A Model with Explicit Solution

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Motivation


- Full equilibrium dynamics of an economy with financial frictions:
  1. Nonlinearity: model will respond very differently to small and large shocks.
  2. Volatility paradox: lower values of exogenous risk may lead to higher levels of endogenous risk.

- Features:
  1. Continuous time.
  2. We have more productive but less patient agents borrowing from less productive but more patient agents. Financial frictions difficult the flow of funds between both groups.
Preferences

- Continuum of infinitely lived, risk-neutral agents:

  1. Experts, $\mathbb{I} = [0, 1]$:
     \[ \mathbb{E}_0 \int e^{-\rho t} \, dc_t \]
     where $c_t$ is cumulative consumption until time $t$. We impose $dc_t \geq 0$.

  2. Households, $\mathbb{J} = [0, 1]$:
     \[ \mathbb{E}_0 \int e^{-rt} \, dc_t \]
     where $c_t$ is cumulative consumption until time $t$. We do not impose $dc_t \geq 0$ (negative consumption can be thought as additional labor effort): hence $r$ is risk-free rate.

- Assumption: $r < \rho$. 
Technology I

Experts with efficiency units of capital $k_t$ produce output:

$$y_t = a k_t$$

Experts can invest:

$$dk_t = \left( \phi (\iota_t) - \delta \right) k_t dt + \sigma k_t dZ_t$$

where

1. $\iota_t$ is investment rate per unit of capital.
2. $\phi (\iota_t)$ is an investment technology with adjustment costs ($\phi (0) = 0$, $\phi' (\cdot) = 0$, and $\phi'' (\cdot) < 0$).
3. We do not impose $\iota_t > 0$. Concavity of $\phi (\cdot)$ imposes large costs to disinvestment.
4. $dZ_t$ is a Brownian motion.
Households with efficiency units of capital $k_t$ produce output:

$$y_t = ak_t$$

where $a < a$.

Households can invest:

$$dk_t = (\phi (\lambda_t) - \delta) k_t dt + \sigma k_t dZ_t$$

where $\delta > \delta$.

We take output as numeraire.
Capital can be traded at price $q_t$, which evolves as:

$$dq_t = \mu_t q_t dt + \sigma_t q_t dZ_t$$

Boundaries for price of capital:

$$q = \max_{\underline{l}} \left( \frac{a - \underline{l}}{r - \phi (\underline{l}) + \delta} \right)$$

$$\bar{q} = \max_{\overline{l}} \left( \frac{a - \overline{l}}{r - \phi (\overline{l}) + \delta} \right)$$
Thus, the value of the capital hold by an expert generates:

1. Capital gains:
\[
\frac{d (k_t q_t)}{k_t q_t} = (\phi (\iota_t) - \delta + \mu^q_t + \sigma^q_t) \, dt + (\sigma + \sigma^q_t) \, dZ_t
\]

2. Dividend
\[
\frac{a - \iota_t}{q_t}
\]

3. Total return:
\[
dr_t^k = \frac{a - \iota_t}{q_t} + (\phi (\iota_t) - \delta + \mu^q_t + \sigma^q_t) \, dt + (\sigma + \sigma^q_t) \, dZ_t
\]

Similarly, return for a household:
\[
dr_t^h = \frac{a - \iota_t}{q_t} + (\phi (\iota_t) - \delta + \mu^q_t + \sigma^q_t) \, dt + (\sigma + \sigma^q_t) \, dZ_t
\]
Two components of risk on returns:

1. $\sigma dZ_t$ is exogenous risk caused by stochastic process for capital efficiency.

2. $\sigma^q_t dZ_t$ is endogenous risk caused by financial frictions. Without them, $\sigma^q_t = 0$ because $q_t = \bar{q}$.

- Even if experts are risk-neutral with respect to consumption, they exhibit risk-averse behavior because the return to investment is time-varying.

- Experts suffer losses when they want to buy more capital: its price is lower.
Because of an agency problem, experts must retain 100 percent of equity and finance the rest of their investment with risk-free debt.

Experts:

\[
\frac{dn_t}{n_t} = x_t dr_t^k + (1 - x_t) rdt - \frac{dc_t}{n_t}
\]

where \( n_t \geq 0 \) is net wealth, a fraction \( x_t \geq 0 \) invested in capital and a fraction \( 1 - x_t \) in the risk-free asset. In general, \( x_t > 1 \). The solvency constrain: \( n_t \geq 0 \).

Households:

\[
\frac{dn_t}{n_t} = x_t dr_t^k + (1 - x_t) rdt - \frac{dc_t}{n_t}
\]

with \( x_t \geq 0 \).
Equilibrium I

Definition

For any initial endowments of capital \(\{k_i^0, k_j^0; i \in \mathbb{I}, j \in \mathbb{J}\}\) such that:

\[
\int_{\mathbb{I}} k_i^0 \, di + \int_{\mathbb{J}} k_j^0 \, dj = K_0
\]

an equilibrium is described by a group of stochastic processes on the filtered probability space defined by the Brownian motion \(\{Z_t, t \geq 0\}\): the price process for capital \(\{q_t\}\), net worths \(\{n_i^t, n_j^t \geq 0\}\), capital holdings \(\{k_i^t, k_j^t \geq 0\}\), investment decisions \(\{\iota_i^t, \iota_j^t\}\), and consumption choices \(\{dc_i^t \geq 0, dc_j^t\}\) of individual agents \(i \in \mathbb{I}, j \in \mathbb{J}\) such that:

1. Given prices, all experts and households maximize.
2. Initial net worths satisfy \(n_i^0 = k_i^0 q_0\) and \(n_j^0 = k_j^0 q_0\) for all \(i \in \mathbb{I}, j \in \mathbb{J}\).
Equilibrium II

Definition

3. Markets clear:

\[
\int_\mathbb{I} k^i_t \, di + \int_\mathbb{J} k^j_t \, dj = K_t \\
\int_\mathbb{I} (dc^i_t) \, di + \int_\mathbb{J} (dc^j_t) \, dj = \left( \int_\mathbb{I} (a - \xi^i_t) \, k^i_t \, di + \int_\mathbb{J} (a - \xi^j_t) \, k^j_t \, dj \right) \, dt \\
k_t \, dt = \left( \int_\mathbb{I} (\phi \, (\xi^i_t) - \delta) \, k^i_t \, di + \int_\mathbb{J} (\phi \, (\xi^j_t) - \delta) \, k^j_t \, dj \right) \, dt + \sigma K_t \, dZ_t
\]
First, to maximize experts return with respect to $\iota_t$, from

$$dr^k_t = \frac{a - \iota_t}{q_t} + (\phi(\iota_t) - \delta + \mu^q_t + \sigma^q_t) \, dt + (\sigma + \sigma^q_t) \, dZ_t$$

set:

$$\phi'(\iota_t) = \frac{1}{q_t} \Rightarrow \iota_t = \iota(q_t)$$

Similarly, for a household:

$$\phi'(\iota_t) = \frac{1}{q_t} \Rightarrow \iota_t = \iota(q_t)$$

Thus:

$$\iota_t = \iota_t = \iota(q_t)$$
Now, define expected excess returns:

\[
E_t \frac{dr^k}{dt} - r = \frac{a - \ell(q_t)}{q_t} + \phi(\ell(q_t)) - \delta + \mu^q_t + \sigma^q_t - r
\]

\[
E_t \frac{dr^k}{dt} - r = \frac{a - \ell(q_t)}{q_t} + \phi(\ell(q_t)) - \delta + \mu^q_t + \sigma^q_t - r
\]
Consider a feasible strategy for the experts \( \{ x_t, d\zeta_t \} \) with payoff:

\[
\theta_t n_t = \mathbb{E}_t \int_t^\infty e^{-\rho(s-t)} dc_s
\]

where \( dc_t = n_t d\zeta_t \)

Then, for a price process for capital \( \{ q_t \} \) that generates a finite maximal payoff, a feasible strategy is optimal if and only if:

\[
\max_{\hat{x}_t \geq 0, \hat{\zeta}_t \geq 0} n_t d\hat{\zeta}_t + \mathbb{E}_t (\theta_t n_t)
\]

s.t.

\[
\frac{dn_t}{n_t} = \hat{x}_t d\kappa_t + (1 - \hat{x}_t) r dt - d\hat{\zeta}_t
\]

\[
\mathbb{E}_t e^{-\rho t} \theta_t n_t \to 0
\]
Moreover, for

\[ \frac{d\theta_t}{\theta_t} = \mu^\theta_t dt + \sigma^\theta_t dZ_t \]

a feasible strategy is optimal if and only if:

1. \( \theta_t \geq 1 \) and \( d\zeta_t > 0 \) only when \( \theta_t = 1 \).

2. \( \mu^\theta_t = \rho - r \).

3. Either:

   \[ x_t > 0 \text{ and } \frac{\mathbb{E}_t dr^k_t}{dt} - r = - (\sigma + \sigma^q_t) \sigma^\theta_t \] (desire to leverage)

   or

   \[ x_t = 0 \text{ and } \frac{\mathbb{E}_t dr^k_t}{dt} - r \leq - (\sigma + \sigma^q_t) \sigma^\theta_t \] (flight to quality)

4. \( \mathbb{E} e^{-\rho t} \theta_t n_t \rightarrow 0 \).
For the households,

\[
\frac{\mathbb{E}_t d r^k_t}{dt} - r \leq 0
\]

with equality if

\[
1 - \psi_t = \frac{1}{K_t} \int \int k^i_t dj > 0
\]

It can be verified that, in equilibrium,

\[
\psi_t q_t K_t > N_t = \int \int n^i_t di
\]
Wealth Distribution I

- Aggregate wealth:

\[ N_t = \int_\mathbb{I} n_t^i \, di \]

\[ q_t K_t - N_t = \int_\mathbb{J} n_t^i \, dj \]

- Hence, experts’ wealth share:

\[ \eta_t = \frac{N_t}{q_t K_t} \in [0, 1] \]
Wealth Distribution II

- Law of motion:

\[
\frac{d\eta_t}{\eta_t} = \left( \frac{\psi_t - \eta_t}{\eta_t} \right) \left( dr^k - rdt - (\sigma + \sigma^q_t)^2 dt \right) + \left( \frac{a - \tau(q_t)}{q_t} + (1 - \psi_t)(\delta - \delta) \right) dt - d\zeta_t
\]

- If \( \psi_t > 0 \Rightarrow x_t = 0 \) and we have

\[
r = \frac{\mathbb{E}_t dr^k_t}{dt} + (\sigma + \sigma^q_t) \sigma^\theta_t \Rightarrow rdt = \mathbb{E}_t dr^k_t + (\sigma + \sigma^q_t) \sigma^\theta_t dt
\]

- Then:

\[
\begin{align*}
dr^k - rdt &= dr^k - \mathbb{E}_t dr^k_t - (\sigma + \sigma^q_t) \sigma^\theta dt \\
&= (\sigma + \sigma^q_t) dZ_t - (\sigma + \sigma^q_t) \sigma^\theta dt \\
&= (\sigma + \sigma^q_t) \left( dZ_t - \sigma^\theta dt \right)
\end{align*}
\]
We can substitute

\[
\frac{d\eta_t}{\eta_t} = \left( \frac{\psi_t - \eta_t}{\eta_t} \right) (\sigma + \sigma^q_t) \left( dZ_t - \sigma^\theta_t dt - (\sigma + \sigma^q_t) dt \right)
\]

\[+ \left( \frac{a - \ell(q_t)}{q_t} + (1 - \psi_t) (\delta - \delta) \right) dt - d\zeta_t \]

\[= \left( -\frac{\psi_t - \eta_t}{\eta_t} (\sigma + \sigma^q_t) (\sigma + \sigma^\theta_t + \sigma^q_t) \right) dt \]

\[+ \frac{a - \ell(q_t)}{q_t} + (1 - \psi_t) (\delta - \delta) \]

\[+ \frac{\psi_t - \eta_t}{\eta_t} (\sigma + \sigma^q_t) dZ_t - d\zeta_t \]
Wealth Distribution IV

- By defining

\[
\mu_t^n = - \frac{\psi_t - \eta_t}{\eta_t} (\sigma + \sigma^q_t) \left( \sigma + \sigma^q_t + \sigma^\theta_t \right) + \frac{a - \iota(q_t)}{q_t} + (1 - \psi_t) (\delta - \delta) \\
\sigma_t^\eta = \frac{\psi_t - \eta_t}{\eta_t} (\sigma + \sigma^q_t)
\]

we get

\[
\frac{d\eta_t}{\eta_t} = \mu_t^n dt + \sigma_t^\eta dZ_t - d\zeta_t
\]
Markov Equilibria

- Search for functions:

\[
q_t = q(\eta_t) \\
\theta_t = \theta(\eta_t) \\
\psi_t = \psi(\eta_t)
\]

- Then, once we know \( \eta_t \), we can get \( q_t, \theta_t, \psi_t \), and from

\[
\frac{d\eta_t}{\eta_t} = \mu_t^n dt + \sigma_t^n dZ_t - d\zeta_t
\]

we get \( d\eta_t \).
Let us suppose that we know \((\eta, q(\eta), q'(\eta), \theta(\eta), \theta'(\eta))\).

1. Find \(\psi \in \left(\eta, \eta + \frac{q'(\eta)}{q(\eta)}\right)\) such that:

\[
\frac{a - a}{q(\eta)} + \delta - \delta + \sigma^\theta_t (\sigma + \sigma^q_t) = 0
\]

where:

\[
\sigma^\eta_t = \frac{1}{\eta} \frac{(\psi - \eta) a}{1 - (\psi - \eta) \frac{q'(\eta)}{q(\eta)}}
\]

\[
\sigma^q_t = \frac{q'(\eta)}{q(\eta)} \sigma^\eta_t \eta
\]

\[
\sigma^\theta_t = \frac{\theta'(\eta)}{\theta(\eta)} \sigma^\eta_t \eta
\]
A Proposition II

2. If $\psi > 1$, set $\psi = 1$ and recalculate $\sigma_t^\eta, \sigma_t^q, \sigma_t^\theta$.

3. Compute:

$$
\mu_t^n = -\sigma_t^\eta \left( \sigma + \sigma_t^q + \sigma_t^\theta \right) + \frac{a - \lambda(q(\eta))}{q(\eta)} + (1 - \psi)(\delta - \delta)
$$

$$
\mu_t^q = r - \frac{a - \lambda(q(\eta))}{q(\eta)} - \phi(q(\eta)) + \delta - \sigma \sigma_t^q - \sigma_t^\theta (\sigma + \sigma_t^q)
$$

$$
\mu_t^\theta = \rho - r
$$

$$
q''(\eta) = \frac{2(\mu_t^q q(\eta) - q'(\eta) \mu_t^n \eta)}{(\sigma_t^\eta)^2 \eta^2}
$$

$$
\theta''(\eta) = \frac{2(\mu_t^\theta \theta(\eta) - \theta'(\eta) \mu_t^n \eta)}{(\sigma_t^\eta)^2 \eta^2}
$$
A Proposition III

4 Use boundary conditions

\[ q' (\eta^*) = 0, \ q (0) = \underline{q} \]
\[ \theta (\eta^*) = 1, \ \theta' (\eta^*) = 0 \]
\[ \lim \theta (\eta) = \infty \]

where \( \eta^* \) is the reflecting boundary when experts consume.
An Algorithm

- Set $q(0) = q$, $\theta(0) = 1$, and $\theta'(0) =$ small number.
- Set $q_L = 0$ and $q_H =$ large number.
- Guess $q'(0) = \frac{q_H + q_L}{2}$ and solve for $q(\eta)$ and $\theta(\eta)$ until the first of the three conditions holds:
  1. $q(\eta) = \bar{q}$.
  2. $\theta'(\eta) = 0$.
  3. $q'(\eta) = 0$.
- If $q'(\eta) = 0$, set $q_L = q'(0)$, otherwise $q_H = q'(0)$.
- Iterate until convergence.
- Check that $q'(\eta)$ and $\theta'(\eta)$ reach 0 at the same point $\eta^*$. 
- Normalize $\theta(\eta) = \frac{\theta(\eta)}{\theta(\eta^*)}$ to match boundary condition.
Calibration

- Parameters

<table>
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<th>$\rho$</th>
<th>$r$</th>
<th>$a$</th>
<th>$\bar{a}$</th>
<th>$\delta$</th>
<th>$\bar{\delta}$</th>
<th>$\sigma$</th>
<th>$\phi(\iota)$</th>
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<td>0.05</td>
<td>0.1</td>
<td>0.05</td>
<td>0.03</td>
<td>0.05</td>
<td>0.4</td>
<td>0.1\sqrt{1+20\iota}</td>
</tr>
</tbody>
</table>

- This implies $q = 0.5858$ and $\bar{q} = 1.3101$. 
Three Inefficiencies

1. Capital misallocation: for low $\eta_t$, households manage part of the capital ($\psi < 1$).

2. Under investment: $\iota(q_t) < \iota(\bar{q})$.

3. Consumption distortions: experts should only consume at time 0.

Note: inefficiencies get worse for low $\eta_t$. 
Adverse Feedback Loop

- Adverse shock
  - $k_t \downarrow$
  - $n_t \downarrow$ due to leverage

- $n_t \downarrow$

- $q_t \downarrow$

- Capital demand $\downarrow$

Diagram shows the flow of how adverse feedback can lead to a decrease in capital and demand.
Figure 9: The peculiar dynamics of VIX: 2004-2012. Source: CBOE