ (l - l_0) &= c_1 (k - k_0) + d_1 z \end{aligned}$$

- Then we show that:

$a_3 = \frac{\gamma}{\zeta} k_0^{\gamma-\zeta} a_1$	$b_3 = \gamma k_0^{\gamma-1} b_1$
$c_3 = \frac{\mu}{\zeta} l_0^{\mu-1} k_0^{1-\zeta} c_1$	$d_3 = \mu l_0^{\mu-1} d_1$

Finding the Parameters γ, ζ, μ and φ

- Minimize over a grid the Euler Error.

- Some optimal results

Table 6.2.2: Euler Equation Errors

γ	ζ	μ	SEE
1	1	1	0.0856279
0.986534	0.991673	2.47856	0.0279944

Sensitivity Analysis

- Different parameter values.

- Most interesting finding is when we change σ:

Table 6.3.3: Optimal Parameters for different σ's

σ	γ	ζ	μ
0.014	0.98140	0.98766	2.47753
0.028	1.04804	1.05265	1.73209
0.056	1.23753	1.22394	0.77869

- A first order approximation corrects for changes in variance!

A Quasi-Optimal Approximation I

- Sensitivity analysis reveals that for different parametrizations

$$\gamma \simeq \zeta$$

- This suggests the quasi-optimal approximation:

$$\begin{aligned} k'^{\gamma} - k_0^{\gamma} &= a_3 \left(k^{\gamma} - k_0^{\gamma}\right) + b_3 z \\ l^{\mu} - l_0^{\mu} &= c_3 \left(k^{\gamma} - k_0^{\gamma}\right) + d_3 z \end{aligned}$$

A Quasi-Optimal Approximation II

- Note that if define $\widehat{k} = k^{\gamma} - k_0^{\gamma}$ and $\widehat{l} = l^{\mu} - l_0^{\mu}$ we get:

$$\begin{aligned} \widehat{k}' &= a_3\widehat{k} + b_3 z \\ \widehat{l} &= c_3\widehat{k} + d_3 z \end{aligned}$$

- Linear system:

  1. Use for analytical study (Campbell, 1994 and Woodford, 2003).

  2. Use for estimation with a Kalman Filter.

References

- General Perturbation theory: *Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory* by Carl M. Bender, Steven A. Orszag.

- Perturbation in Economics:

  1. "Perturbation Methods for General Dynamic Stochastic Models" by Hehui Jin and Kenneth Judd.

  2. "Perturbation Methods with Nonlinear Changes of Variables" by Kenneth Judd.

- A gentle introduction: "Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function" by Martín Uribe and Stephanie Schmitt-Grohe.

Figure 5.1.1: Labor Supply at z = 0, τ = 2 / σ = 0.007

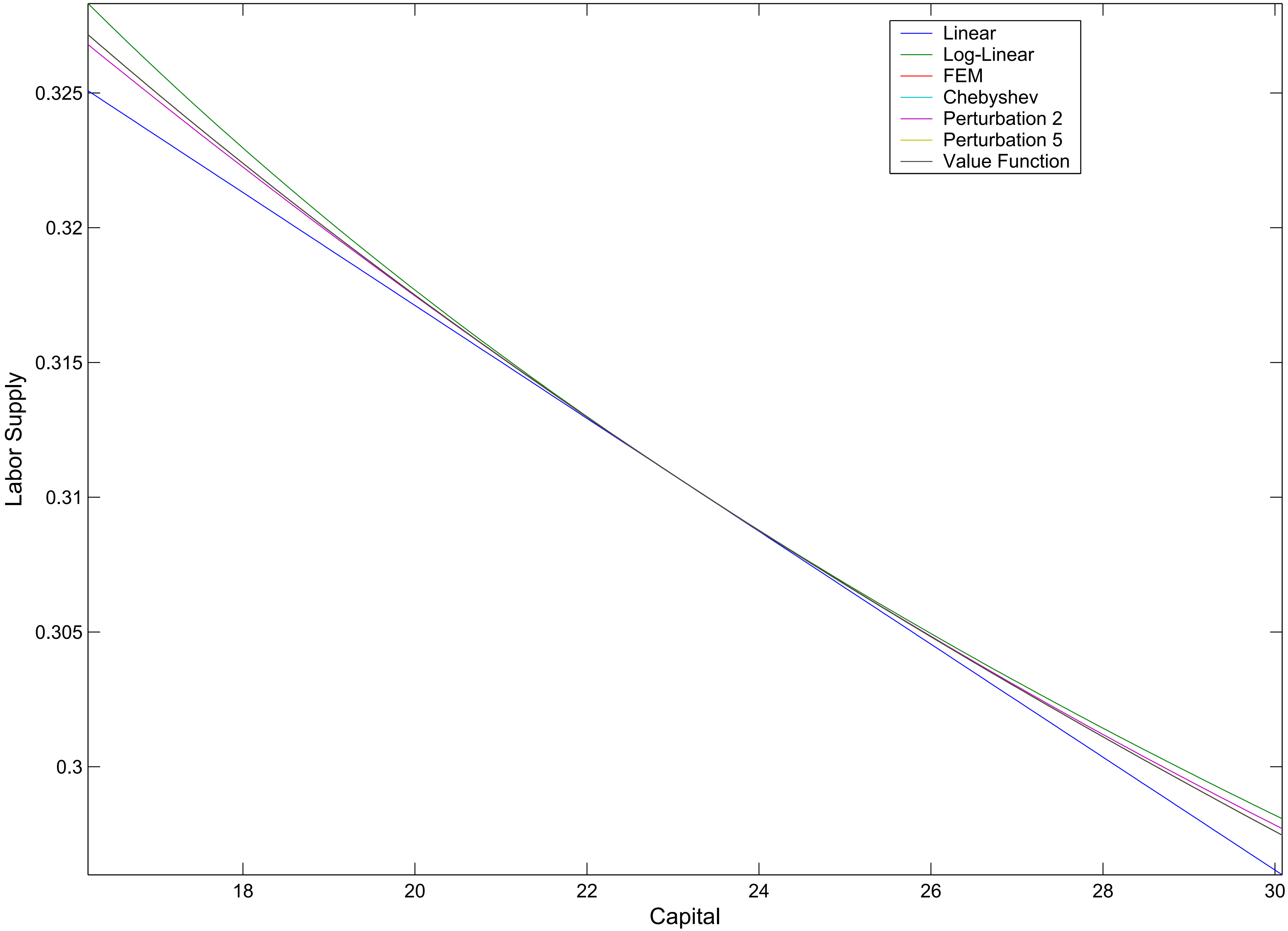


Figure 5.1.2: Investment at z = 0, τ = 2 / σ = 0.007

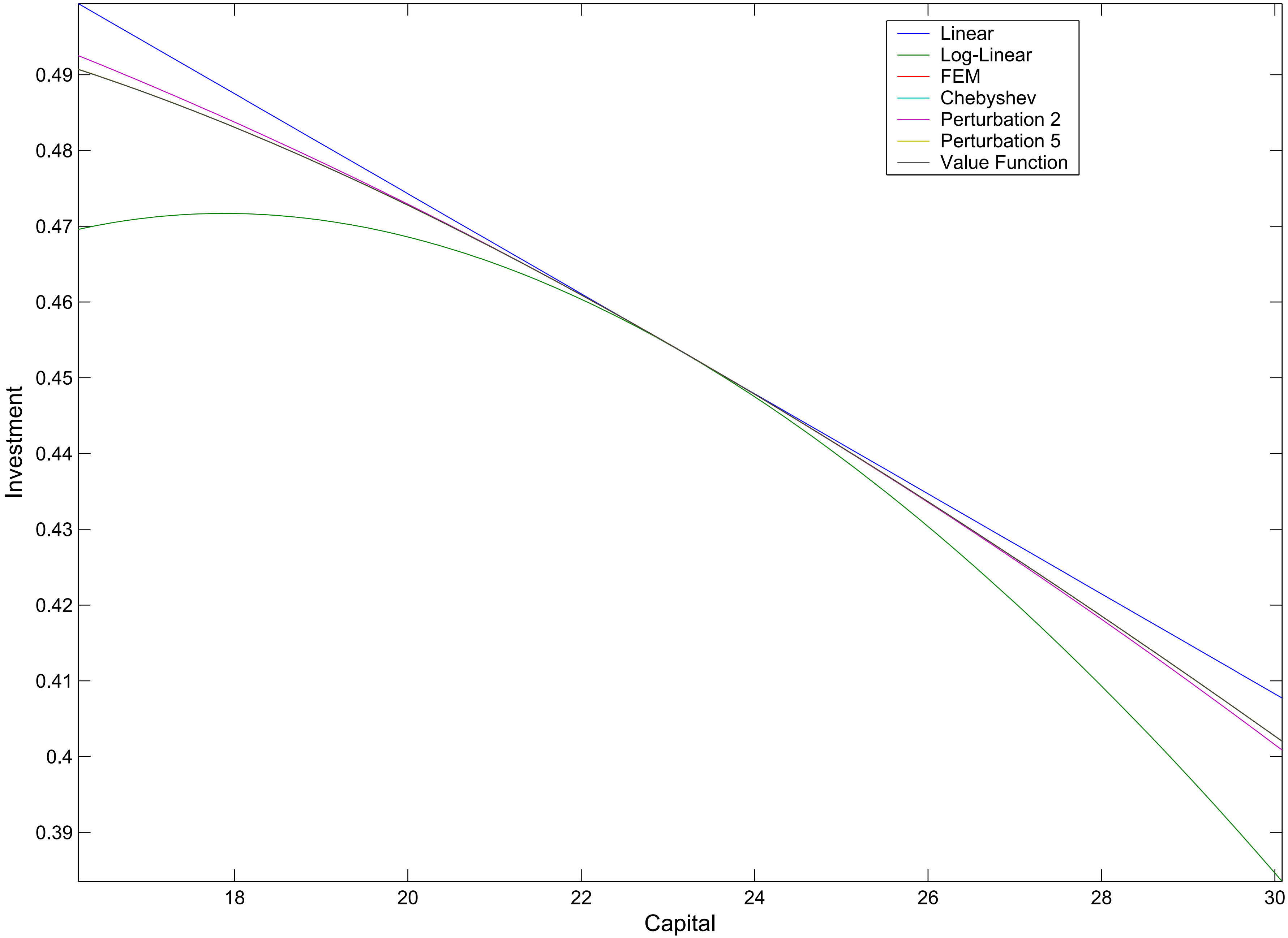


Figure 5.1.3 : Labor Supply at z = 0, τ = 50 / σ = 0.035

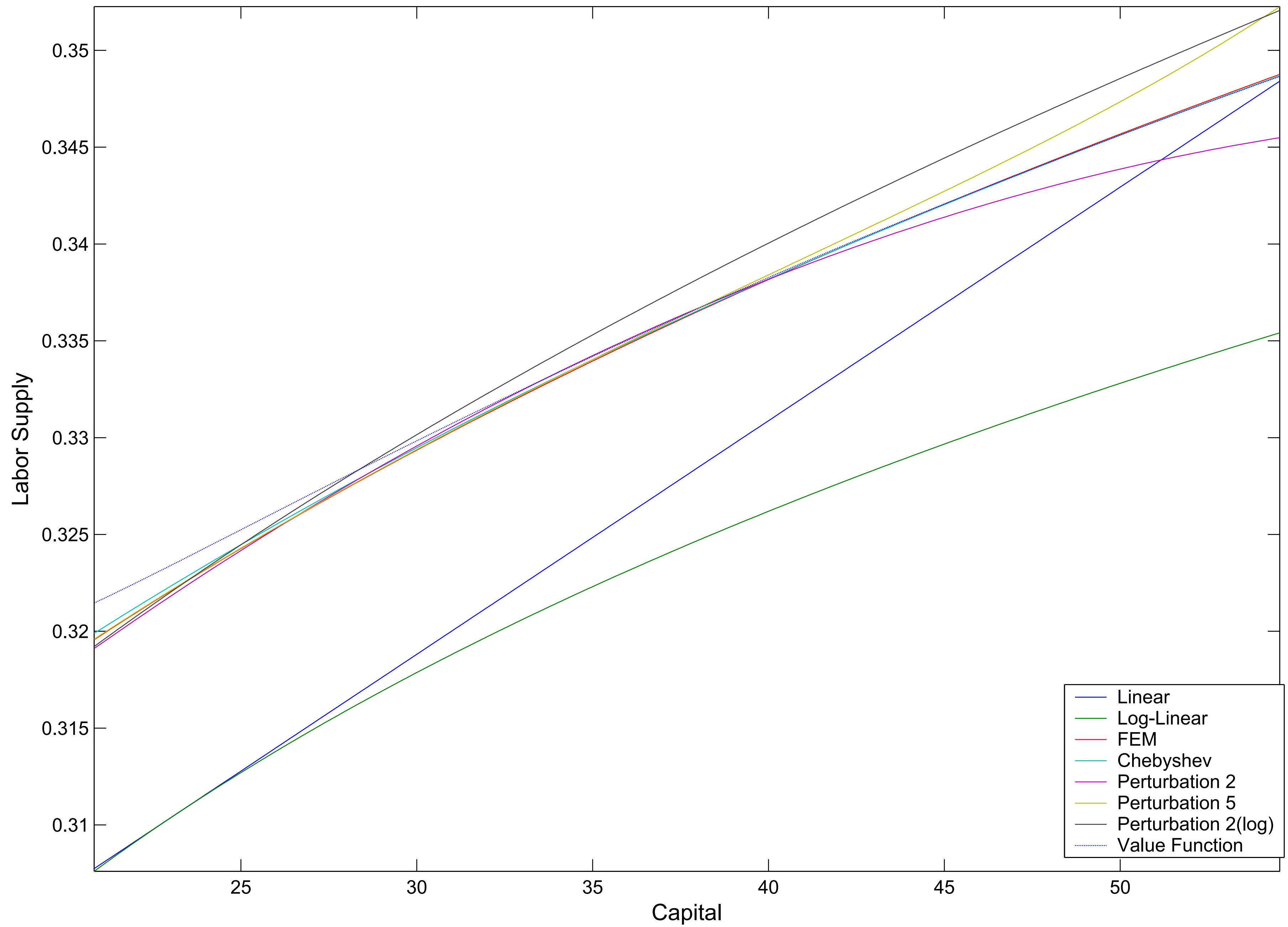


Figure 5.1.4 : Investment at z = 0, τ = 50 / σ = 0.035

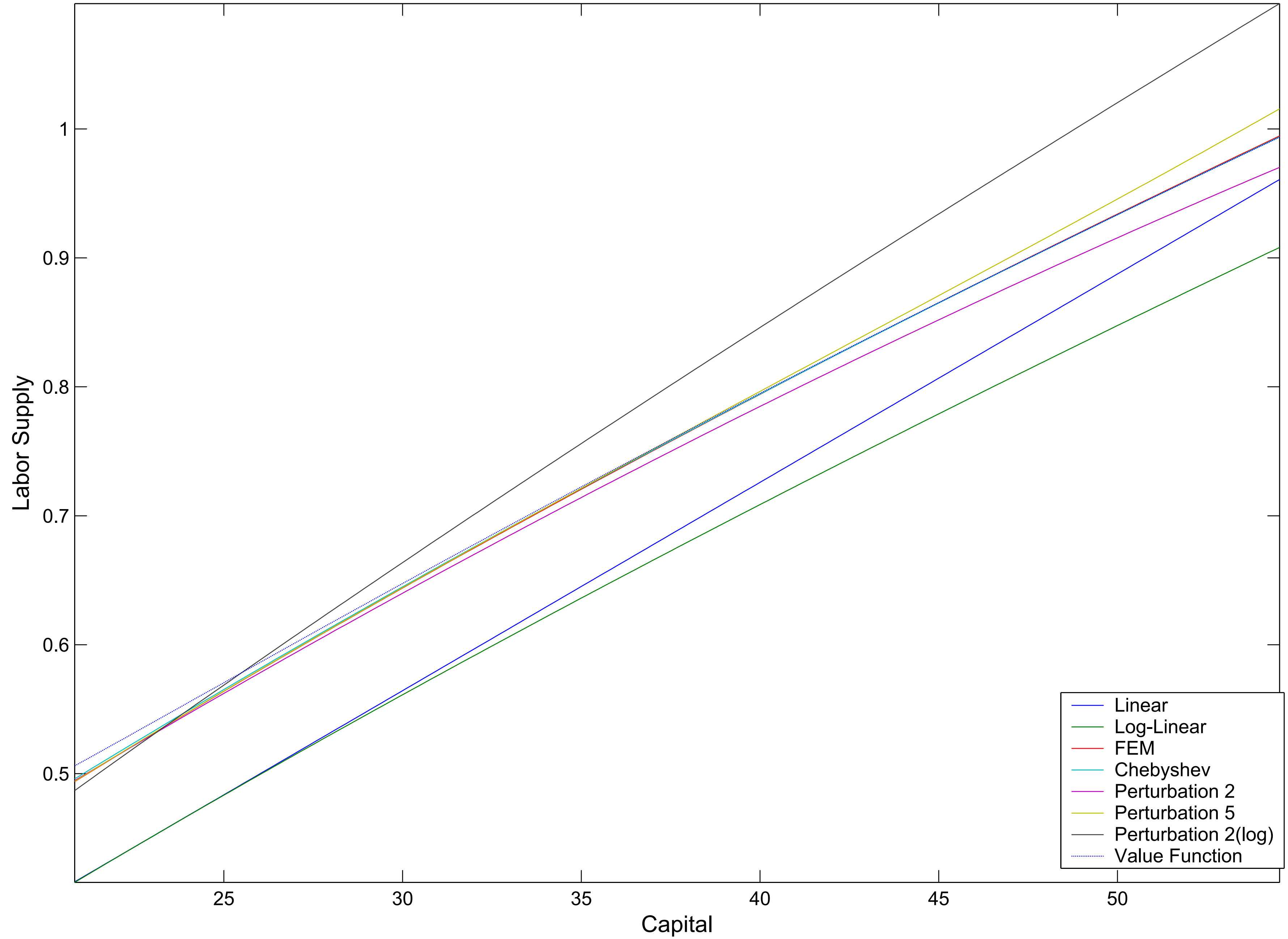


Figure 5.2.1 : Density of Output, τ = 2 / σ = 0.007

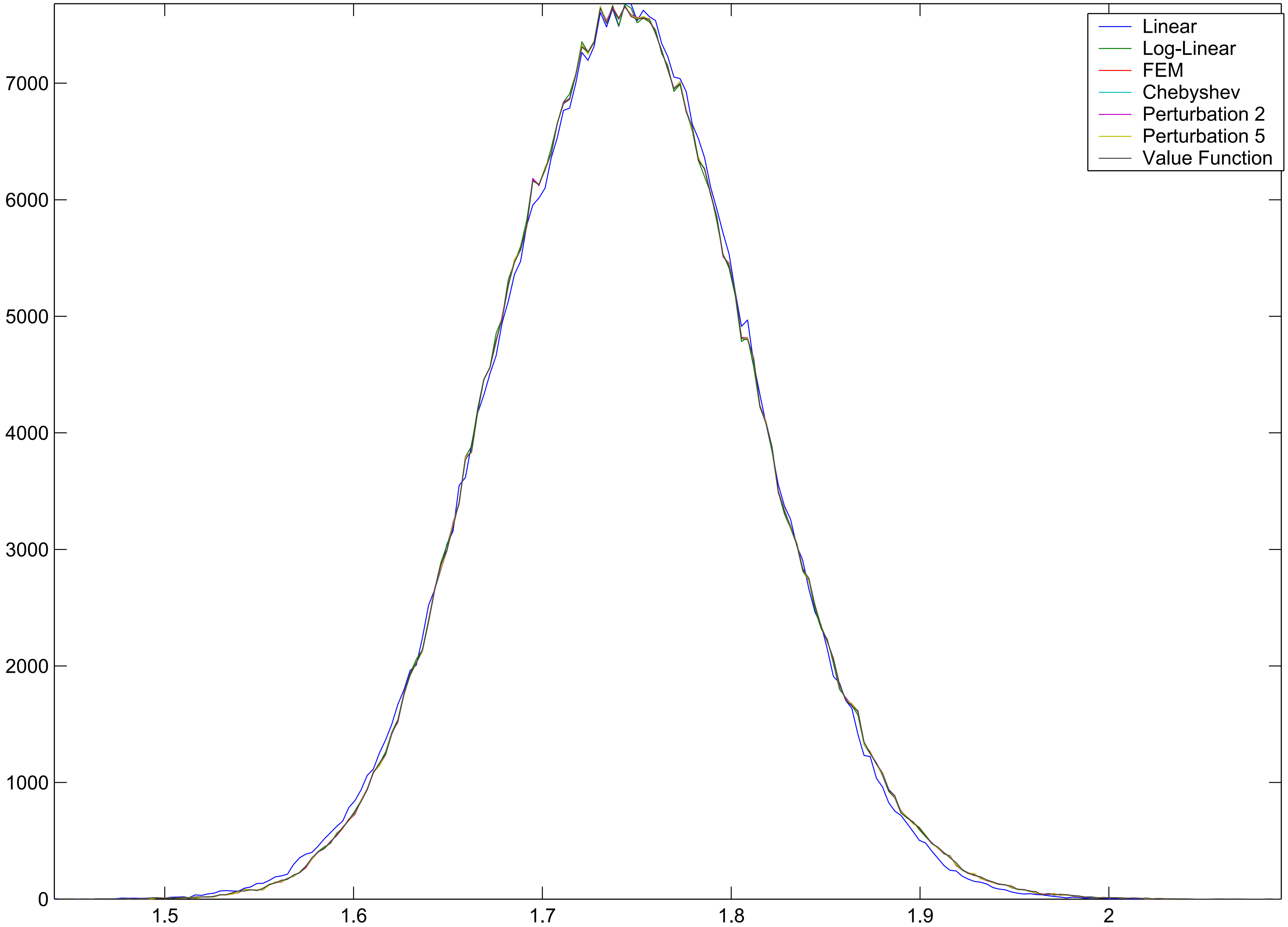


Figure 5.2.2 : Density of Capital, $\tau = 2$ / $\sigma = 0.007$

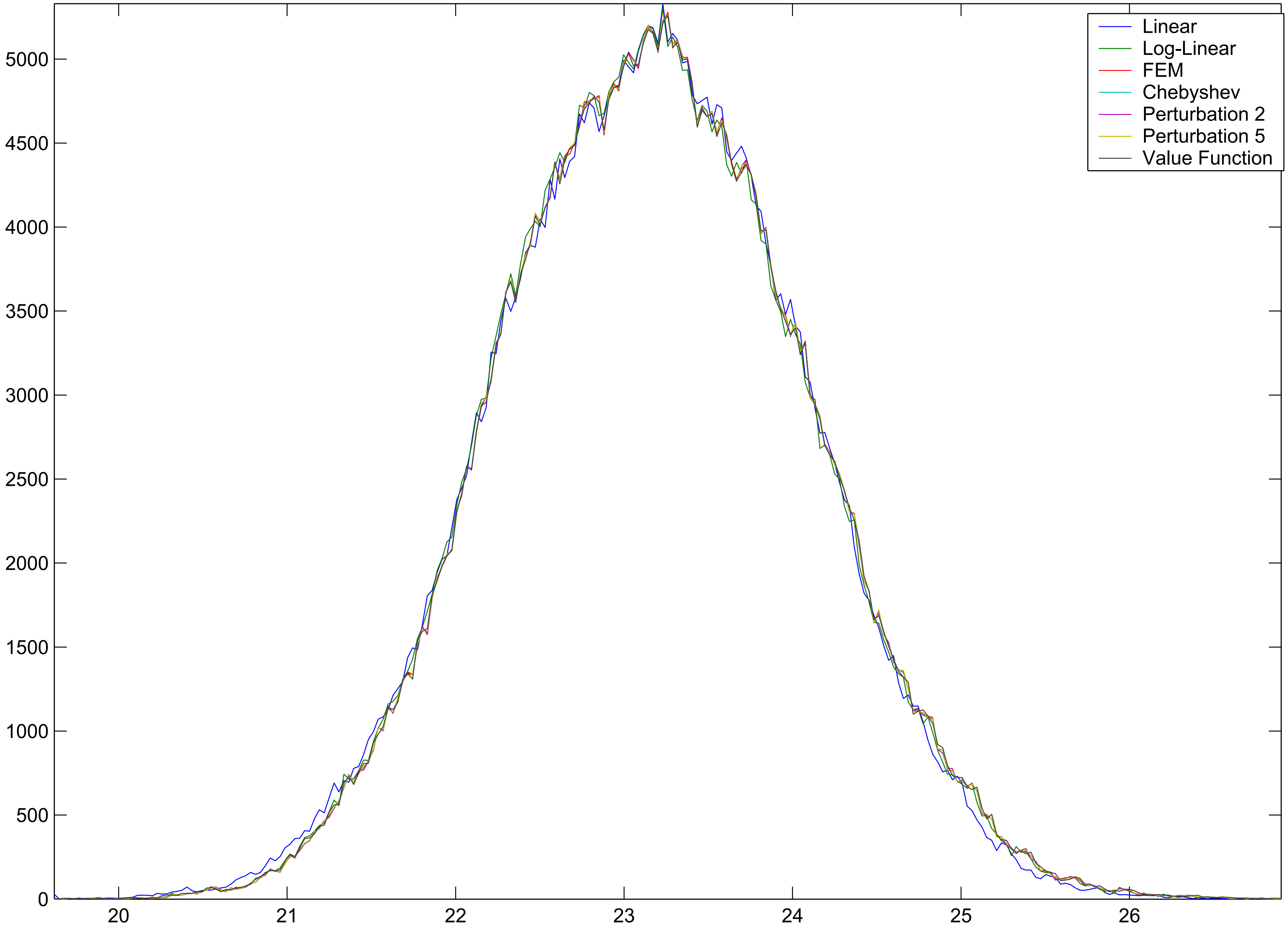


Figure 5.2.3 : Density of Labor, $\tau = 2$ / $\sigma = 0.007$

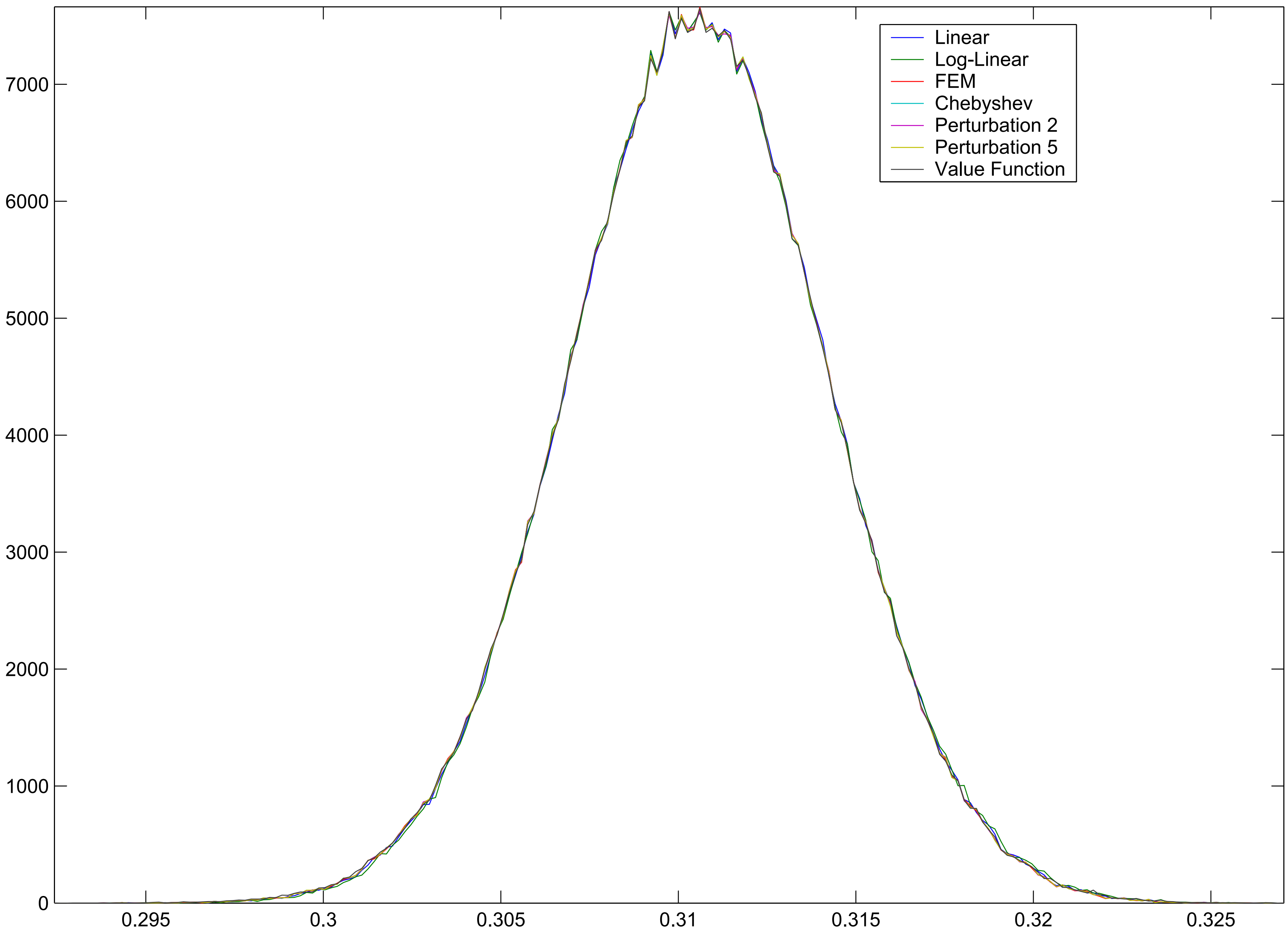


Figure 5.2.4 : Density of Consumption, τ = 2 / σ = 0.007

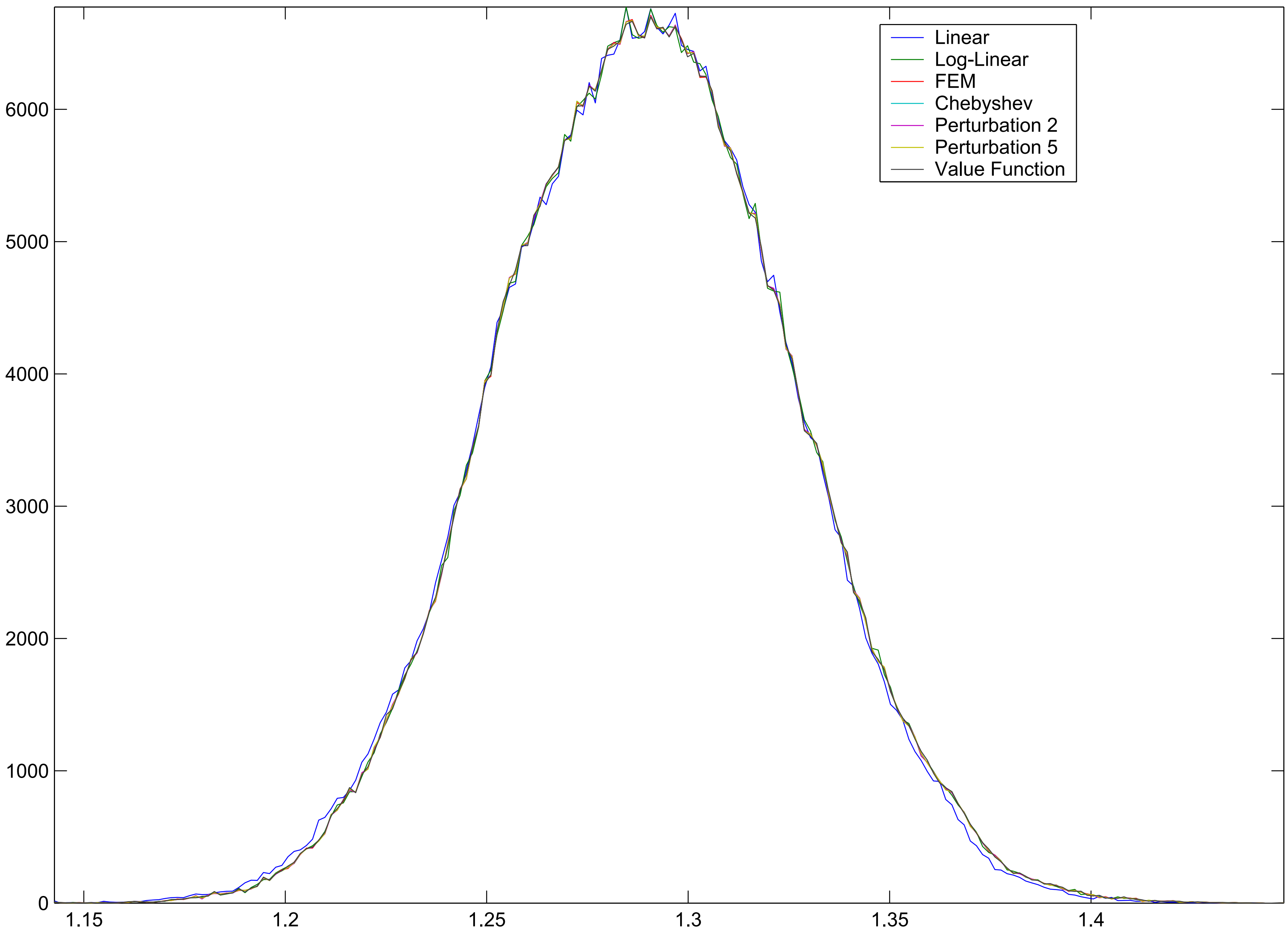


Figure 5.2.5: Time Series for Output, $\tau = 2$ / $\sigma = 0.007$

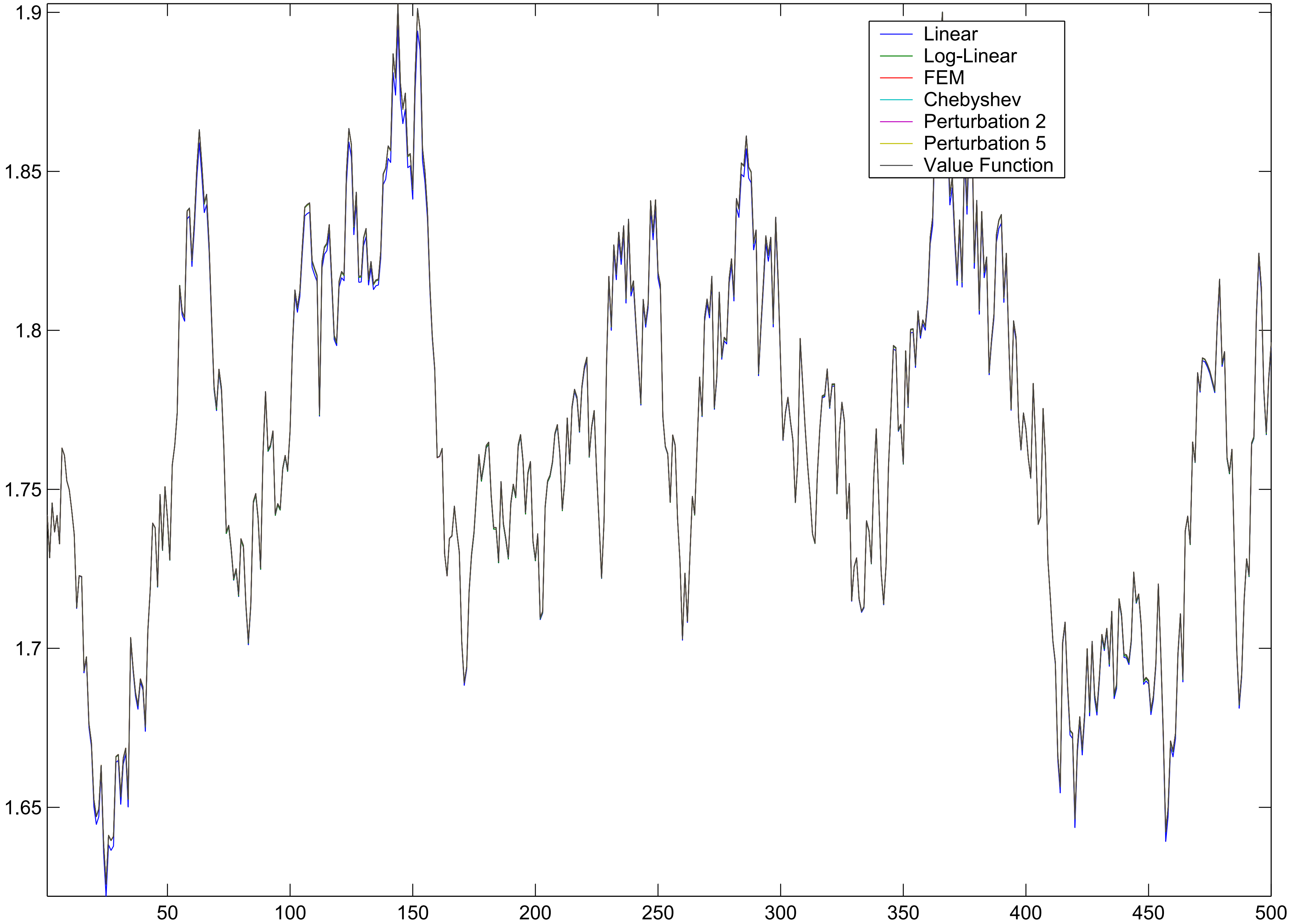


Figure 5.2.6: Time Series for Capital, $\tau = 2$ / $\sigma = 0.007$

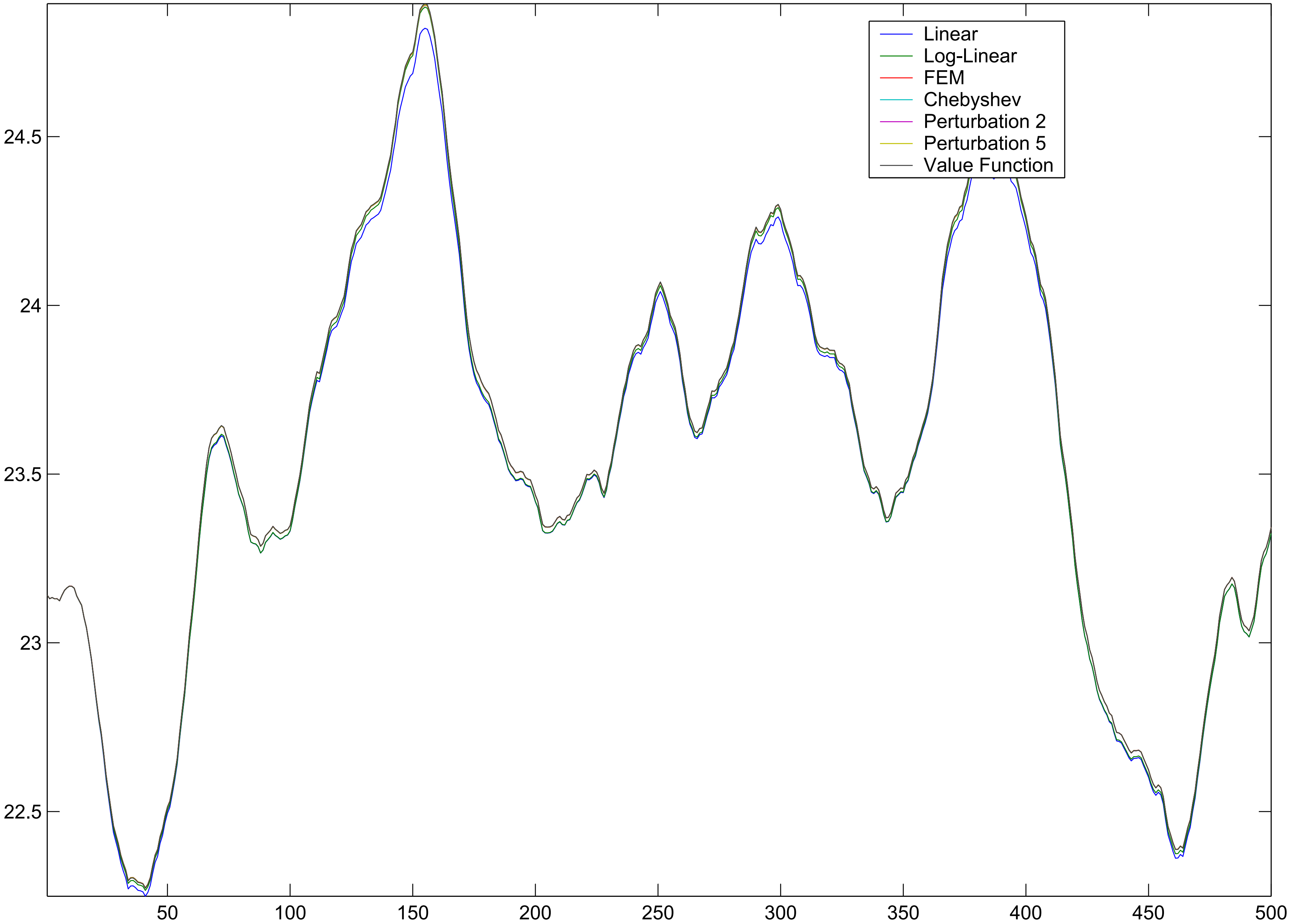


Figure 5.2.7 : Density of Output, τ = 50 / σ = 0.035

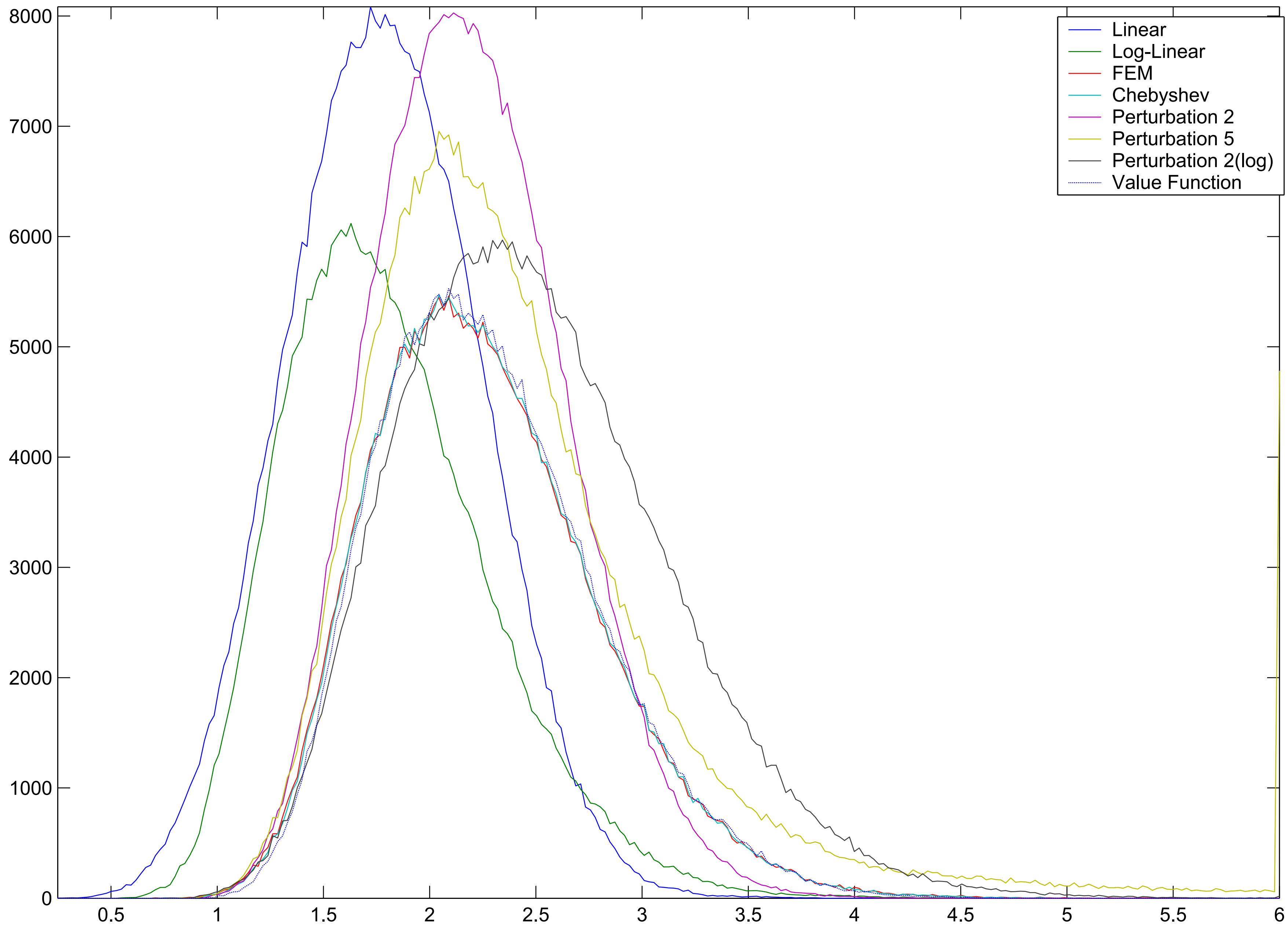


Figure 5.2.8 : Density of Capital, τ = 50 / σ = 0.035

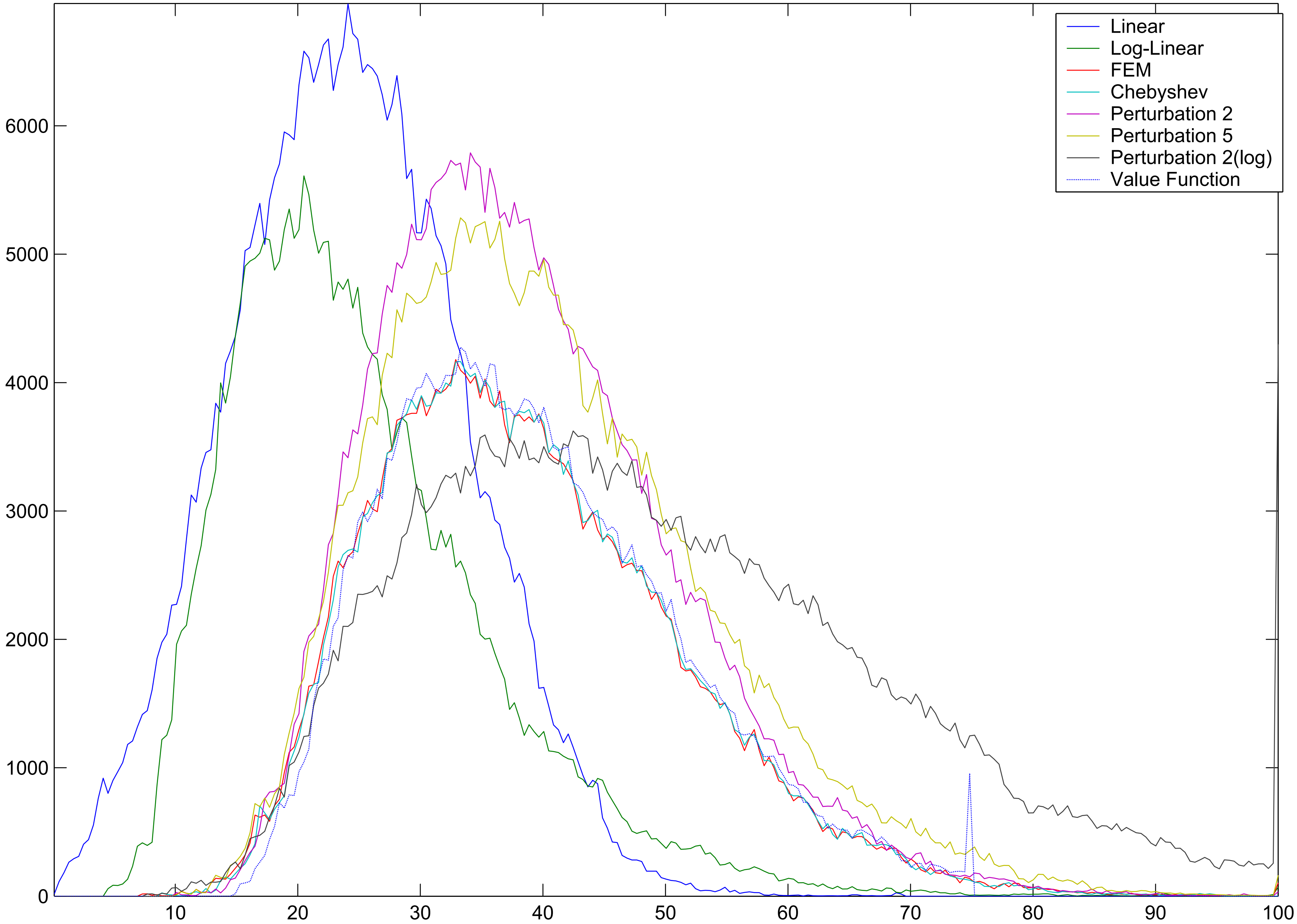


Figure 5.2.9 : Density of Labor, τ = 50 / σ = 0.035

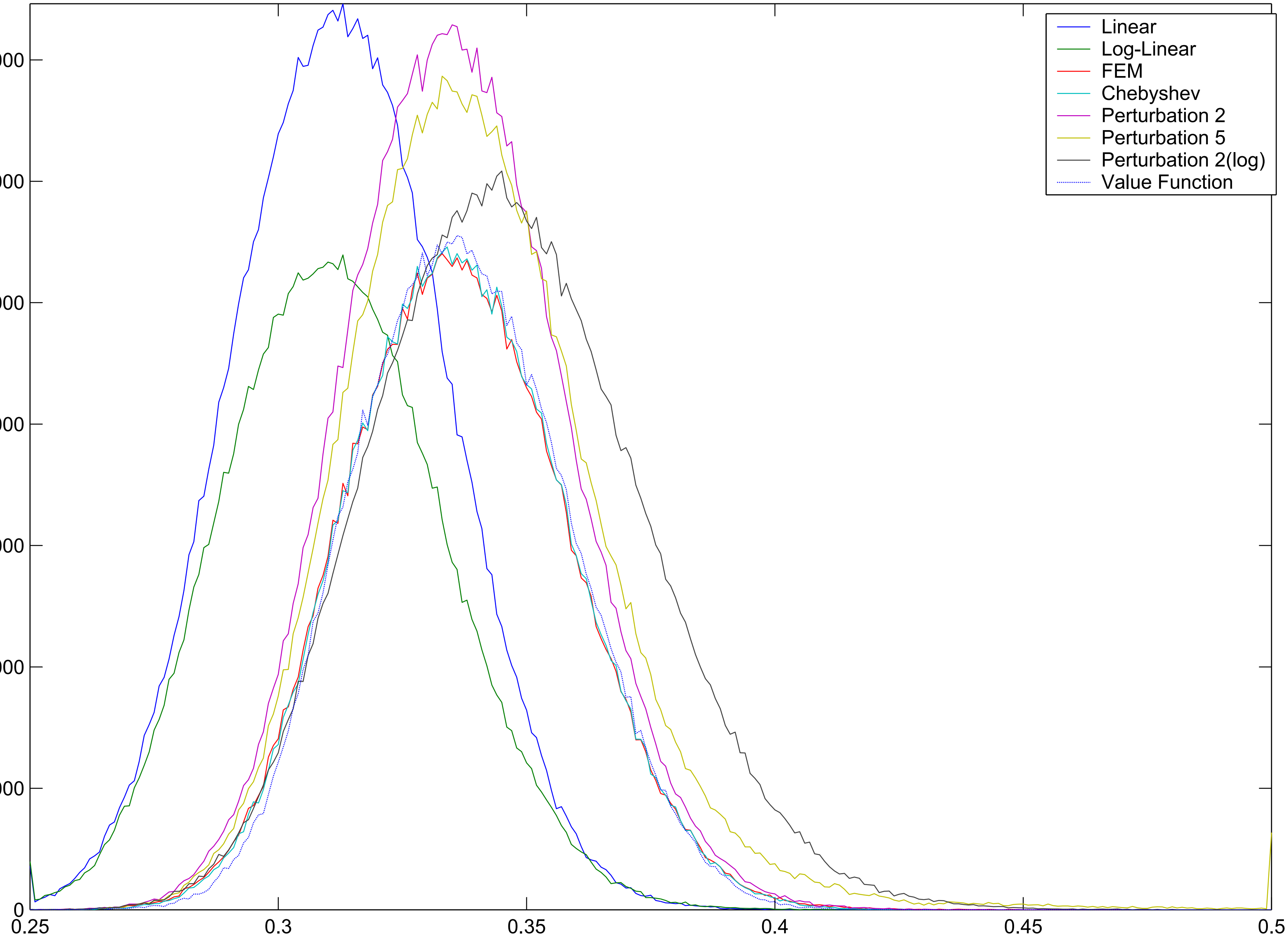


Figure 5.2.10 : Density of Consumption, τ = 50 / σ = 0.035

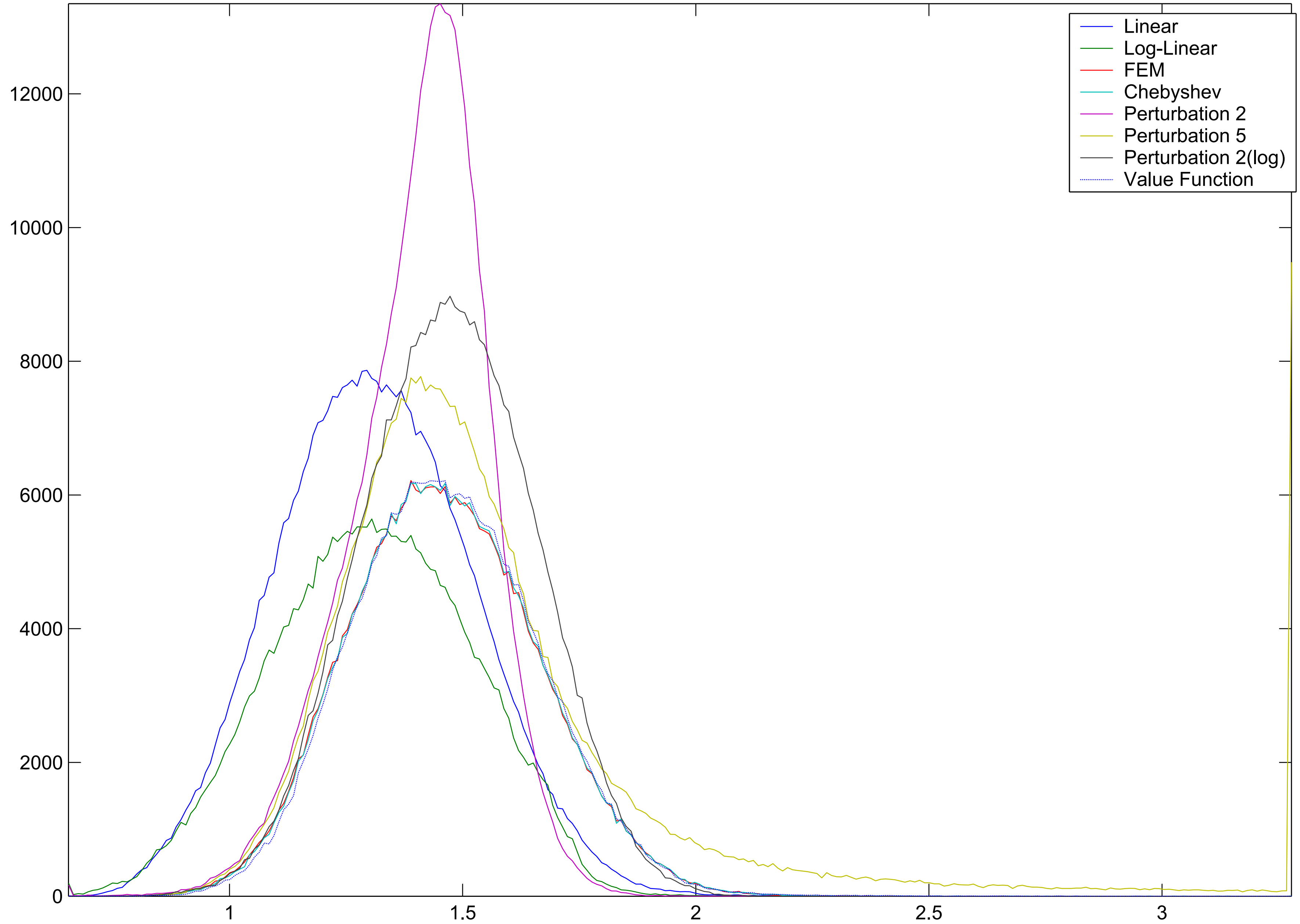


Figure 5.2.11 : Time Series for Output, τ = 50 / σ = 0.035

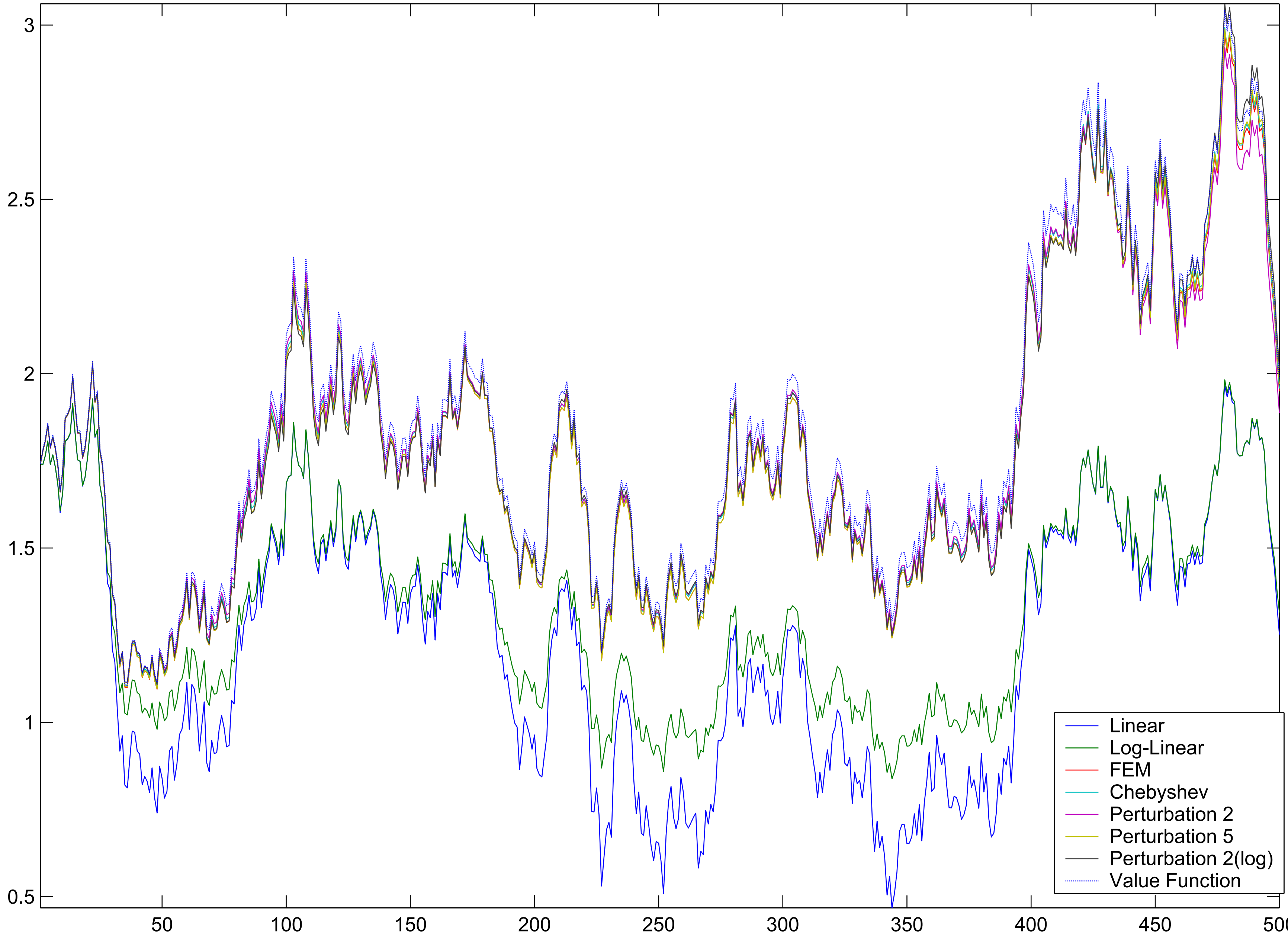


Figure 5.2.12 : Time Series for Capital, τ = 50 / σ = 0.035

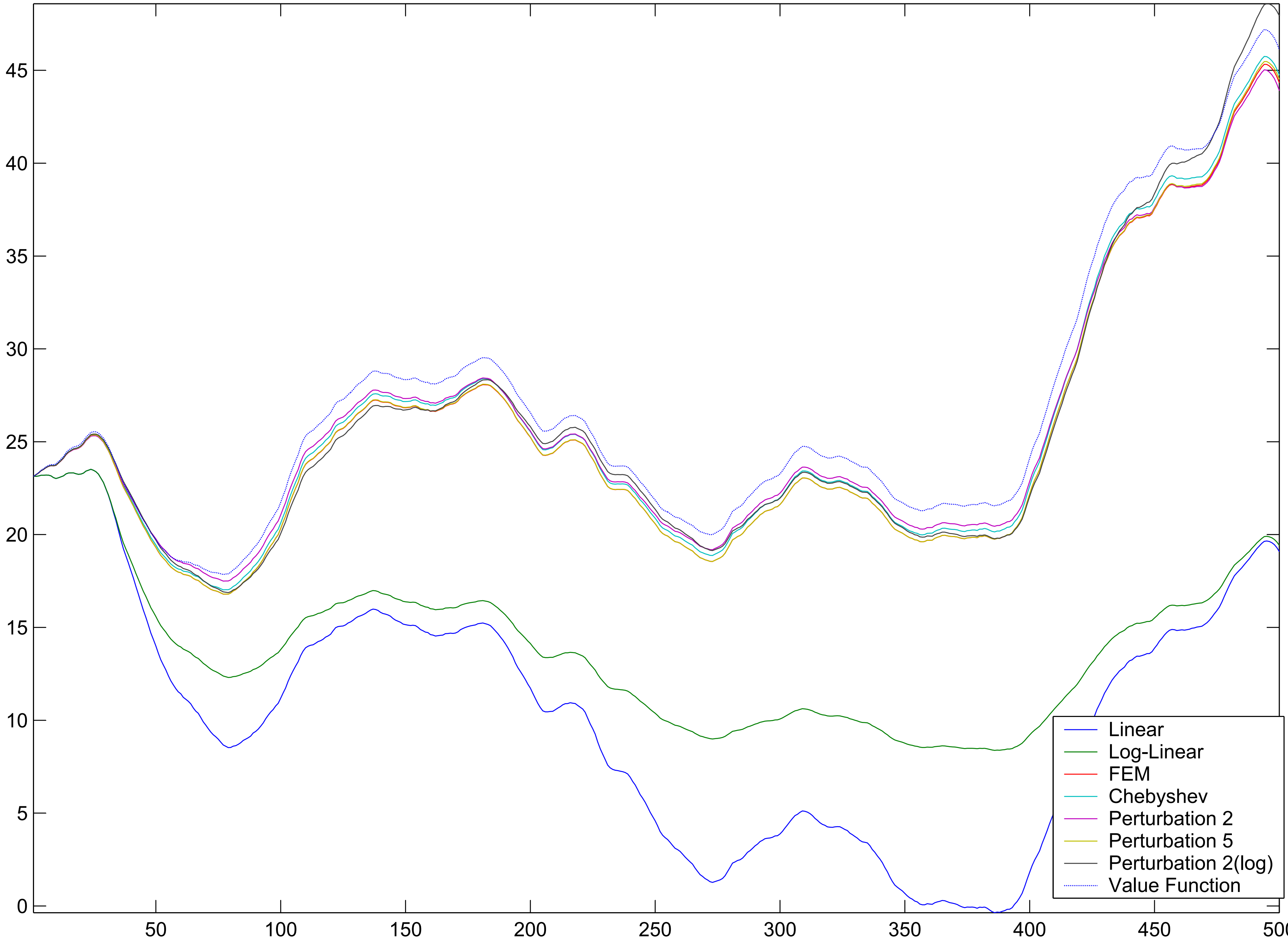


Figure 5.3.1 : Empirical CDF of den Haan Marcet Tests, $\tau = 50$ / $\sigma = 0.035$

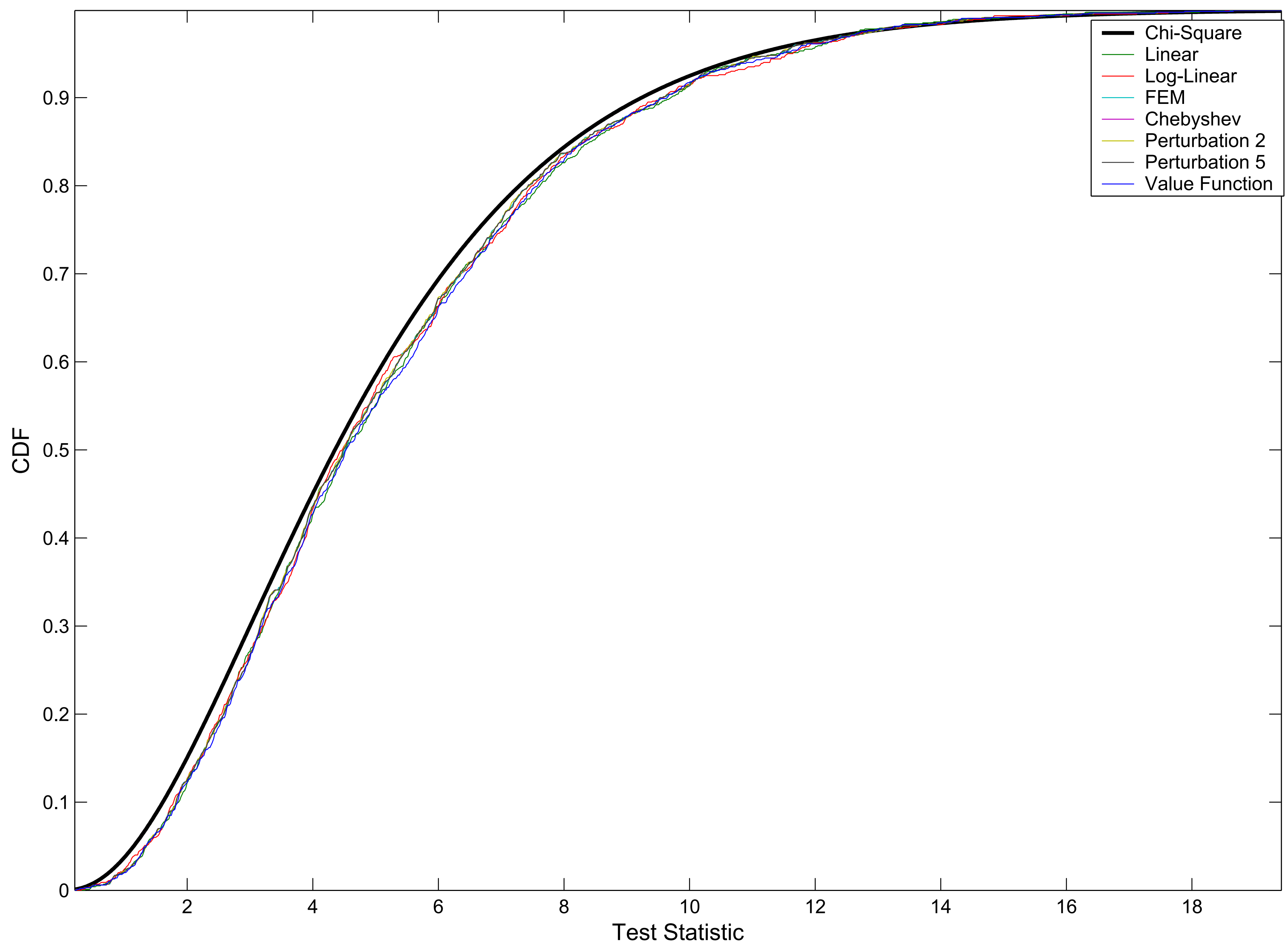


Figure 5.3.2 : Empirical CDF of den Haan Marcet Tests, τ = 50 / σ = 0.035

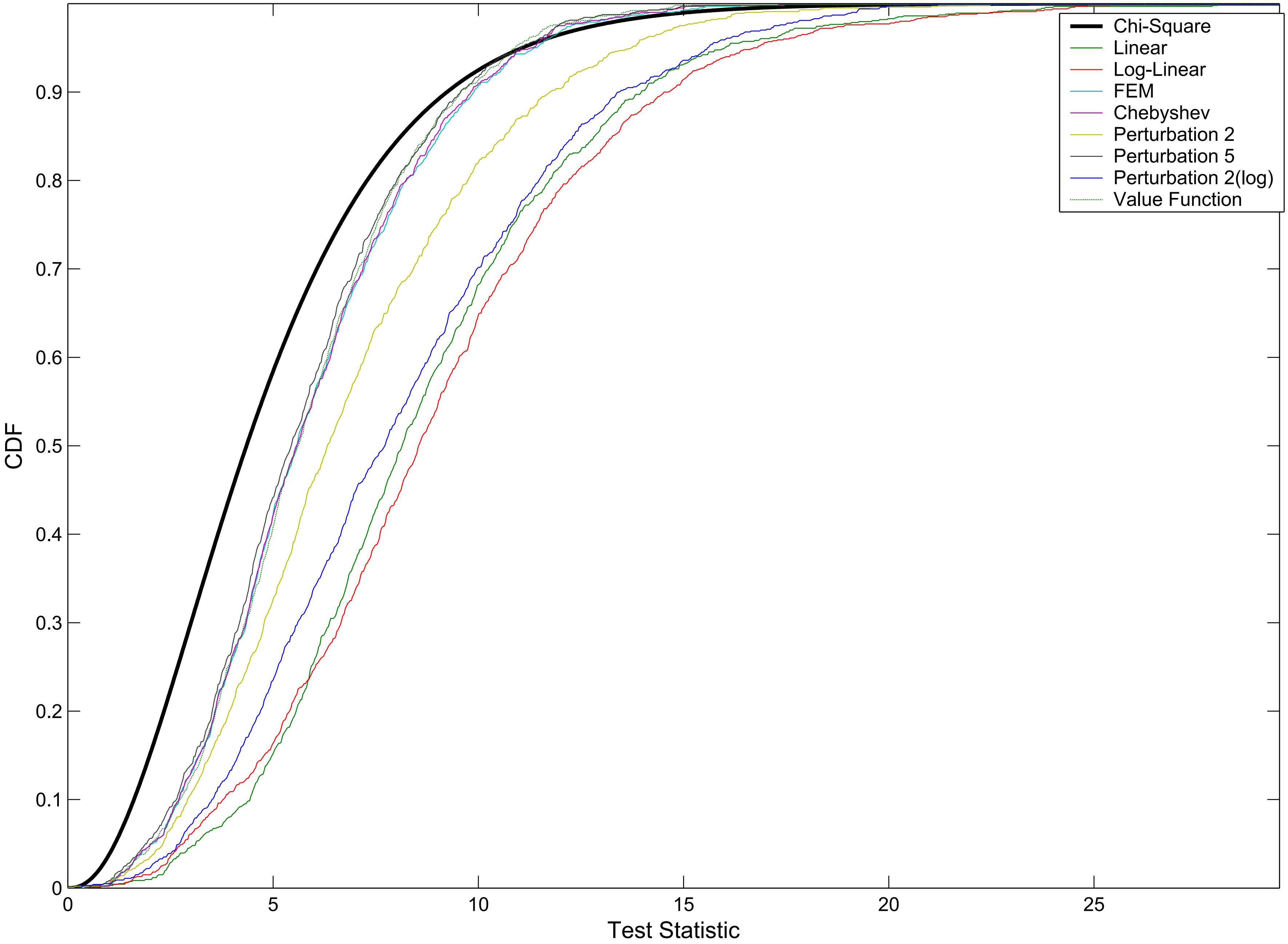


Figure 5.4.1: Euler Equation Errors, Linear Approximation, $\tau = 2 / \sigma = 0.007$

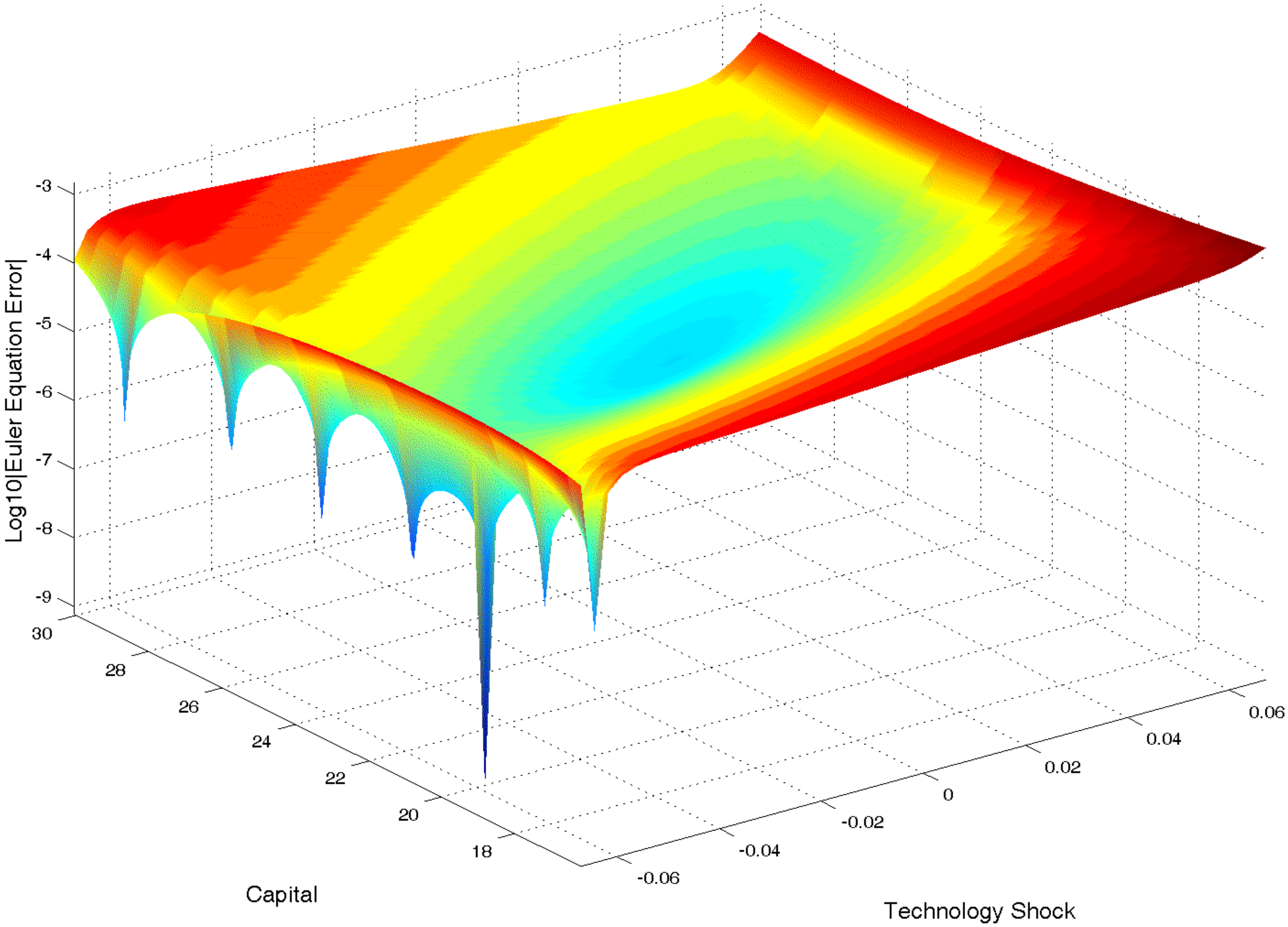


Figure 5.4.2: Euler Equation Errors, Log-Linear Approximation, $\tau = 2$ / $\sigma = 0.007$

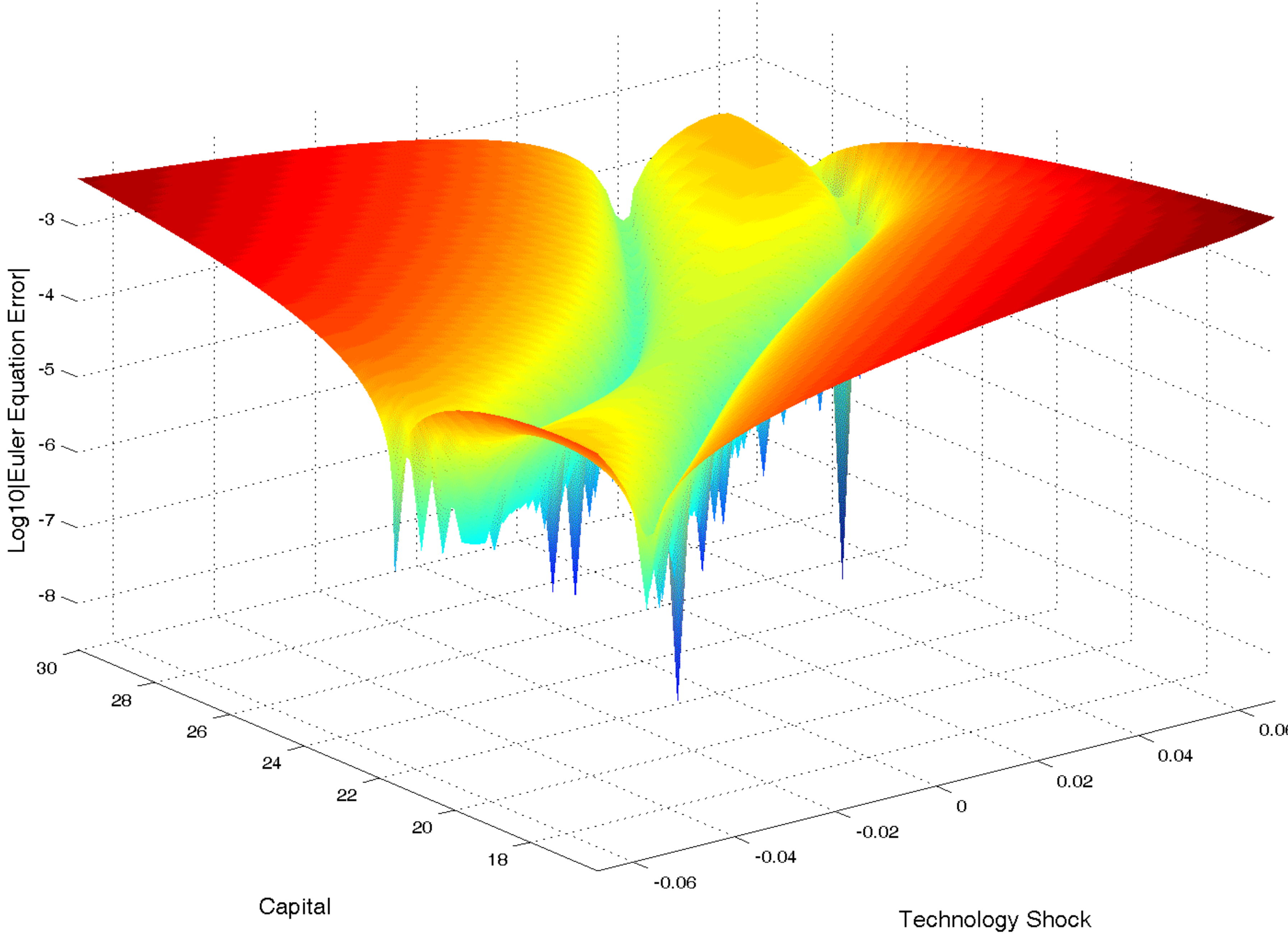


Figure 5.4.3 : Euler Equation Errors, Finite Elements Approximation, $\tau = 2$ / $\sigma = 0.007$

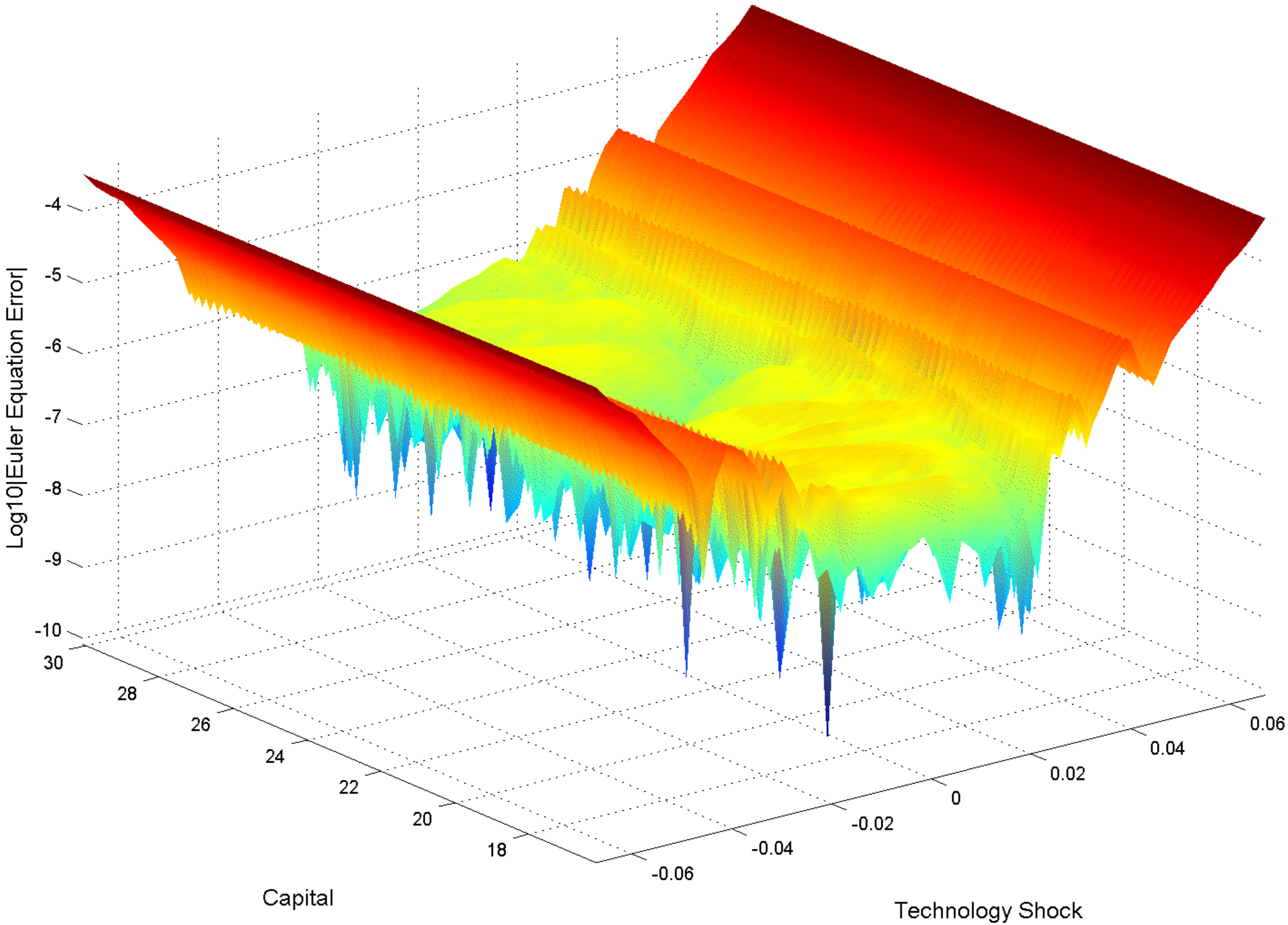


Figure 5.4.4 : Euler Equation Errors, Chebyshev Approximation, $\tau = 2 / \sigma = 0.007$

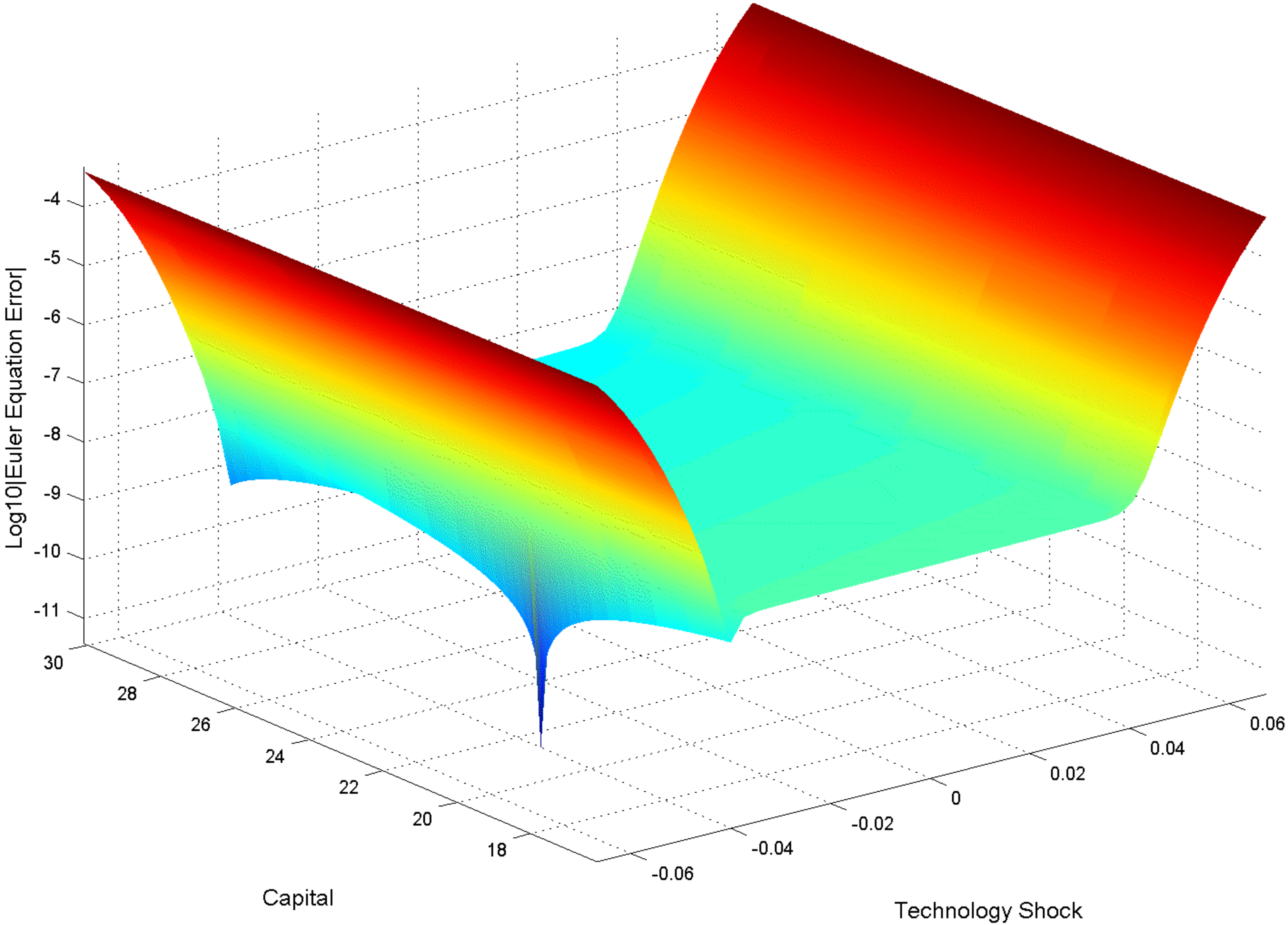


Figure 5.4.5: Euler Equation Errors, 2nd Order Perturbation Approximation, $\tau = 2$ / $\sigma = 0.007$

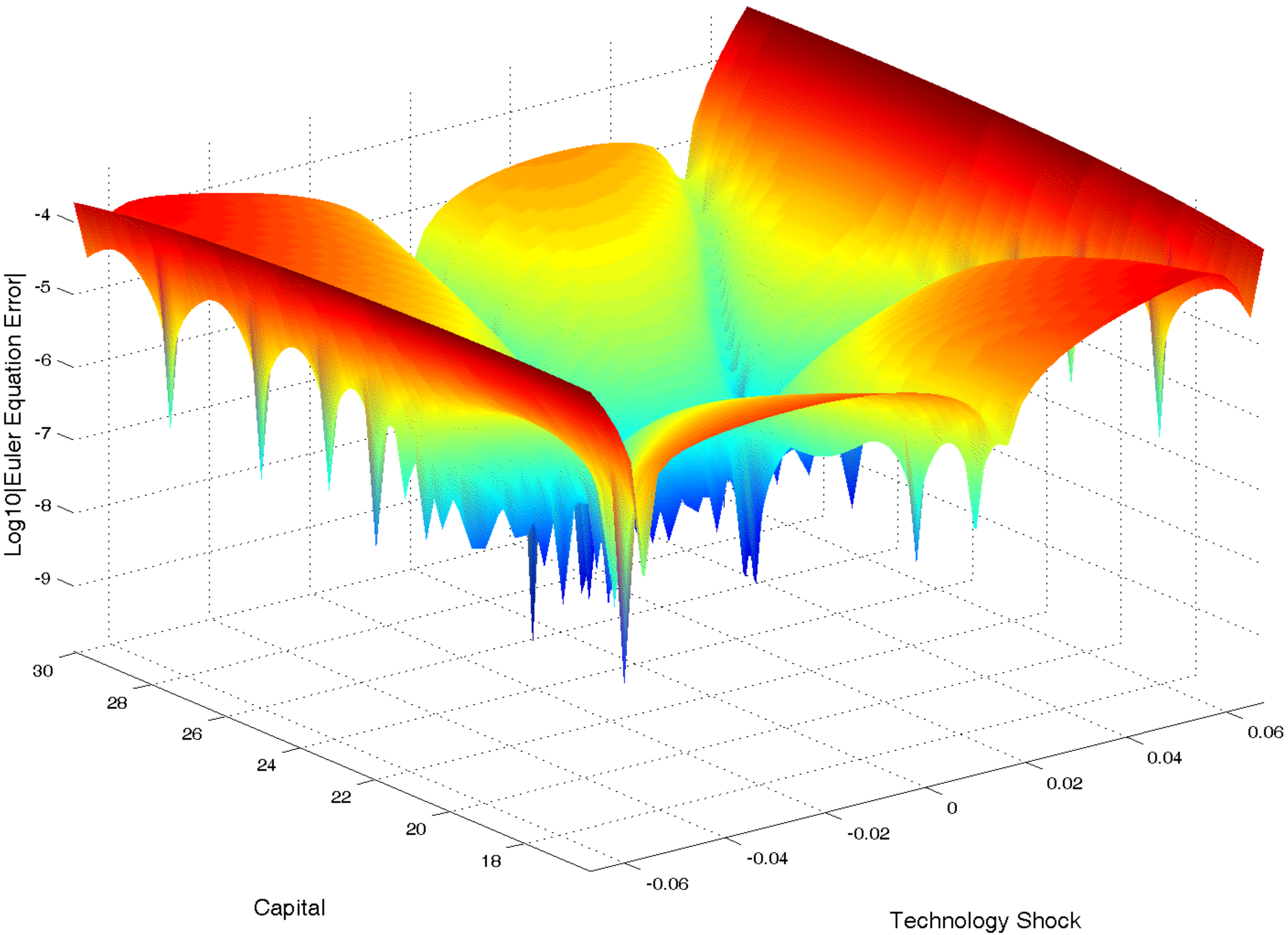


Figure 5.4.6: Euler Equation Errors, 5th Other Perturbation Approximation, $\tau = 2 / \sigma = 0.007$

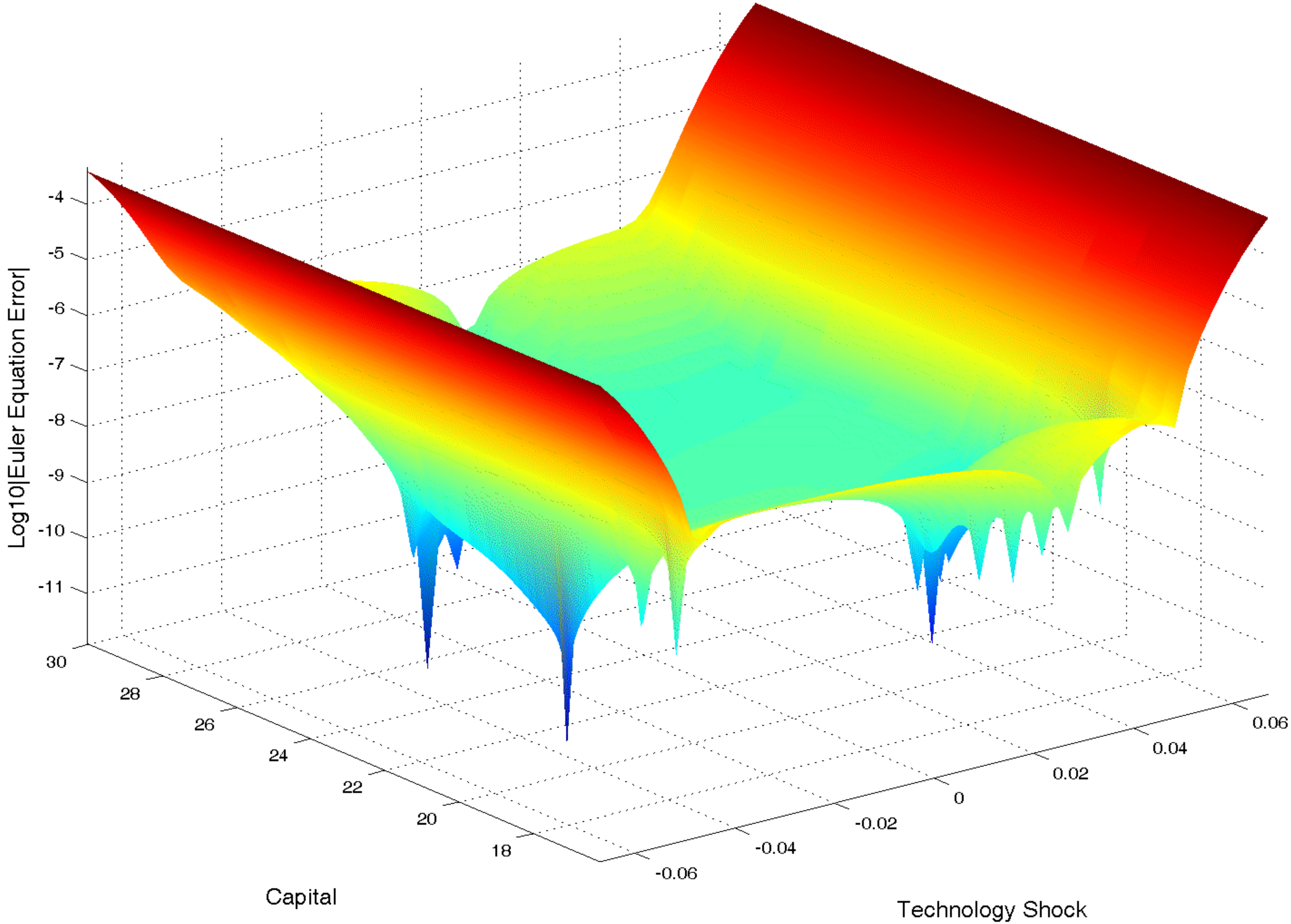


Figure 5.4.7 : Euler Equation Errors, Value Function Approximation, $\tau = 2$ / $\sigma = 0.007$

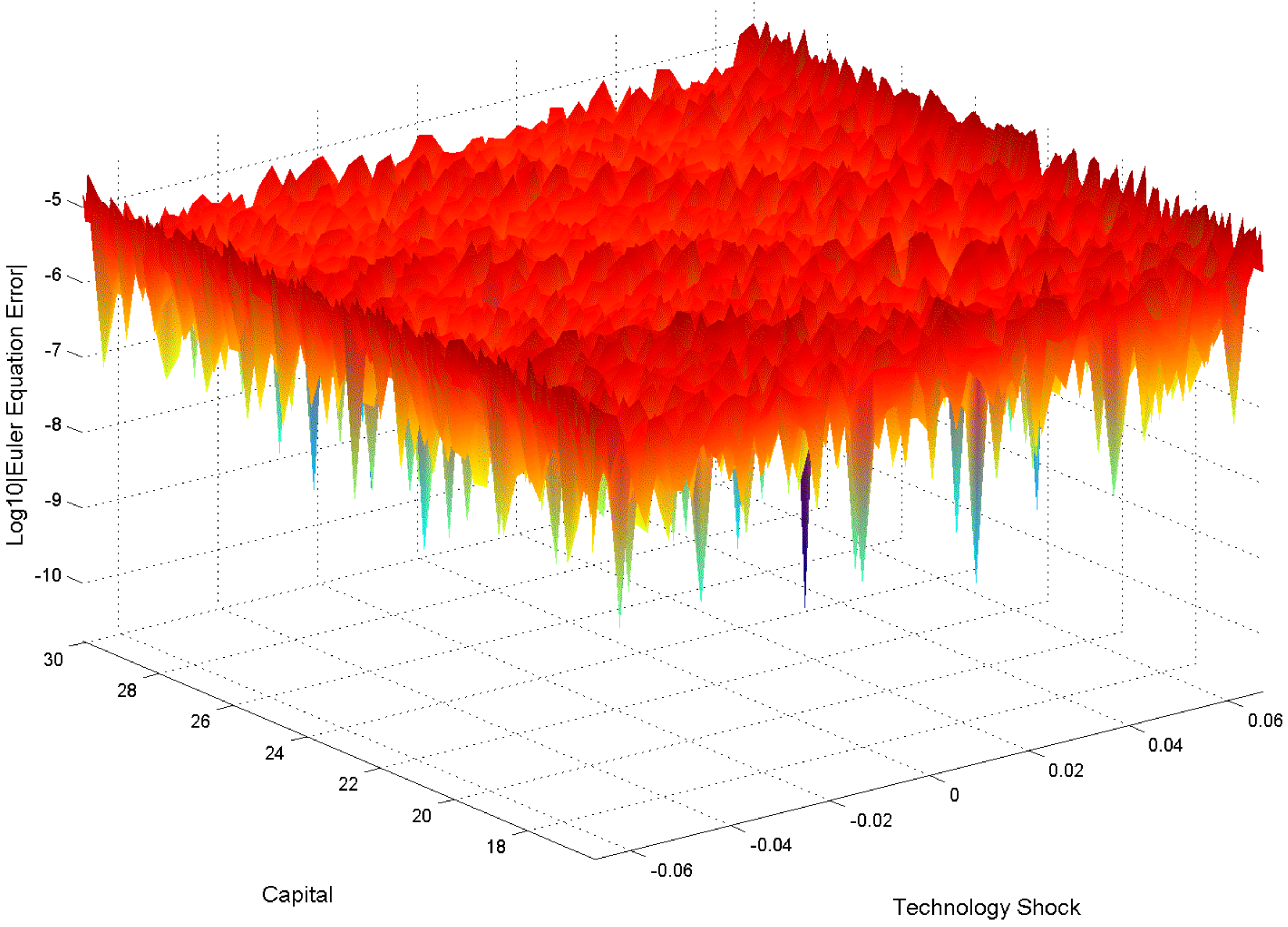


Figure 5.4.8 : Euler Equation Errors at z = 0, τ = 2 / σ = 0.007

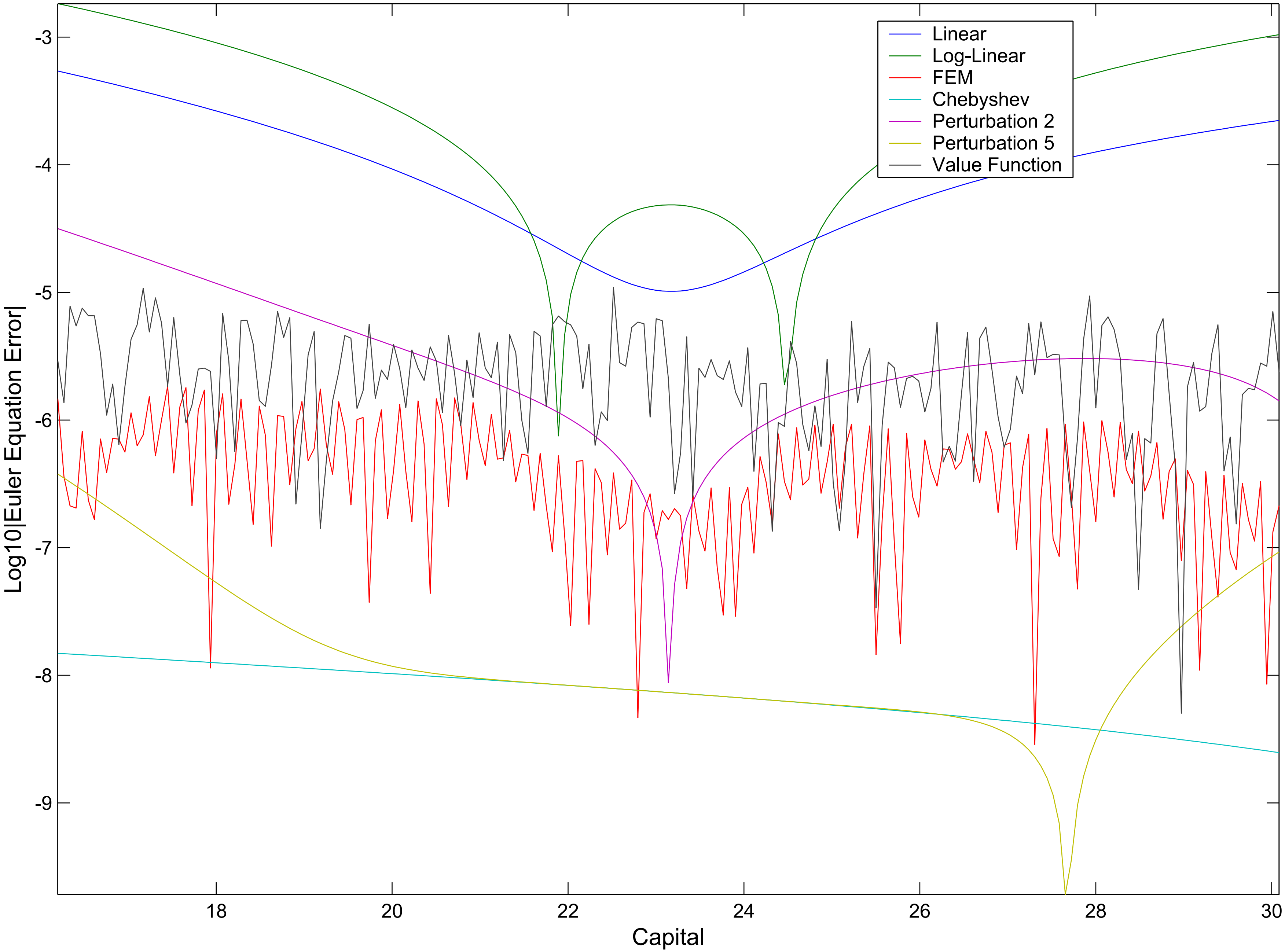
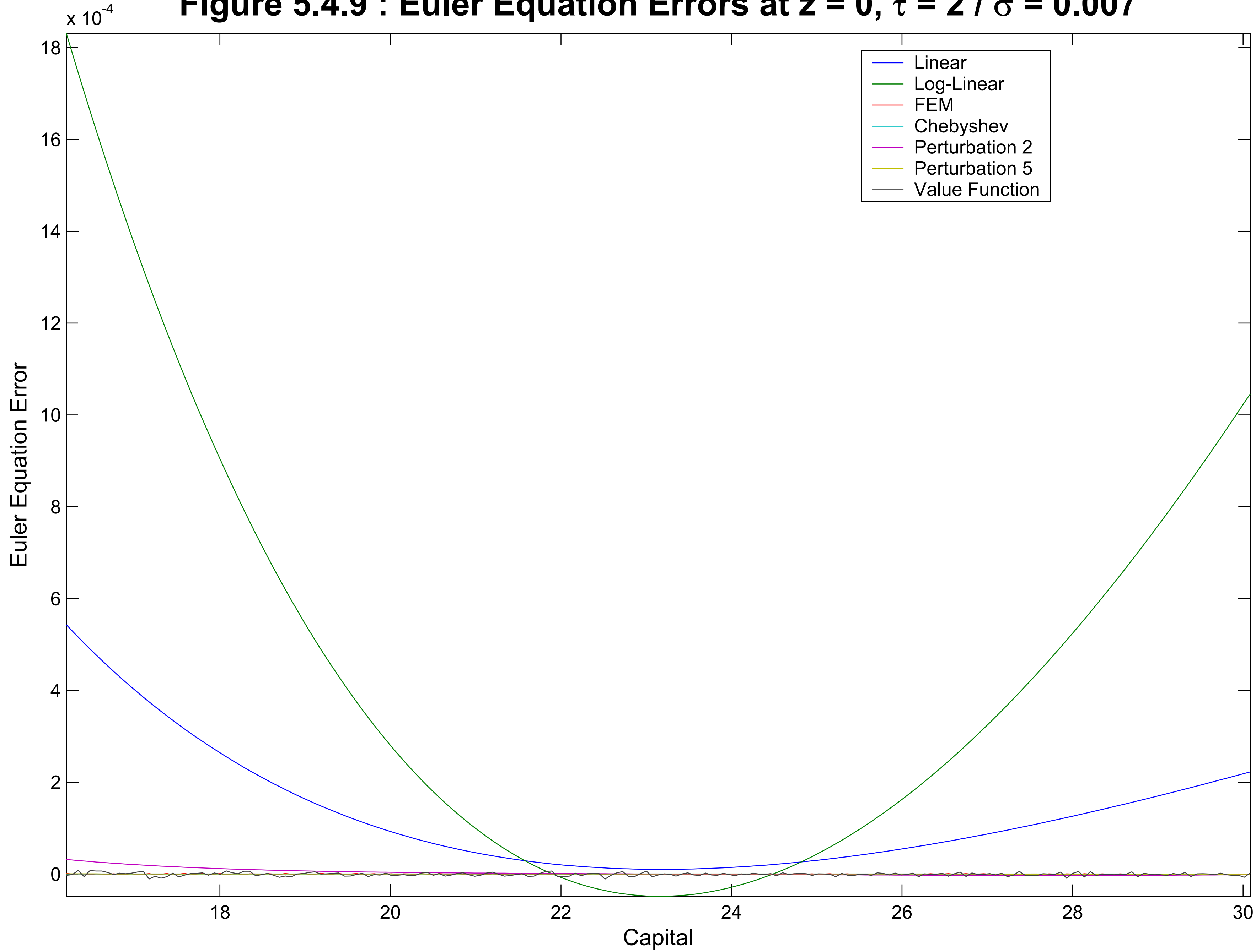


Figure 5.4.9 : Euler Equation Errors at z = 0, τ = 2 / σ = 0.007



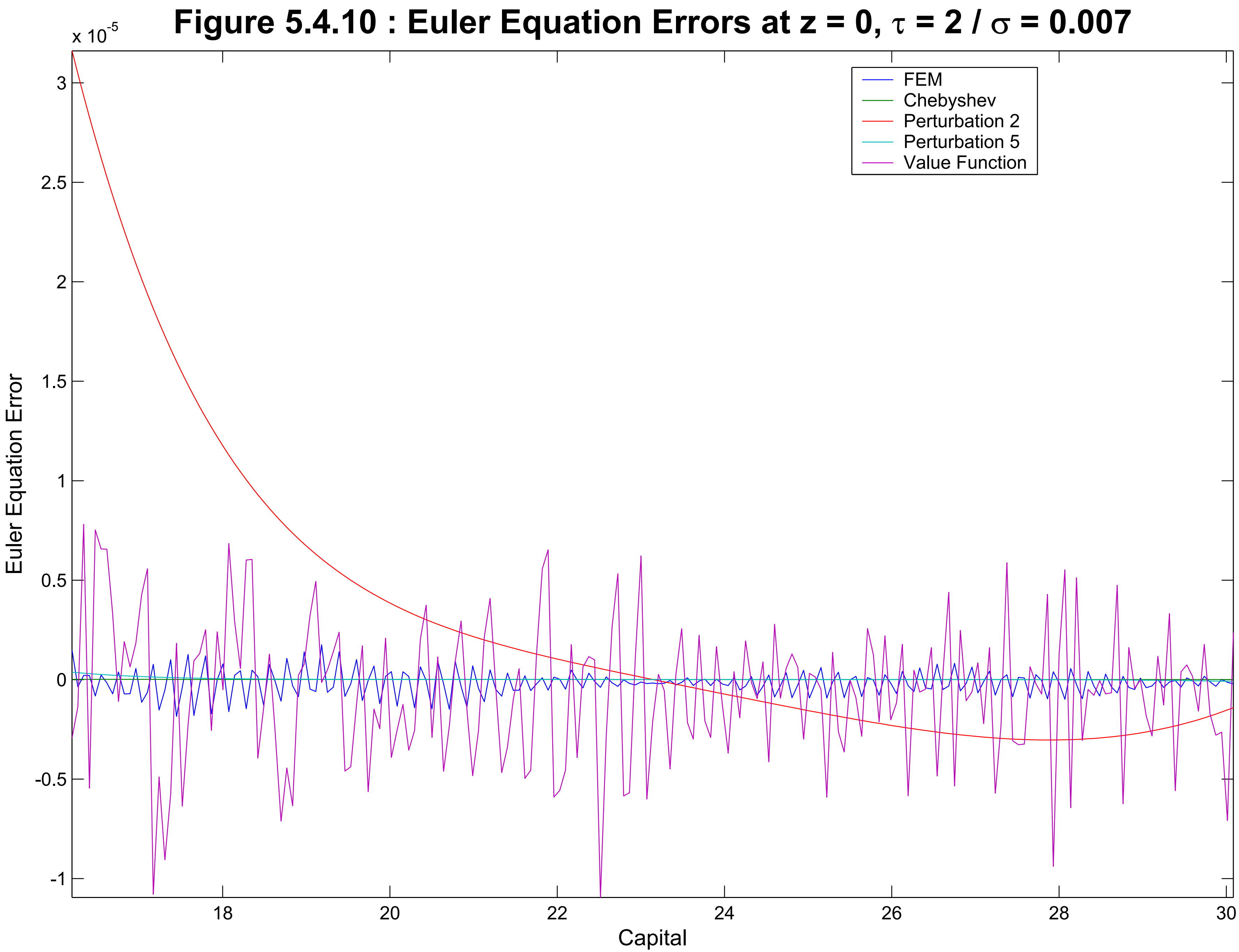


Figure 5.4.10 : Euler Equation Errors at z = 0, τ = 2 / σ = 0.007
x 10^-5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
Euler Equation Error
18
20
22
24
26
28
30
Capital
FEM
Chebyshev
Perturbation 2
Perturbation 5
Value Function

Figure 5.4.11 : Marginal Density of Capital versus Euler Errors at z=0, τ = 2 / σ = 0.007

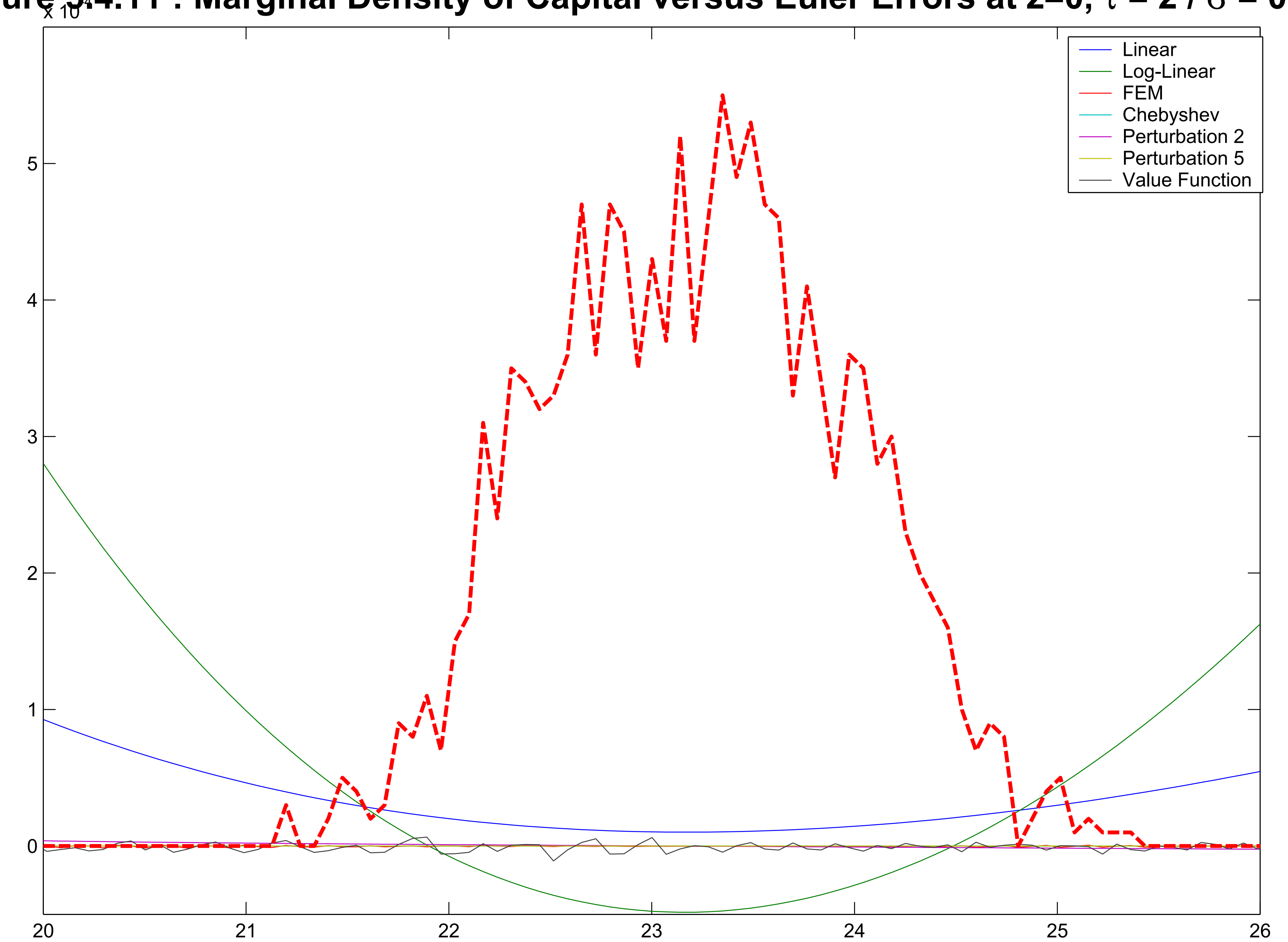


Figure 5.4.12 : Euler Equation Errors, Linear Approximation, $\tau = 50 / \sigma = 0.035$

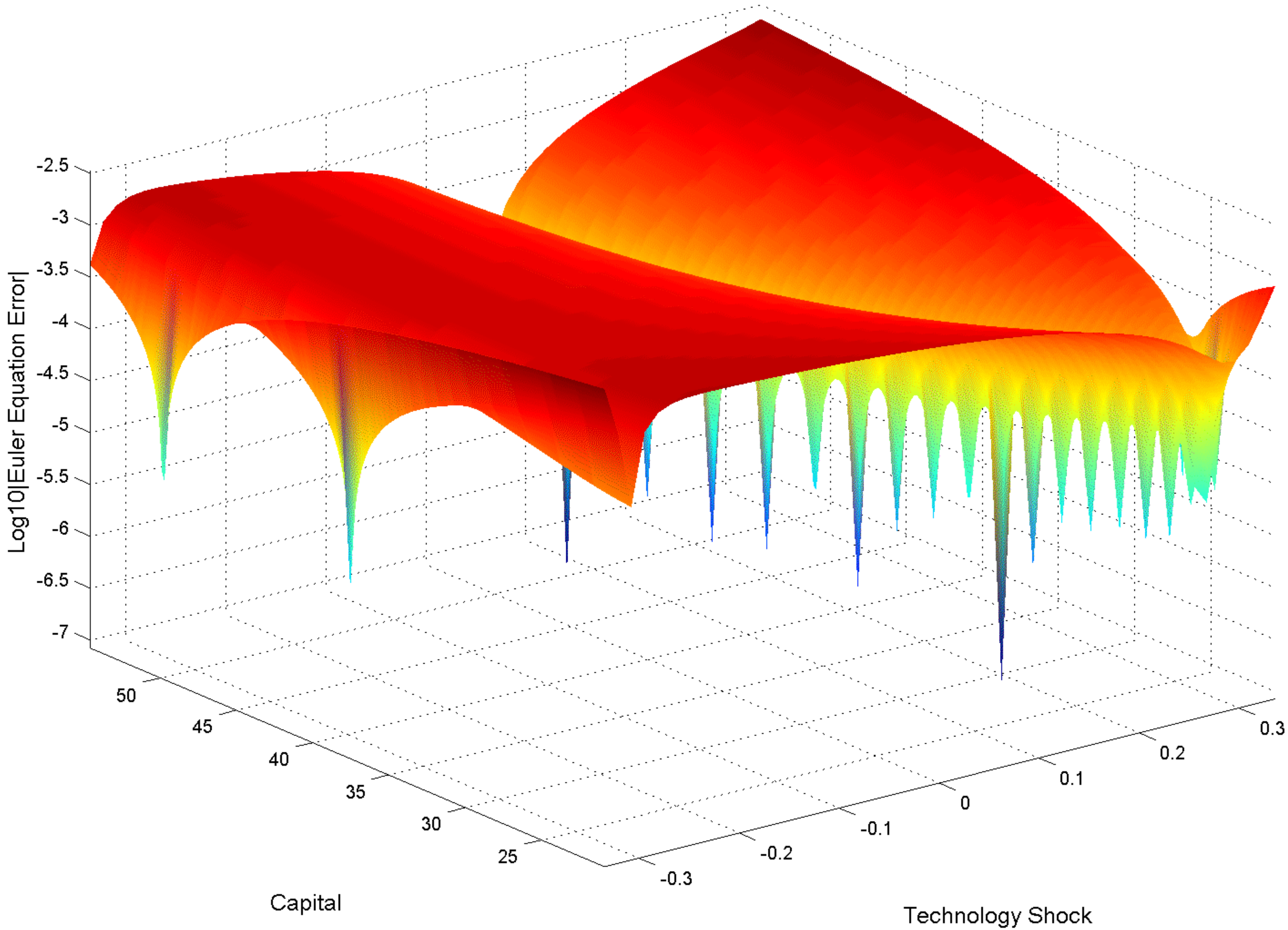


Figure 5.4.13 : Euler Equation Errors, Log-Linear Approximation, $\tau = 50$ / $\sigma = 0.035$

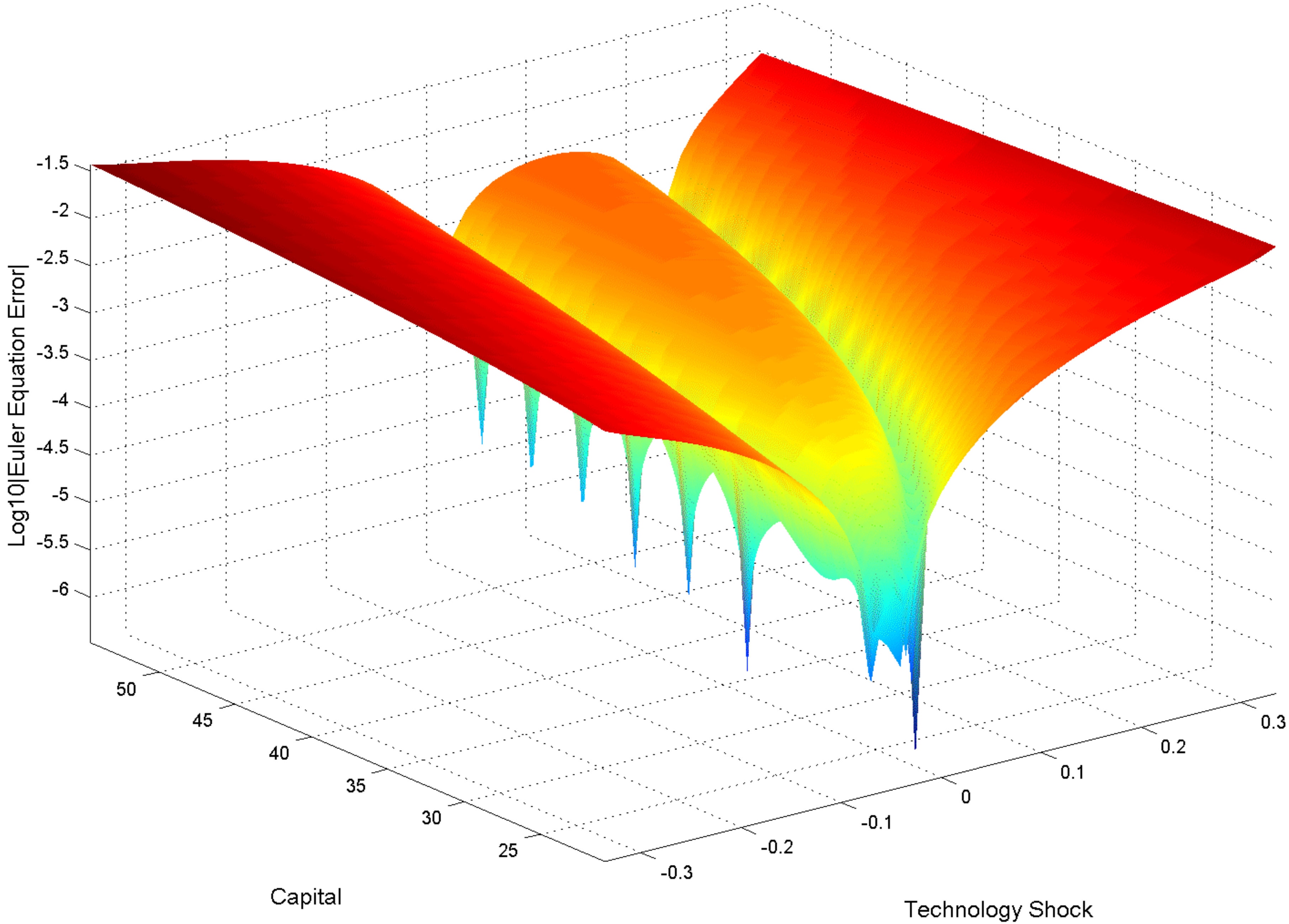


Figure 5.4.14 : Euler Equation Errors, Finite Elements Approximation, $\tau = 50 / \sigma = 0.035$

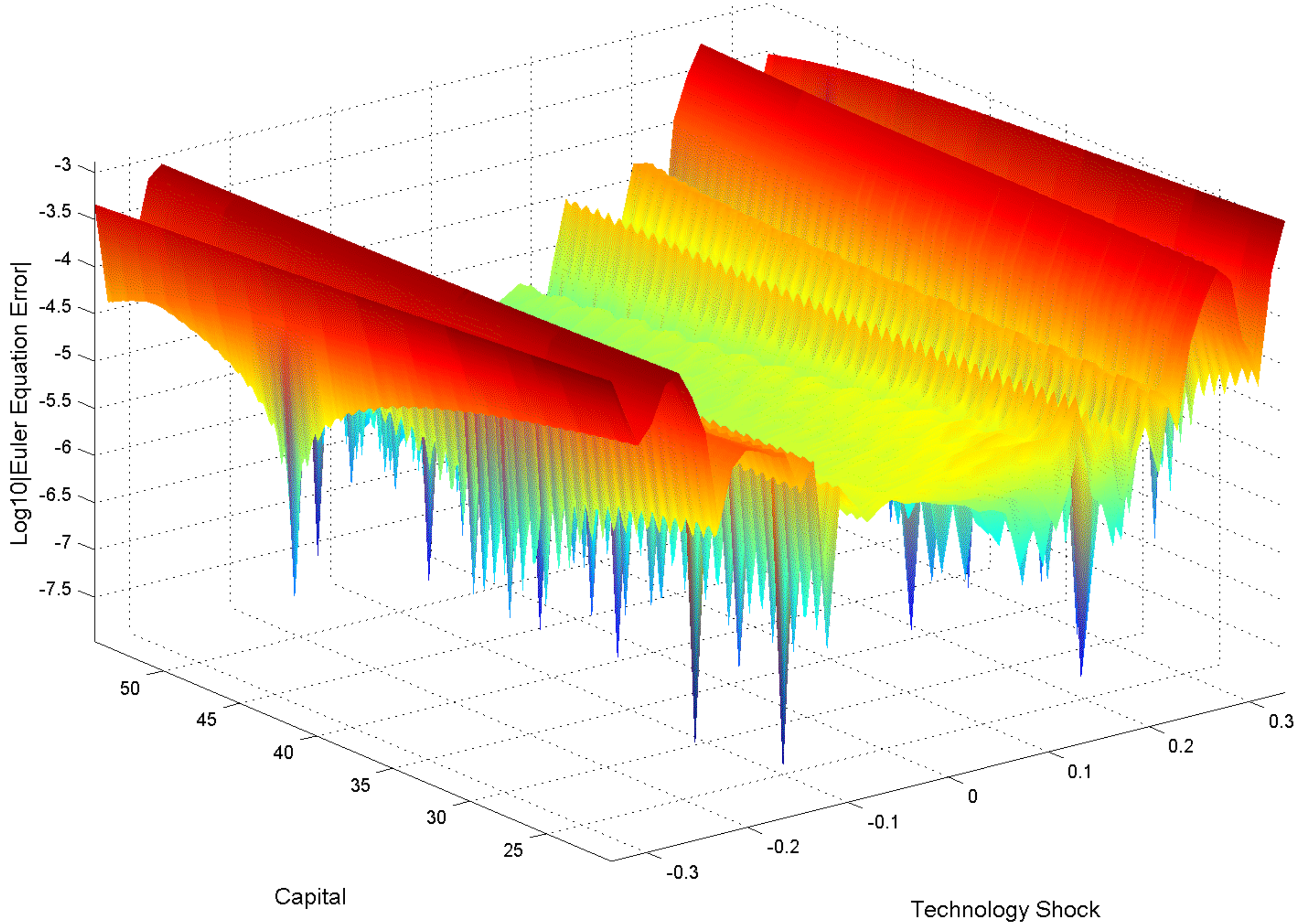


Figure 5.4.15 : Euler Equation Errors, Chebyshev Appr., $\tau = 50$ / $\sigma = 0.035$

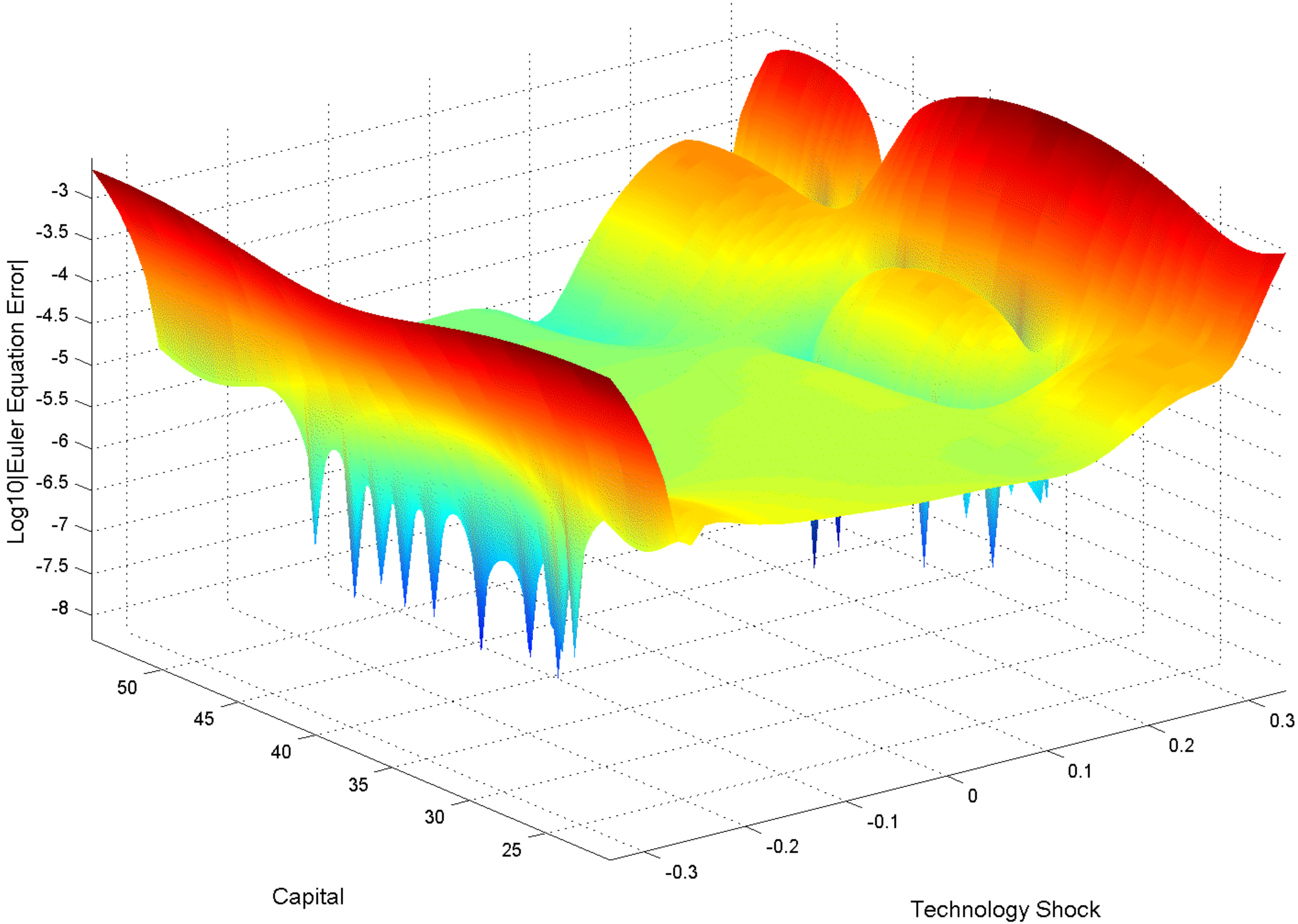


Figure 5.4.16 : Euler Equation Errors, 2nd Order Perturbation Appr., τ = 50 / σ = 0.035

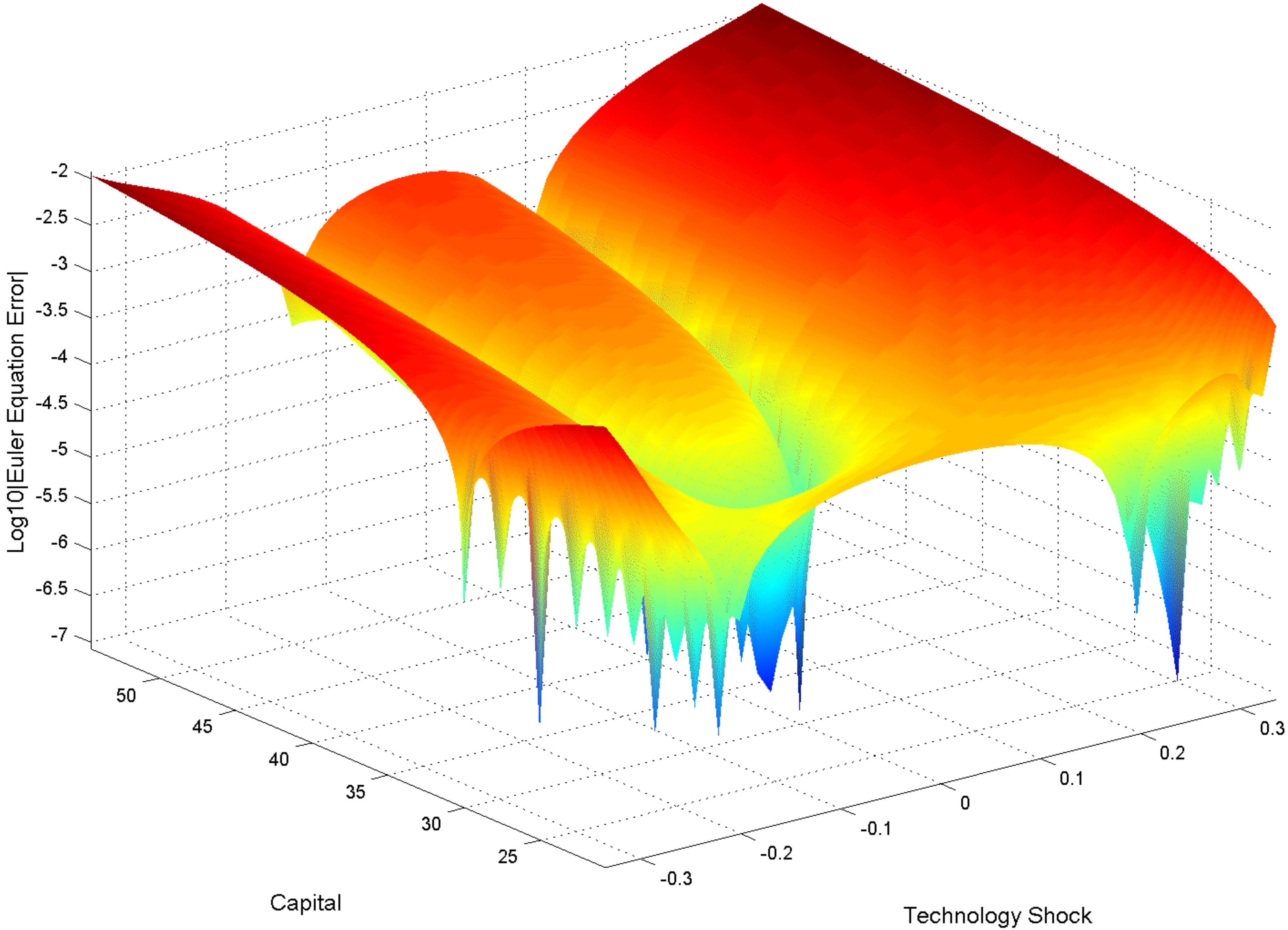


Figure 5.4.17 : Euler Equation Errors, 5th Order Perturbation Appr., $\tau = 50$ / $\sigma = 0.035$

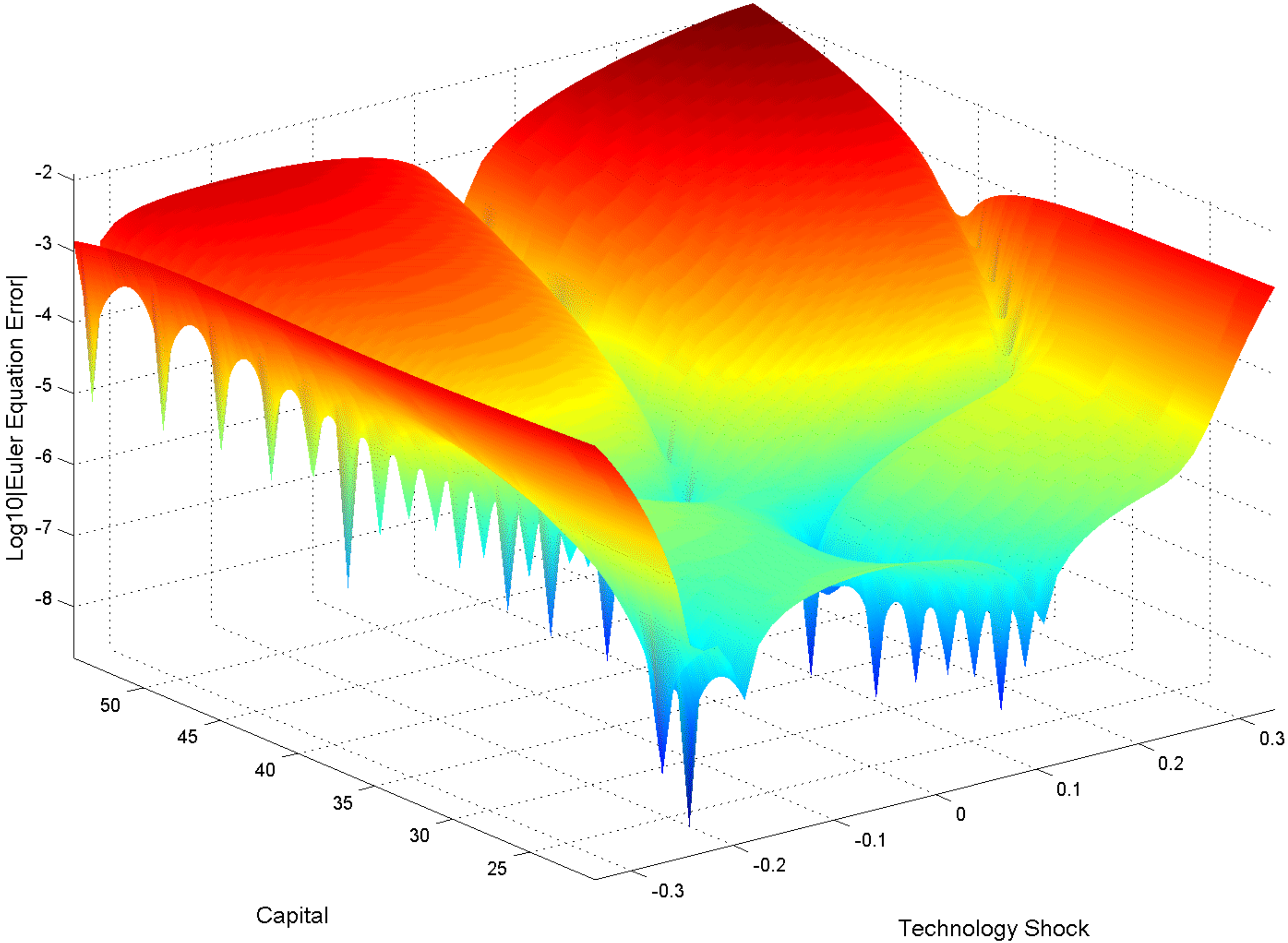


Figure 5.4.18 : Euler Equation Errors, Value Function Appr., $\tau = 50 / \sigma = 0.035$

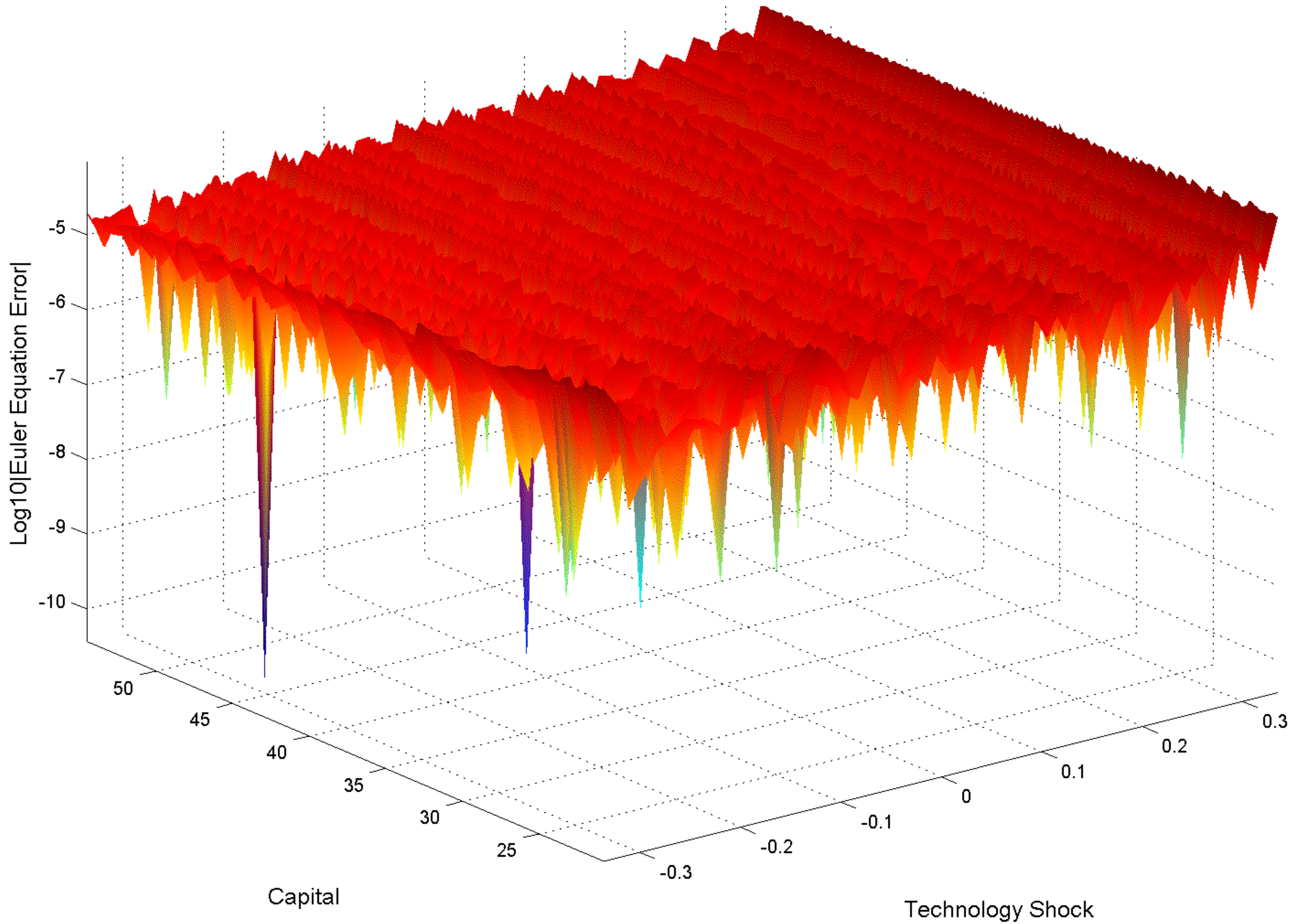
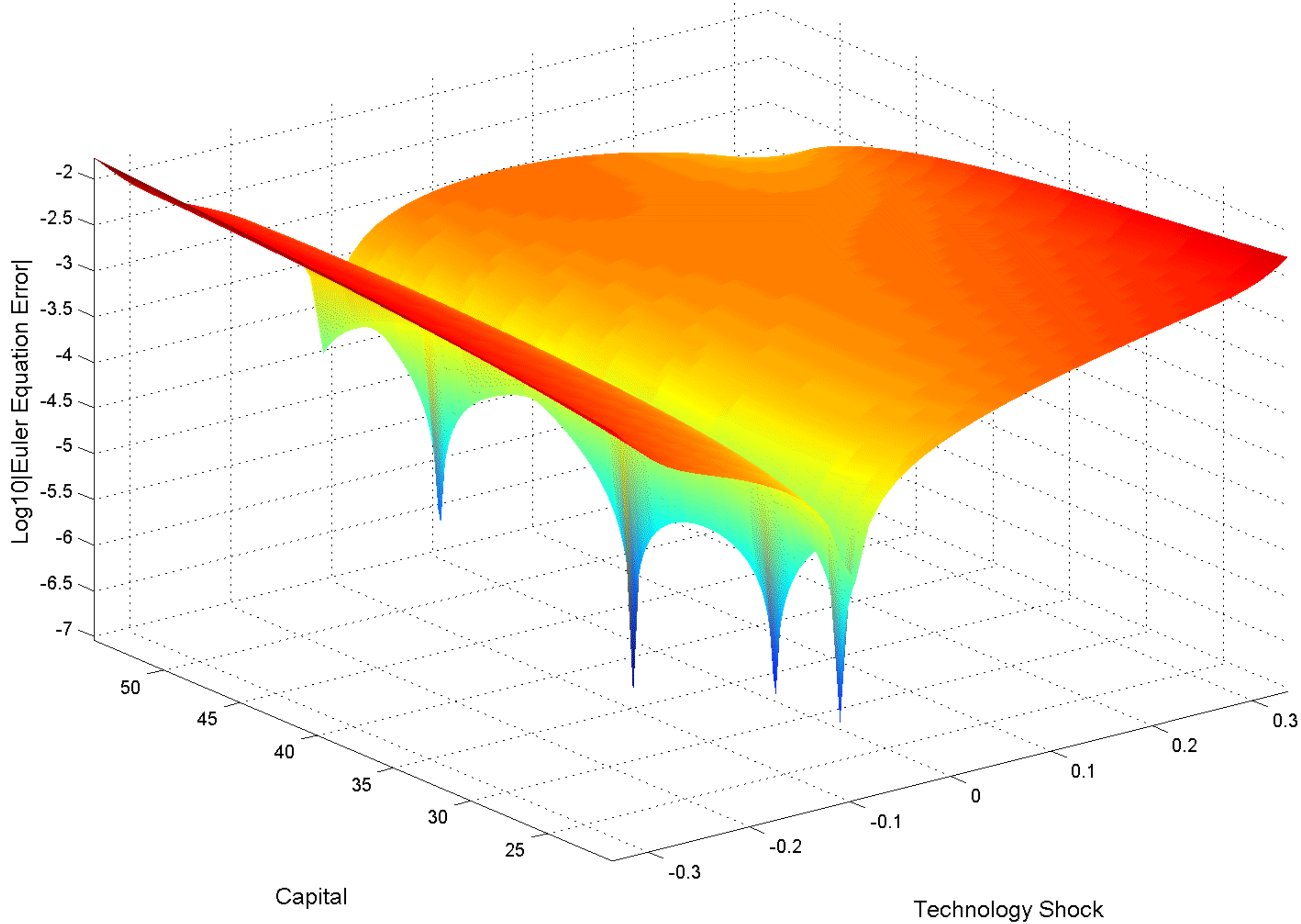


Figure 5.4.19 : Euler Eq. Errors, 2nd Order Log-Linear Perturbation Appr., τ = 50 / σ = 0.035



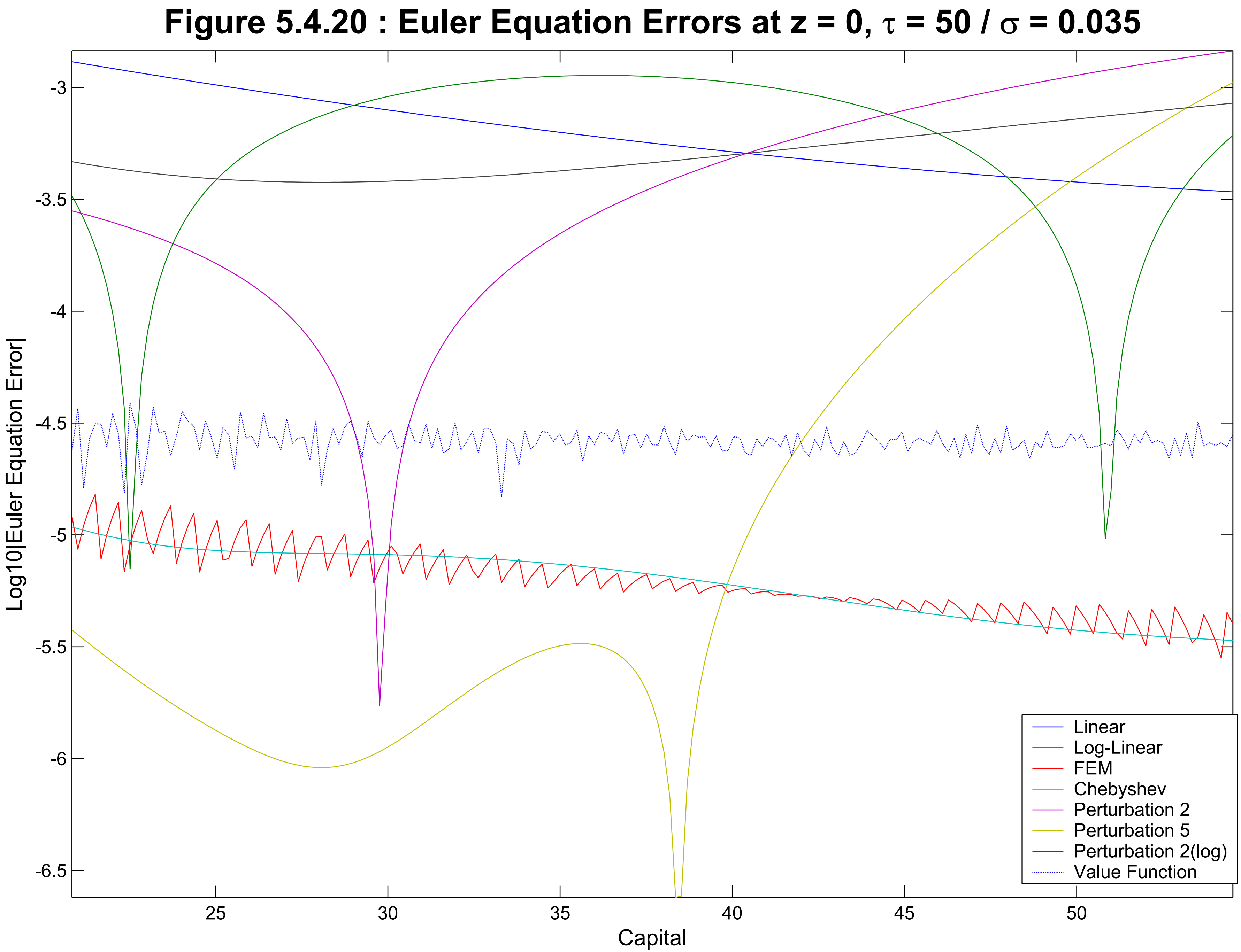


Figure 5.4.20 : Euler Equation Errors at z = 0, τ = 50 / σ = 0.035
Log10|Euler Equation Error|
Capital
-3
-3.5
-4
-4.5
-5
-5.5
-6
-6.5
25
30
35
40
45
50
Linear
Log-Linear
FEM
Chebyshev
Perturbation 2
Perturbation 5
Perturbation 2(log)
Value Function

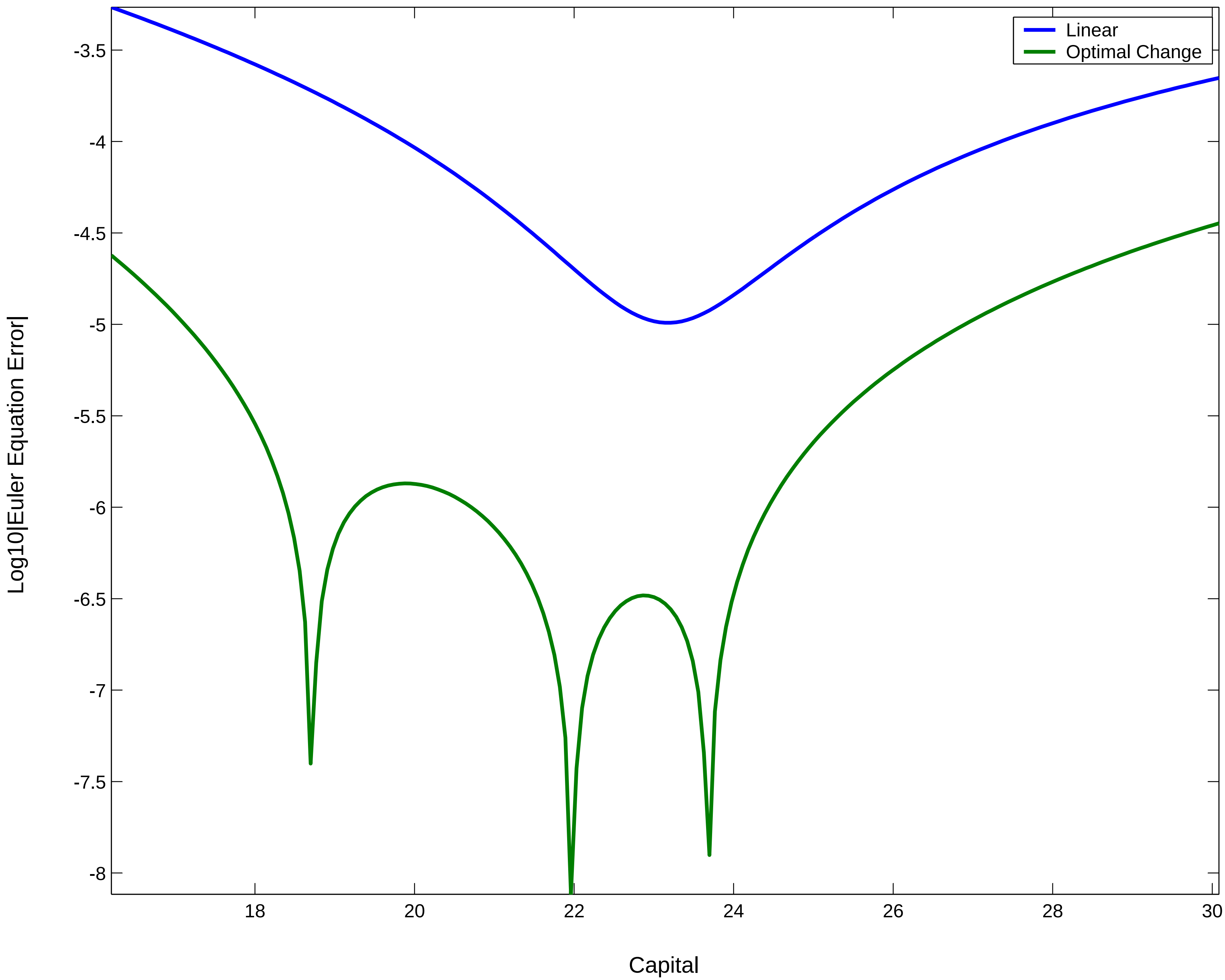
Table 5.4.1: Integral of the Euler Errors ($x10^{-4}$)

Linear	0.2291
Log-Linear	0.6306
Finite Elements	0.0537
Chebyshev	0.0369
Perturbation 2	0.0481
Perturbation 5	0.0369
Value Function	0.0224

Table 5.4.2: Integral of the Euler Errors ($x10^{-4}$)

Linear	7.12
Log-Linear	24.37
Finite Elements	0.34
Chebyshev	0.22
Perturbation 2	7.76
Perturbation 5	8.91
Perturbation 2 (log)	6.47
Value Function	0.32

Figure 6.2.1 : Euler Equation Errors at z = 0, τ = 2 / σ = 0.007



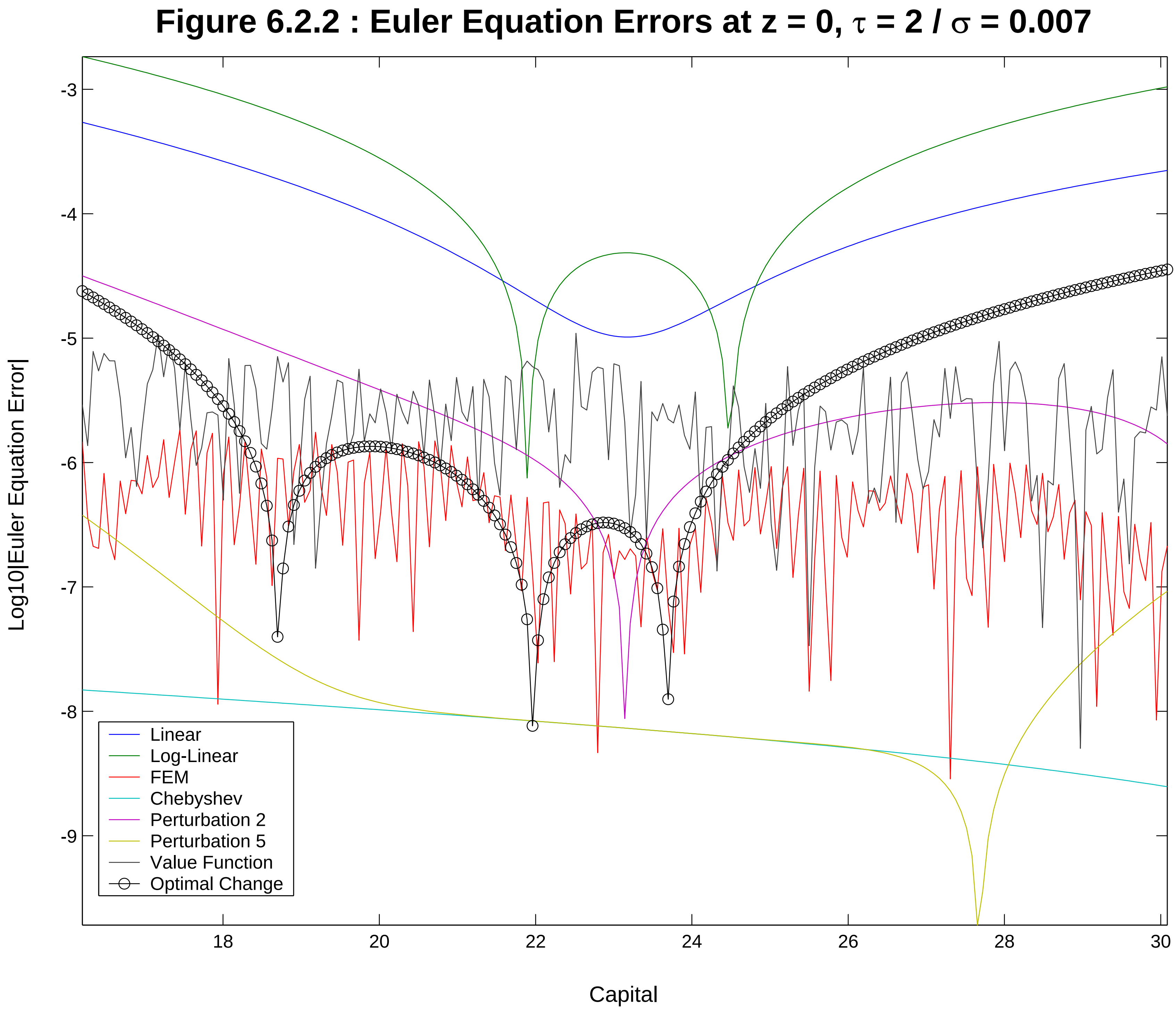


Figure 6.2.2 : Euler Equation Errors at z = 0, τ = 2 / σ = 0.007
-3
-4
-5
-6
-7
-8
-9
18
20
22
24
26
28
30
Log10|Euler Equation Error|
Capital
Linear
Log-Linear
FEM
Chebyshev
Perturbation 2
Perturbation 5
Value Function
Optimal Change

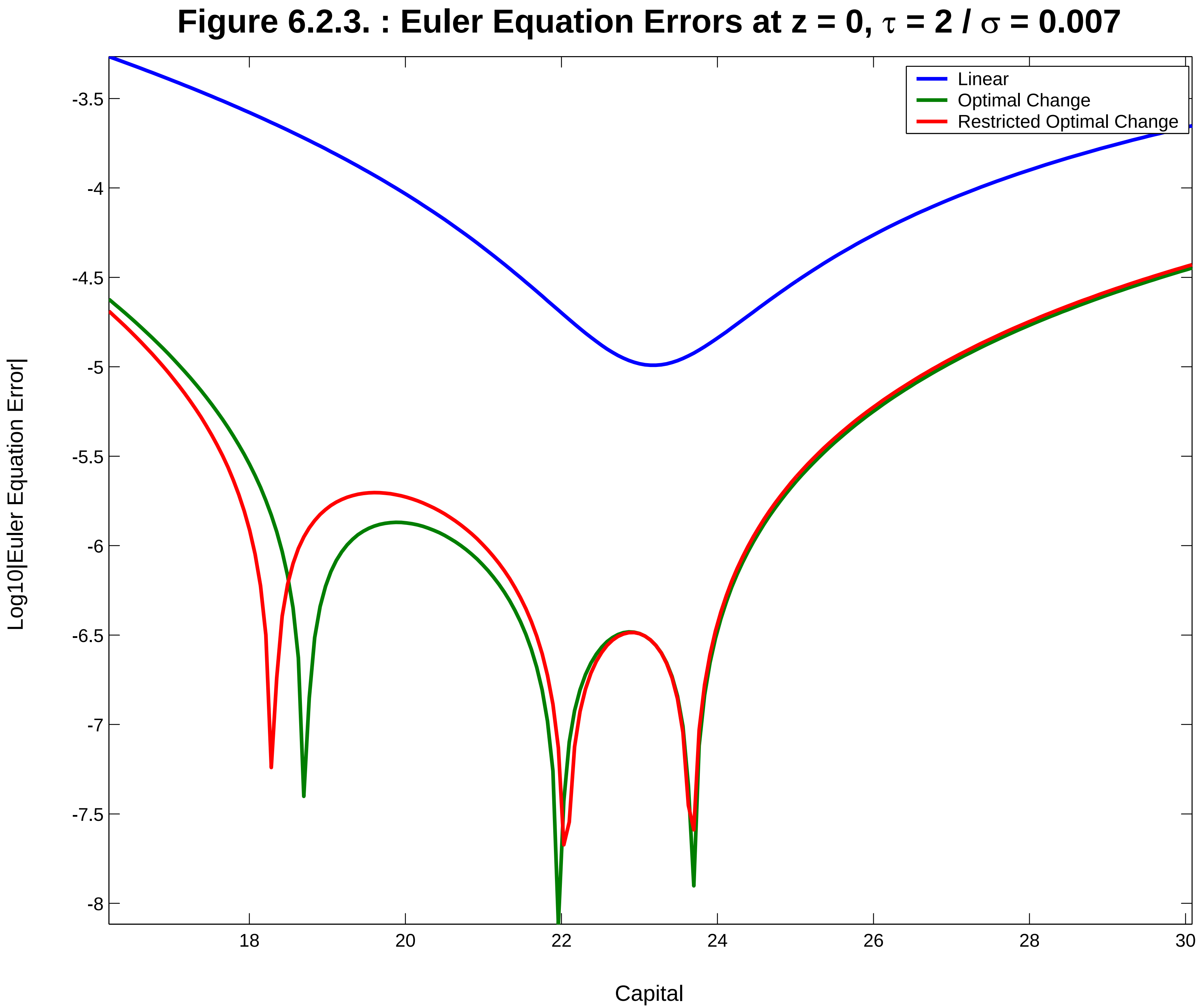


Figure 6.2.3. : Euler Equation Errors at z = 0, τ = 2 / σ = 0.007
Linear
Optimal Change
Restricted Optimal Change
Log10|Euler Equation Error|
-3.5
-4
-4.5
-5
-5.5
-6
-6.5
-7
-7.5
-8
18
20
22
24
26
28
30
Capital

Computing Time and Reproducibility

- How methods compare?

- Web page:

`www.econ.upenn.edu/~jesusfv/companion.htm`