A Model with Collateral Constraints

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Motivation

- Kiyotaki and Moore, 1997.
- Main ideas:
  1. Feedback loop between financial constraints and economic activity.
  2. Dual role of assets as factors of production and as collateral (fire sale Shleifer and Vishny, 1992).
- Simple model:
  1. Discrete time.
  2. Perfect foresight.
  3. Little heterogeneity.

Preferences

- Continuum of infinitely lived, risk-neutral agents:

  1. Mass 1 of farmers:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t x_t$$

  2. Mass m of gatherers:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{*t} x_t^*$$

- Assumption A: $\beta < \beta^*$.

Goods and Markets

- Two goods:

  1. Durable asset (land): does not depreciate, fixed supply $\overline{K}$.

  2. Nondurable commodity (fruit): x_t and x_t^*.

- Fruit is the numeraire.

- Competitive spot market for land in each period t: price of 1 unit of land q_t.

- Credit market: one unit of fruit at period t is exchanged for R_t units of fruit at period $t+1$.

Technology for Farmers

- Linear production function that uses land k_t to produce fruit:

$$y_{t+1} = (a + c)\, k_t$$

- Two parts of output:

  1. a: tradable output.
  2. c: non-tradable output (basically to induce some current consumption).

- Assumption B: non-tradable output is big enough

$$c > \left(\frac{1}{\beta} - 1\right) a$$

Budget Constraint for the Farmer

- Farmers buy (net) land $k_t - k_{t-1}$ at price q_t.
- Farmers borrow a quantity b_t at interest rate R_t.
- Farmers consume x_t at cost $x_t - ck_{t-1}$ (total consumption less non-tradable output).
- Farmers sell output ak_{t-1}.
- Therefore:

$$q_t \left(k_t - k_{t-1}\right) + R_t b_{t=1} + x_t - ck_{t-1} = ak_{t-1} + b_t$$

Borrowing Constraint

- Hart and Moore, 1994
- Farmer labor input is necessary and lot-specific once production has started.
- Farmer labor cannot be precommitted.
- Hence:

$$\text{outside value} = q_{t+1} k_t < (a + c) k_t = \text{inside value}$$

- Under renegotiation after a default, the farmer can never get less than $q_{t+1} k_t$.
- A farmer can, then borrow a quantity b_t such that (secure debt):

$$R_t b_t \leq q_{t+1} k_t$$

Technology for Gatherers

- Concave production function that uses land k_t to produce fruit:

$$y_{t+1} = G\left(k_{t-1}^{*}\right).$$

- Assumption C: to avoid corner solutions:

$$G'(0) > aR_t > G'\left(\frac{\overline{K}}{m}\right)$$

that is, marginal productivity of gatherers is such that, in equilibrium both farmers and gatherers hold some land (easy because we will see below that R_t is constant).

- Budget constraint:

$$q_t\left(k_t^{*} - k_{t-1}^{*}\right) + R_t b_{t=1}^{*} + x_t^{*} = G\left(k_{t-1}^{*}\right) + b_t^{*}$$

- No specific skill in production: no borrowing constraint.

Equilibrium I

Definition

An equilibrium is an allocation $\{k_t, k_t^*, x_t, x_t^*\}_{t=0}^{\infty}$, debt $\{b_t, b_t^*\}_{t=0}^{\infty}$, and prices $\{q_t, R_t\}_{t=0}^{\infty}$ such that:

1. Given prices $\{q_t, R_t\}_{t=0}^{\infty}$, farmers solve their problem:

$$\max_{\{k_t, x_t, b_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t x_t$$

$$\text{s.t. } q_t (k_t - k_{t-1}) + R_t b_{t-1} + x_t - c k_{t-1} = a k_{t-1} + b_t$$

$$R_t b_t \leq q_{t+1} k_t$$

Equilibrium II

Definition

2. Given prices $\{q_t, R_t\}_{t=0}^{\infty}$, gatherers solve their problem:

$$\max_{\{k_t^*, x_t^*, b_t^*\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{*t} x_t^*$$

$$\text{s.t. } q_t \left(k_t^* - k_{t-1}^*\right) + R_t b_{t-1}^* + x_t^* = G\left(k_{t-1}^*\right) + b_t^*$$

3. Markets clear:

$$x_t + m x_t^* = (a + c) k_{t-1} + mG\left(k_{t-1}^*\right)$$

$$k_t + m k_t^* = \overline{K}$$

$$b_t + m b_t^* = 0$$

Characterizing the Equilibrium I

- Since they are never constrained, gatherers satisfy Euler equation:

$$1 = \beta^* R_{t+1} \Rightarrow R = R_t = \frac{1}{\beta^*}$$

- Also, by non-arbitrage, gatherers are indifferent at the margin about buying one unit more of land

$$-q_t + \frac{1}{R}\left(G'\left(k_t^*\right) + q_{t+1}\right) = 0$$

  or, rearranging terms, equate the rate of return of land to its user cost u_t:

$$G'\left(k_t^*\right) = R\left(q_t - \frac{1}{R}q_{t+1}\right) = Ru_t$$

- Hence, all gatherers have the same amount of land.

Characterizing the Equilibrium II

- $R_t = \frac{1}{\beta^*}$ together with assumption A$\Rightarrow$farmers are always constrained.
- Decision rule of farmers (close to the steady state, role of assumption B):

1. Borrow the maximum amount possible:

$$b_t = \frac{q_{t+1} k_t}{R}$$

2. Consume just their non-tradable fruit:

$$x_t = c k_{t-1}$$

Characterizing the Equilibrium III

- Using 1. and 2.: farmers buy as much land as possible:

$$q_t\left(k_t - k_{t-1}\right) + Rb_{t-1} + x_t - ck_{t-1} = ak_{t-1} + b_t \Rightarrow$$

$$q_t\left(k_t - k_{t-1}\right) + q_t k_{t-1} = ak_{t-1} + \frac{q_{t+1}k_t}{R} \Rightarrow$$

$$\left(Rq_t - q_{t+1}\right)k_t = Rak_{t-1} \Rightarrow$$

$$k_t = \frac{1}{q_t - \frac{1}{R}q_{t+1}} ak_{t-1} \Rightarrow$$

$$k_t = \frac{1}{u_t} ak_{t-1} = \frac{1}{u_t}\left(\left(a + q_t\right)k_{t-1} - Rb_{t-1}\right)$$

- Last equation: farmer leverages his net wealth $\left(a + q_t\right)k_{t-1} - Rb_{t-1}$ given the down payment u_t (which is also the user cost of land!)
- Decision rules are linear: distribution of land among farmers is irrelevant.

Steady State I

- First,

$$k = \frac{1}{u} ak \Rightarrow u = a$$

- Second,

$$u = q - \frac{1}{R} q = (1 - \beta^*) q \Rightarrow q = \frac{a}{1 - \beta^*}$$

- Third,

$$k^* = G'^{-1}(Ru) = G'^{-1}(Ra)$$
$$k = \overline{K} - mk^*$$

Steady State II

- Fourth,

$$b = \frac{qk}{R} = \frac{\beta^*}{1-\beta^*} ak$$

$$b^* = -\frac{b}{m}$$

- Fifth,

$$x = ck$$

$$x^* = \frac{a}{m} k + G(k^*)$$

Comparison with Social Planner

- Optimal allocation of land:

$$G'\left(k_{sp}^*\right) = a + c \Rightarrow k_{sp}^* = G'^{-1}\left(a + c\right)$$

  (same marginal productivity in both sectors).

- In the market allocation:

$$k^* = G'^{-1}\left(a\right)$$

- Thus:

$$k^* > k_{sp}^*$$

- Intuition.
- Consumption: it would depend on the social planner's objective function.

Computation of the Equilibrium I

- Given some initial k_t^* and k_t, we find:

$$u_t = \frac{1}{R} G' \left(k_t^* \right)$$

$$k_{t+1} = \frac{1}{u_t} a k_t$$

$$k_{t+1}^* = \frac{1}{m} \left(\overline{K} - k_{t+1} \right)$$

- By imposing a transversality condition to run out bubbles, we can find $\{q_t\}_{t=0}^{\infty}$ that satisfies:

$$u_t = q_t - \frac{1}{R} q_{t+1}$$

Computation of the Equilibrium II

- To close the model:

$$b_t = \frac{q_{t+1} k_t}{R}$$

$$b_t^* = -\frac{b_t}{m}$$

$$x_t = c k_{t-1}$$

$$x_t^* = \frac{a}{m} k_{t-1} + G\left(k_{t-1}^*\right)$$

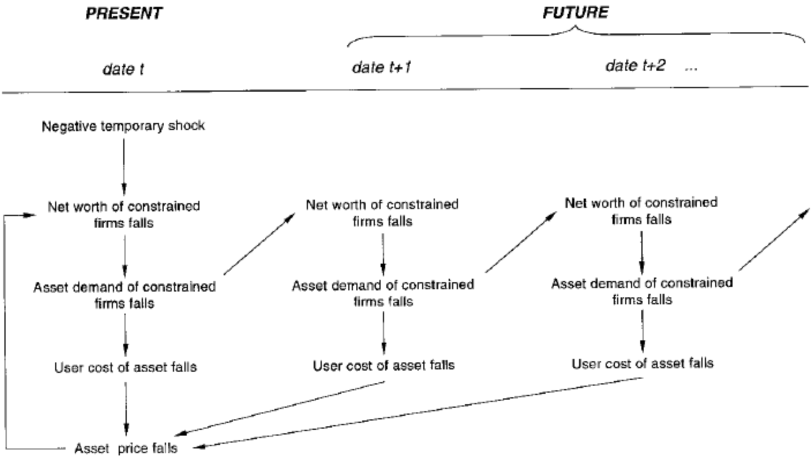
A Shock the Economy

- Think about the case where, unanticipatedly, farmers produce at time t

$$y_t = (a - \Delta + c)\, k_{t-1}$$

- Farmers are poorer.
- Farmers demand less land: q_t goes down, u_t goes down, land moves from farmers to gatherers$\Rightarrow$fall in production.
- But a lower q_t means farmers can borrow less (they are leveraged, and hence their net wealth goes down more than proportionally to the shock) and get even less land.
- Comparison with social planner's response.

Feedback Loop



An Extended Model

- Two modifications:

    1. Opportunities to invest arrive randomly.

    2. Trees in addition to land.

- We will explore them later.

- Main idea.

Computation

- Basic model where:

$$y_t = (a + \varepsilon_t + c)\, k_{t-1}$$

where

$$\varepsilon_t \sim \mathcal{N}(0, \sigma)$$

- `Dynare`.

Main Idea

- We can think about equilibrium conditions as a system of functional equations of the form:

$$\mathbb{E}_t \mathcal{H}(d) = \mathbf{0}$$

  for an unknown decision rule d.

- Perturbation solves the problem by specifying:

$$d^n(x, \theta) = \sum_{i=0}^{n} \theta_i (x - x_0)^i$$

- We use implicit-function theorems to find coefficients θ_i's.

- Inherently local approximation. However, often good global properties.

Motivation

- Many complicated mathematical problems have:

  1. either a particular case
  2. or a related problem.

  that is easy to solve.

- Often, we can use the solution of the simpler problem as a building block of the general solution.

- Very successful in physics.

- Sometimes perturbation is known as asymptotic methods.

Applications to Economics

- Judd and Guu (1993) showed how to apply it to economic problems.

- Recently, perturbation methods have been gaining much popularity.

- In particular, second- and third-order approximations are easy to compute and notably improve accuracy.

- Perturbation theory is the generalization of the well-known linearization strategy.

- Hence, we can use much of what we already know about linearization.

References

- General:
  1. *A First Look at Perturbation Theory* by James G. Simmonds and James E. Mann Jr.
  2. *Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory* by Carl M. Bender, Steven A. Orszag.

- Economics:
  1. Perturbation Methods for General Dynamic Stochastic Models" by Hehui Jin and Kenneth Judd.
  2. Perturbation Methods with Nonlinear Changes of Variables" by Kenneth Judd.
  3. A gentle introduction: "Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function" by Martín Uribe and Stephanie Schmitt-Grohe.

Asymptotic Expansion

$$
\begin{aligned}
y_t = \;& y\left(s_t, \varepsilon_t; \sigma\right)\big|_{k,0,0} = y\left(s,0;0\right) \\
& + y_s\left(s,0;0\right)\left(s_t - s\right) + y_\varepsilon\left(s,0;0\right)\varepsilon_t + y_\sigma\left(s,0;0\right)\sigma \\
& + \frac{1}{2} y_{ss}\left(s,0;0\right)\left(s_t - s\right)^2 + \frac{1}{2} y_{s\varepsilon}\left(s,0;0\right)\left(s_t - s\right)\varepsilon_t \\
& + \frac{1}{2} y_{s\sigma}\left(s,0;0\right)\left(s_t - s\right)\sigma + \frac{1}{2} y_{\varepsilon s}\left(s,0;0\right) z_t\left(k_t - k\right) \\
& + \frac{1}{2} y_{\varepsilon\varepsilon}\left(s,0;0\right)\varepsilon_t^2 + \frac{1}{2} y_{\varepsilon\sigma}\left(s,0;0\right)\varepsilon_t\sigma \\
& + \frac{1}{2} y_{\sigma s}\left(s,0;0\right)\sigma\left(k_t - k\right) + \frac{1}{2} y_{\sigma\varepsilon}\left(s,0;0\right)\sigma\varepsilon_t \\
& + \frac{1}{2} y_{\sigma^2}\left(s,0;0\right)\sigma^2 + \ldots
\end{aligned}
$$

The General Case

- Most of previous argument can be easily generalized.

- The set of equilibrium conditions of many DSGE models can be written as (note recursive notation)

$$\mathbb{E}_t \mathcal{H}(y, y', x, x') = 0,$$

  where y_t is a $n_y \times 1$ vector of controls and $x_t = (s_t, \varepsilon_t)$ is a $n_x \times 1$ vector of states.

- Define $n = n_x + n_y$.

- Then $\mathcal{H}$ maps $R^{n_y} \times R^{n_y} \times R^{n_x} \times R^{n_x}$ into R^n.

Partitioning the State Vector

- The state vector x_t can be partitioned as $x = [x_1; x_2]^t$.
- $x_1 = s_t$ is a $(n_x - n_\epsilon) \times 1$ vector of endogenous state variables.
- $x_2 = \varepsilon_t$ is a $n_\epsilon \times 1$ vector of exogenous state variables.
- Why do we want to partition the state vector?

Exogenous Stochastic Process

$$x_2' = \Lambda x_2 + \sigma \eta_\epsilon \epsilon'$$

- Process with 3 parts:
  1. The deterministic component Λx_2:
     1. Λ is a $n_\epsilon \times n_\epsilon$ matrix, with all eigenvalues with modulus less than one.
     2. More general: $x_2' = \Gamma(x_2) + \sigma \eta_\epsilon \epsilon'$, where Γ is a non-linear function satisfying that all eigenvalues of its first derivative evaluated at the non-stochastic steady state lie within the unit circle.
  2. The scaled innovation $\eta_\epsilon \epsilon'$ where:
     1. η_ϵ is a known $n_\epsilon \times n_\epsilon$ matrix.
     2. ϵ is a $n_\epsilon \times 1$ i.i.d innovation with bounded support, zero mean, and variance/covariance matrix I.
  3. The perturbation parameter σ.
- We can accommodate very general structures of x_2 through changes in the definition of the state space: i.e. stochastic volatility.
- Note we do not impose gaussianity.

The Perturbation Parameter

- The scalar $\sigma \geq 0$ is the perturbation parameter.

- If we set $\sigma = 0$ we have a deterministic model.

- Important: there is only ONE perturbation parameter. The matrix η_ϵ takes account of relative sizes of different shocks.

- Why bounded support? Samuelson (1970) and Jin and Judd (2002).

Solution of the Model

- The solution to the model is of the form:

$$y = g(x; \sigma)$$
$$x' = h(x; \sigma) + \sigma \eta \epsilon'$$

where g maps $R^{n_x} \times R^{+}$ into R^{n_y} and h maps $R^{n_x} \times R^{+}$ into R^{n_x}.

- The matrix η is of order $n_x \times n_\epsilon$ and is given by:

$$\eta = \begin{bmatrix} \emptyset \\ \eta_\epsilon \end{bmatrix}$$

Perturbation

- We wish to find a perturbation approximation of the functions g and h around the non-stochastic steady state, $x_t = \bar{x}$ and $\sigma = 0$.

- We define the non-stochastic steady state as vectors $(\bar{x}, \bar{y})$ such that:

$$\mathcal{H}(\bar{y}, \bar{y}, \bar{x}, \bar{x}) = 0.$$

- Note that $\bar{y} = g(\bar{x}; 0)$ and $\bar{x} = h(\bar{x}; 0)$. This is because, if $\sigma = 0$, then $\mathbb{E}_t \mathcal{H} = \mathcal{H}$.

Plugging-in the Proposed Solution

- Substituting the proposed solution, we define:

$$F(x;\sigma) \equiv \mathbb{E}_t \mathcal{H}(g(x;\sigma), g(h(x;\sigma)+\eta\sigma\epsilon',\sigma), x, h(x;\sigma)+\eta\sigma\epsilon') = 0$$

- Since $F(x;\sigma) = 0$ for any values of x and σ, the derivatives of any order of F must also be equal to zero.

- Formally:

$$F_{x^k\sigma^j}(x;\sigma) = 0 \quad \forall x, \sigma, j, k,$$

where $F_{x^k\sigma^j}(x,\sigma)$ denotes the derivative of F with respect to x taken k times and with respect to σ taken j times.

First-Order Approximation

- We are looking for approximations to g and h around $(x, \sigma) = (\bar{x}, 0)$ of the form:

$$\begin{aligned} g(x;\sigma) &= g(\bar{x};0) + g_x(\bar{x};0)(x-\bar{x}) + g_\sigma(\bar{x};0)\sigma \\ h(x;\sigma) &= h(\bar{x};0) + h_x(\bar{x};0)(x-\bar{x}) + h_\sigma(\bar{x};0)\sigma \end{aligned}$$

- As explained earlier,

$$\begin{aligned} g(\bar{x};0) &= \bar{y} \\ h(\bar{x};0) &= \bar{x} \end{aligned}$$

- The remaining four unknown coefficients of the first-order approximation to g and h are found by using the fact that:

$$\begin{aligned} F_x(\bar{x};0) &= 0 \\ F_\sigma(\bar{x};0) &= 0 \end{aligned}$$

- Before doing so, I need to introduce the tensor notation.

Tensors

- General trick from physics.
- An n^{th}-rank tensor in a m-dimensional space is an operator that has n indices and m^n components and obeys certain transformation rules.
- $[\mathcal{H}_y]^i_\alpha$ is the (i, α) element of the derivative of $\mathcal{H}$ with respect to y:
  1. The derivative of $\mathcal{H}$ with respect to y is an $n \times n_y$ matrix.
  2. Thus, $[\mathcal{H}_y]^i_\alpha$ is the element of this matrix located at the intersection of the i-th row and α-th column.
  3. Thus, $[\mathcal{H}_y]^i_\alpha [g_x]^\alpha_\beta [h_x]^\beta_j = \sum_{\alpha=1}^{n_y} \sum_{\beta=1}^{n_x} \frac{\partial \mathcal{H}^i}{\partial y^\alpha} \frac{\partial g^\alpha}{\partial x^\beta} \frac{\partial h^\beta}{\partial x^j}$.
- $[\mathcal{H}_{y'y'}]^i_{\alpha\gamma}$:
  1. $\mathcal{H}_{y'y'}$ is a three dimensional array with n rows, n_y columns, and n_y pages.
  2. Then $[\mathcal{H}_{y'y'}]^i_{\alpha\gamma}$ denotes the element of $\mathcal{H}_{y'y'}$ located at the intersection of row i, column α and page γ.

Solving the System I

- g_x and h_x can be found as the solution to the system:

$$\begin{array}{rcl} [F_x(\bar{x};0)]^i_j & = & [\mathcal{H}_{y'}]^i_\alpha [g_x]^\alpha_\beta [h_x]^\beta_j + [\mathcal{H}_y]^i_\alpha [g_x]^\alpha_j + [\mathcal{H}_{x'}]^i_\beta [h_x]^\beta_j + [\mathcal{H}_x]^i_j = 0 \\ i & = & 1, \ldots, n; \quad j, \beta = 1, \ldots, n_x; \quad \alpha = 1, \ldots, n_y \end{array}$$

- Note that the derivatives of $\mathcal{H}$ evaluated at $(y, y', x, x') = (\bar{y}, \bar{y}, \bar{x}, \bar{x})$ are known.

- Then, we have a system of $n \times n_x$ quadratic equations in the $n \times n_x$ unknowns given by the elements of g_x and h_x.

- We can solve with a standard quadratic matrix equation solver.

Solving the System II

- g_σ and h_σ are the solution to the n equations:

$$[F_\sigma(\bar{x};0)]^i =$$

$$\mathbb{E}_t\{[\mathcal{H}_{y'}]^i_\alpha[g_x]^\alpha_\beta[h_\sigma]^\beta + [\mathcal{H}_{y'}]^i_\alpha[g_x]^\alpha_\beta[\eta]^\beta_\phi[\epsilon']^\phi + [\mathcal{H}_{y'}]^i_\alpha[g_\sigma]^\alpha$$

$$+[\mathcal{H}_y]^i_\alpha[g_\sigma]^\alpha + [\mathcal{H}_{x'}]^i_\beta[h_\sigma]^\beta + [\mathcal{H}_{x'}]^i_\beta[\eta]^\beta_\phi[\epsilon']^\phi\}$$

$$i = 1,\ldots,n; \quad \alpha = 1,\ldots,n_y; \quad \beta = 1,\ldots,n_x; \quad \phi = 1,\ldots,n_\epsilon.$$

- Then:

$$[F_\sigma(\bar{x};0)]^i$$

$$= [\mathcal{H}_{y'}]^i_\alpha[g_x]^\alpha_\beta[h_\sigma]^\beta + [\mathcal{H}_{y'}]^i_\alpha[g_\sigma]^\alpha + [\mathcal{H}_y]^i_\alpha[g_\sigma]^\alpha + [f_{x'}]^i_\beta[h_\sigma]^\beta = 0;$$

$$i = 1,\ldots,n; \quad \alpha = 1,\ldots,n_y; \quad \beta = 1,\ldots,n_x; \quad \phi = 1,\ldots,n_\epsilon.$$

- Certainty equivalence: linear and homogeneous equation in g_σ and h_σ. Thus, if a unique solution exists, it satisfies:

$$h_\sigma = \mathbf{0}$$

Second-Order Approximation I

The second-order approximations to g around $(x;\sigma) = (\bar{x};0)$ is

$$\begin{aligned}
[g(x;\sigma)]^i &= [g(\bar{x};0)]^i + [g_x(\bar{x};0)]^i_a [(x-\bar{x})]_a + [g_\sigma(\bar{x};0)]^i [\sigma] \\
&+ \frac{1}{2}[g_{xx}(\bar{x};0)]^i_{ab}[(x-\bar{x})]_a[(x-\bar{x})]_b \\
&+ \frac{1}{2}[g_{x\sigma}(\bar{x};0)]^i_a[(x-\bar{x})]_a[\sigma] \\
&+ \frac{1}{2}[g_{\sigma x}(\bar{x};0)]^i_a[(x-\bar{x})]_a[\sigma] \\
&+ \frac{1}{2}[g_{\sigma\sigma}(\bar{x};0)]^i[\sigma][\sigma]
\end{aligned}$$

where $i = 1, \ldots, n_y$, $a, b = 1, \ldots, n_x$, and $j = 1, \ldots, n_x$.

Second-Order Approximation II

The second-order approximations to h around $(x;\sigma) = (\bar{x};0)$ is

$$\begin{aligned} [h(x;\sigma)]^j &= [h(\bar{x};0)]^j + [h_x(\bar{x};0)]^j_a[(x-\bar{x})]_a + [h_\sigma(\bar{x};0)]^j[\sigma] \\ &+\frac{1}{2}[h_{xx}(\bar{x};0)]^j_{ab}[(x-\bar{x})]_a[(x-\bar{x})]_b \\ &+\frac{1}{2}[h_{x\sigma}(\bar{x};0)]^j_a[(x-\bar{x})]_a[\sigma] \\ &+\frac{1}{2}[h_{\sigma x}(\bar{x};0)]^j_a[(x-\bar{x})]_a[\sigma] \\ &+\frac{1}{2}[h_{\sigma\sigma}(\bar{x};0)]^j[\sigma][\sigma], \end{aligned}$$

where $i = 1, \ldots, n_y$, $a, b = 1, \ldots, n_x$, and $j = 1, \ldots, n_x$.

Second-order Approximation III

- The unknowns of these expansions are $[g_{xx}]^i_{ab}$, $[g_{x\sigma}]^i_a$, $[g_{\sigma x}]^i_a$, $[g_{\sigma\sigma}]^i$, $[h_{xx}]^j_{ab}$, $[h_{x\sigma}]^j_a$, $[h_{\sigma x}]^j_a$, $[h_{\sigma\sigma}]^j$.

- These coefficients can be identified by taking the derivative of $F(x;\sigma)$ with respect to x and σ twice and evaluating them at $(x;\sigma) = (\bar{x};0)$.

- By the arguments provided earlier, these derivatives must be zero.

Solving the System I

We use $F_{xx}(\bar{x};0)$ to identify $g_{xx}(\bar{x};0)$ and $h_{xx}(\bar{x};0)$:

$$
\begin{gathered}
[F_{xx}(\bar{x};0)]^i_{jk} = \\
\left([\mathcal{H}_{y'y'}]^i_{\alpha\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [\mathcal{H}_{y'y}]^i_{\alpha\gamma}[g_x]^\gamma_k + [\mathcal{H}_{y'x'}]^i_{\alpha\delta}[h_x]^\delta_k + [\mathcal{H}_{y'x}]^i_{\alpha k}\right)[g_x]^\alpha_\beta[h_x]^\beta_j \\
+[\mathcal{H}_{y'}]^i_\alpha[g_{xx}]^\alpha_{\beta\delta}[h_x]^\delta_k[h_x]^\beta_j + [\mathcal{H}_{y'}]^i_\alpha[g_x]^\alpha_\beta[h_{xx}]^\beta_{jk} \\
+\left([\mathcal{H}_{yy'}]^i_{\alpha\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [\mathcal{H}_{yy}]^i_{\alpha\gamma}[g_x]^\gamma_k + [\mathcal{H}_{yx'}]^i_{\alpha\delta}[h_x]^\delta_k + [\mathcal{H}_{yx}]^i_{\alpha k}\right)[g_x]^\alpha_j \\
+[\mathcal{H}_y]^i_\alpha[g_{xx}]^\alpha_{jk} \\
+\left([\mathcal{H}_{x'y'}]^i_{\beta\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [\mathcal{H}_{x'y}]^i_{\beta\gamma}[g_x]^\gamma_k + [\mathcal{H}_{x'x'}]^i_{\beta\delta}[h_x]^\delta_k + [\mathcal{H}_{x'x}]^i_{\beta k}\right)[h_x]^\beta_j \\
+[\mathcal{H}_{x'}]^i_\beta[h_{xx}]^\beta_{jk} \\
+[\mathcal{H}_{xy'}]^i_{j\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [\mathcal{H}_{xy}]^i_{j\gamma}[g_x]^\gamma_k + [\mathcal{H}_{xx'}]^i_{j\delta}[h_x]^\delta_k + [\mathcal{H}_{xx}]^i_{jk} = 0; \\
i = 1,\dots n, \quad j,k,\beta,\delta = 1,\dots n_x; \quad \alpha,\gamma = 1,\dots n_y.
\end{gathered}
$$

Solving the System II

- We know the derivatives of $\mathcal{H}$.
- We also know the first derivatives of g and h evaluated at $(y, y', x, x') = (\bar{y}, \bar{y}, \bar{x}, \bar{x})$.
- Hence, the above expression represents a system of $n \times n_x \times n_x$ linear equations in then $n \times n_x \times n_x$ unknowns elements of g_{xx} and h_{xx}.

Solving the System III

Similarly, $g_{\sigma\sigma}$ and $h_{\sigma\sigma}$ can be obtained by solving:

$$
\begin{aligned}
[F_{\sigma\sigma}(\bar{x};0)]^i \quad = \quad & [\mathcal{H}_{y'}]^i_\alpha [g_x]^\alpha_\beta [h_{\sigma\sigma}]^\beta \\
& +[\mathcal{H}_{y'y'}]^i_{\alpha\gamma}[g_x]^\gamma_\delta [\eta]^\delta_\xi [g_x]^\alpha_\beta [\eta]^\beta_\phi [I]^\phi_\xi \\
& +[\mathcal{H}_{y'x'}]^i_{\alpha\delta}[\eta]^\delta_\xi [g_x]^\alpha_\beta [\eta]^\beta_\phi [I]^\phi_\xi \\
& +[\mathcal{H}_{y'}]^i_\alpha [g_{xx}]^\alpha_{\beta\delta}[\eta]^\delta_\xi [\eta]^\beta_\phi [I]^\phi_\xi + [\mathcal{H}_{y'}]^i_\alpha [g_{\sigma\sigma}]^\alpha \\
& +[\mathcal{H}_{y}]^i_\alpha [g_{\sigma\sigma}]^\alpha + [\mathcal{H}_{x'}]^i_\beta [h_{\sigma\sigma}]^\beta \\
& +[\mathcal{H}_{x'y'}]^i_{\beta\gamma}[g_x]^\gamma_\delta [\eta]^\delta_\xi [\eta]^\beta_\phi [I]^\phi_\xi \\
& +[\mathcal{H}_{x'x'}]^i_{\beta\delta}[\eta]^\delta_\xi [\eta]^\beta_\phi [I]^\phi_\xi = 0; \\
i \quad = \quad & 1,\ldots,n;\alpha,\gamma = 1,\ldots,n_y;\beta,\delta = 1,\ldots,n_x;\phi,\xi = 1,\ldots,n_\epsilon
\end{aligned}
$$

a system of n linear equations in the n unknowns given by the elements of $g_{\sigma\sigma}$ and $h_{\sigma\sigma}$.

Cross Derivatives

- The cross derivatives $g_{x\sigma}$ and $h_{x\sigma}$ are zero when evaluated at $(\bar{x}, 0)$.
- Why? Write the system $F_{\sigma x}(\bar{x}; 0) = 0$ taking into account that all terms containing either g_{σ} or h_{σ} are zero at $(\bar{x}, 0)$.
- Then:

$$[F_{\sigma x}(\bar{x};0)]^i_j = [\mathcal{H}_{y'}]^i_\alpha [g_x]^\alpha_\beta [h_{\sigma x}]^\beta_j + [\mathcal{H}_{y'}]^i_\alpha [g_{\sigma x}]^\alpha_\gamma [h_x]^\gamma_j +$$
$$[\mathcal{H}_y]^i_\alpha [g_{\sigma x}]^\alpha_j + [\mathcal{H}_{x'}]^i_\beta [h_{\sigma x}]^\beta_j = 0;$$
$$i = 1, \ldots n; \quad \alpha = 1, \ldots, n_y; \quad \beta, \gamma, j = 1, \ldots, n_x.$$

- This is a system of $n \times n_x$ equations in the $n \times n_x$ unknowns given by the elements of $g_{\sigma x}$ and $h_{\sigma x}$.
- The system is homogeneous in the unknowns.
- Thus, if a unique solution exists, it is given by:

$$g_{\sigma x} = 0$$
$$h_{\sigma x} = 0$$

Structure of the Solution

- The perturbation solution of the model satisfies:

$$\begin{aligned} g_{\sigma}(\bar{x};0) &= 0 \\ h_{\sigma}(\bar{x};0) &= 0 \\ g_{x\sigma}(\bar{x};0) &= 0 \\ h_{x\sigma}(\bar{x};0) &= 0 \end{aligned}$$

- Standard deviation only appears in:
    1. A constant term given by $\frac{1}{2}g_{\sigma\sigma}\sigma^2$ for the control vector y_t.
    2. The first $n_x - n_\epsilon$ elements of $\frac{1}{2}h_{\sigma\sigma}\sigma^2$.
- Correction for risk.
- Quadratic terms in endogenous state vector x_1.
- Those terms capture non-linear behavior.

Higher-Order Approximations

- We can iterate this procedure as many times as we want.
- We can obtain n-th order approximations.
- Problems:
  1. Existence of higher order derivatives (Santos, 1992).
  2. Numerical instabilities.
  3. Computational costs.