Aiyagari Models

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A trio of models

- Different GE heterogenous household models with incomplete markets make different assumptions about how to interpret the assets in the household consumption-savings problem, and how they are supplied:
  1. *Huggett model*: private IOUs in zero net supply (Huggett, 1993).
  2. *Bewley model*: money or bonds in positive net supply (Imrohoroğlu, 1989).
  3. *Aiyagari model*: capital in positive net supply (Aiyagari, 1994).
- That is why the model is sometimes called the Bewley-Huggett-Aiyagari model.
- We will mainly focus on the (canonical) Aiyagari model.
- Later, we will say a few things about the other two models.

Model

Building blocks

- Continuum of households (vs. models with finite number/types of agents).
- One firm renting aggregate capital.
- No aggregate uncertainty.
- Individuals are subject to idiosyncratic shocks to their labor income.
- Incomplete markets.

Households

- Continuum of measure 1 of households.
- Preferences for household i:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

- Budget constraint:

$$c_t + a_{t+1} = w_t y_t + (1 + r_t) a_t$$

- We could consider a hand-to-mouth (i.e., autarky) variation: $c_t = w_t y_t$.
- Initial conditions y_0, $a_0 \geq 0$.
- Borrowing constraint $a_{t+1} \geq 0$.

Labor endowment

- Stochastic labor endowment process $\{y_t\}_{t=0}^{\infty}$:

$$y_t \in Y = \{y_1, y_2, \dots y_N\}$$

- Markov process with transitions $\pi(y'|y) > 0$.
- Interpretation.
- Common for all households, but realizations are specific for each individual.
- Law of large numbers: $\pi(y'|y)$ is also the deterministic fraction of the population that has this particular transition (Uhlig, 1996).
- Unique stationary distribution associated with π, denoted by Π.
- Total labor endowment in the economy at each point of time:

$$L = \sum_y \Pi(y) y$$

Firm

- Perfectly competitive firm with neoclassical technology:

$$Y_t = F(K_t, L_t)$$

- Depreciation rate: $0 < \delta < 1$.
- Aggregate resource constraint:

$$C_t + K_{t+1} - (1 - \delta)K_t = F(K_t, L_t)$$

- The only net asset in economy is physical capital.
- No state-contingent claims (i.e. incomplete markets).
- Remark: ownership of the firm.

Recursive formulation, I

- (a, y): household state.
- $\Phi(a, y)$: aggregate state variable.
- $A = [0, \infty)$: set of possible asset holdings.
- $B(A)$: Borel σ-algebra of A.
- Y: set of possible labor endowment realizations.
- $P(Y)$: power set of Y.
- $Z = A \times Y$ and $B(Z) = P(Y) \times B(A)$.
- $\mathcal{M}$ the set of all probability measures on the measurable space $(Z, B(Z))$.

Recursive formulation, II

- Household problem in recursive formulation:

$$v(a,y;\Phi) = \max_{c\geq 0, a'\geq 0} u(c) + \beta \sum_{y'\in Y} \pi(y'|y) v(a',y';\Phi')$$

$$\begin{aligned} \text{s.t. } c + a' &= w(\Phi)y + (1 + r(\Phi))a \\ \Phi' &= H(\Phi) \end{aligned}$$

- Function $H : \mathcal{M} \rightarrow \mathcal{M}$ is called the aggregate "law of motion."
- Note the complexity of the operator.

Recursive competitive equilibrium

A RCE is value function $v : Z \times \mathcal{M} \rightarrow R$, household policy functions $a', c : Z \times \mathcal{M} \rightarrow R$, firm policy functions $K, L : \mathcal{M} \rightarrow R$, pricing functions $r, w : \mathcal{M} \rightarrow R$ and law of motion $H : \mathcal{M} \rightarrow \mathcal{M}$ s.t.

1. v, a', c are measurable with respect to $\mathcal{B}(Z)$, v satisfies Bellman equation and a', c are the policy functions, given $r()$ and $w()$.
2. K, L satisfy, given $r()$ and $w()$

$$\begin{aligned} r(\Phi) &= F_K(K(\Phi), L(\Phi)) - \delta \\ w(\Phi) &= F_L(K(\Phi), L(\Phi)) \end{aligned}$$

3. For all $\Phi \in \mathcal{M}$, $L(\Phi) = \int y d\Phi$ and

$$K'(\Phi') = K(H(\Phi)) = \int a'(a, y; \Phi) d\Phi$$

$$\int c(a, y; \Phi) d\Phi + \int a'(a, y; \Phi) d\Phi = F(K(\Phi), L(\Phi)) + (1 - \delta) K(\Phi)$$

4. Aggregate law of motion H is generated by π and a'.

Transition functions

- Define transition function $Q_\Phi : Z \times \mathcal{B}(Z) \to [0,1]$ by

$$Q_\Phi((a,y),(\mathcal{A},\mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \begin{cases} \pi(y'|y) \text{ if } a'(a,y;\Phi) \in \mathcal{A} \\ 0 \text{ else} \end{cases}$$

  for all $(a,y) \in Z$ and all $(\mathcal{A},\mathcal{Y}) \in \mathcal{B}(Z)$.

- $Q_\Phi((a,y),(\mathcal{A},\mathcal{Y}))$ is the probability that an agent with current assets a and current income y ends up with assets a' in $\mathcal{A}$ tomorrow and income y' in $\mathcal{Y}$ tomorrow.

- Hence

$$\begin{aligned} \Phi'(\mathcal{A},\mathcal{Y}) &= (H(\Phi))(\mathcal{A},\mathcal{Y}) \\ &= \int Q_\Phi((a,y),(\mathcal{A},\mathcal{Y}))\Phi(da \times dy) \end{aligned}$$

A stationary recursive competitive equilibrium

A stationary RCE is value function $v : Z \to R$, household policy functions $a', c : Z \to R$, firm policies K, L, prices r, w and a measure $\Phi \in \mathcal{M}$ such that

1. v, a', c are measurable with respect to $B(Z)$, v satisfies the household's Bellman equation and a', c are associated policy functions, given r, w.
2. K, L satisfy, given r, w:

$$r = F_k(K, L) - \delta$$
$$w = F_L(K, L)$$

3. $L = \int y d\Phi$ and $K = \int a'(a, y) d\Phi$ and

$$\int c(a, y) d\Phi + \int a'(a, y) d\Phi = F(K, L) + (1 - \delta)K$$

4. Let Q be transition function induced by π and a'. $\forall (\mathcal{A}, \mathcal{Y}) \in \mathcal{B}(Z)$

$$\Phi(\mathcal{A}, \mathcal{Y}) = \int Q((a, y), (\mathcal{A}, \mathcal{Y})) d\Phi$$

Characterizing the stationary RCE

- Recall that L is exogenously given.
- Thus, from

$$r = F_k(K, L) - \delta$$

$$w = F_L(K, L)$$

  we can get w as a function of r (with $w'(r) < 0$).
- Example:

$$Y = K^{\alpha} L^{1-\alpha}$$

  with:

$$r = \alpha K^{\alpha-1} L^{1-\alpha} - \delta \Rightarrow K = \left(\frac{r+\delta}{\alpha}\right)^{\frac{1}{\alpha-1}} L$$

  and

$$w = (1-\alpha) K^{\alpha} L^{1-\alpha} = (1-\alpha)\, \alpha^{\frac{\alpha}{1-\alpha}} (r+\delta)^{\frac{\alpha}{\alpha-1}} L$$

Existence and uniqueness

- By Walras' law, we can forget about goods market and we only need to check input market clearing.
- Define asset market clearing condition:

$$K = K(r) = \int a'(a, y) d\Phi \equiv Ea(r)$$

- Then:

$$r = F_k(K(r), L) - \delta$$

- Existence and uniqueness of stationary RCE boils down to one equation in one unknown.
- From assumptions on production function, $K(r)$ is continuous, strictly decreasing function on $r \in (-\delta, \infty)$ with

$$\begin{aligned} \lim_{r \to -\delta} K(r) &= \infty \\ \lim_{r \to \infty} K(r) &= 0 \end{aligned}$$

A useful result

Theorem (Huggett, 1993)
For $\beta < 1$, $r > -1$, $y_1 > 0$, and CRRA utility with $\sigma > 1$, the functional equation has a unique solution v which is strictly increasing, strictly concave, and continuously differentiable in its first argument. The optimal policies are continuous functions that are strictly increasing (for $c(a, y)$) or increasing or constant at zero (for $a'(a, y)$).

Similar results can be proved for the *iid* case and arbitrary bounded U with $\rho > r$ and $\rho > 0$, see Aiyagari (1994).

Boundedness of the state space: requires $\frac{1}{\beta} > 1 + r$ and additional assumptions (*iid* and limiting exponent of u_c or Huggett's assumptions). Let $\bar{a}$ denote upper bound.

A fixed point problem, I

- From now on assume $\exists \bar{a}$ s.t. $a'(\bar{a}, y_N) = \bar{a}$ and $a'(a, y) \leq \bar{a}$ for all $y \in Y$ and all $a \in [0, \bar{a}]$. State space $Z = [0, \bar{a}] \times Y$ and optimal policy $a'_r(a, y)$ defined on Z, indexed by r.
- Asset demand

$$Ea(r) = \int a'_r(a, y) d\Phi_r$$

- Need Φ_r that satisfies

$$\Phi_r(\mathcal{A}, \mathcal{Y}) = \int Q_r((a, y), (\mathcal{A}, \mathcal{Y})) d\Phi_r$$

where Q_r is the Markov transition function defined by a_r as

$$Q_r((a, y), (\mathcal{A}, \mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \begin{cases} \pi(y'|y) \text{ if } a'_r(a, y) \in \mathcal{A} \\ 0 \text{ else} \end{cases}$$

A fixed point problem, II

- Need to establish that operator $T_r^* : \mathcal{M} \to \mathcal{M}$ defined by

$$(T_r^*(\Phi))(\mathcal{A}, \mathcal{Y}) = \int Q_r((a, y), (\mathcal{A}, \mathcal{Y}))d\Phi$$

has a unique fixed point.

Stationary distributions

Theorem (Hopenhayn and Prescott, 1992)
If the state space Z is compact and

1. *Q_r is a transition function,*
2. *Q_r is increasing,*
3. *there exists $z^* \in Z$, $\varepsilon > 0$ and N such that*

$$P^N(d, \{z : z \leq z^*\}) > \varepsilon \text{ and } P^N(c, \{z : z \geq z^*\}) > \varepsilon$$

   where d is maximal element of Z and c is minimal element of Z,

then, the operator T_r^* has a unique fixed point Φ_r and for all $\Phi_0 \in M$ the sequence of measures defined by

$$\Phi_n = (T^*)^n \Phi_0$$

converges weakly to Φ_r.

Existence of stationary distributions, I

- Assumption 1 requires that Q_r is transition function, i.e., $Q_r(z,.)$ is probability measure on $(Z, B(Z))$ for all $z \in Z$ and $Q_r(., Z)$ is $B(Z)$-measurable $\forall Z \in B(Z)$. Use that $a'(a,y)$ is continuous.

- The assumption that Q_r is increasing requires that for any nondecreasing function $f : Z \to R$ we have that
$$(Tf)(z) = \int f(z')Q_r(z, dz')$$
is also nondecreasing. Note that $a'(a,y)$ is increasing in (a,y).

- Monotone mixing condition 3. satisfied? Pick $z^* = (\frac{1}{2}(a'(0,y_N) + \bar{a}), y_1)$. Start at d with a sequence of bad shocks y_1 and from c with a sequence of good shocks y_N.

Existence of stationary distributions, II

- Conclusion of the theorem assures existence of a unique invariant measure Φ_r which can be found by iterating on the operator T^*.
- Convergence is in the weak sense: for every continuous and bounded real-valued function f on Z, we have

$$\lim_{n \to \infty} \int f(z) d\Phi_n = \int f(z) d\Phi_r$$

Existence of equilibrium

- From previous results, function $Ea(r)$ is well-defined on $r \in [-\delta, \frac{1}{\beta} - 1)$.
- Since $a'_r(a, y)$ is continuous jointly in (r, a) and Φ_r is continuous in r (weak convergence), the function $Ea(r)$ is a continuous function of r on $[-\delta, \frac{1}{\beta} - 1)$.
- $\lim_{r \to -\delta} Ea(r) < \infty$ is fine, but what about

$$\lim_{r \to \frac{1}{\beta} - 1} Ea(r) > K(\frac{1}{\beta} - 1)$$

- If both satisfied, then there exists r^* such that

$$K(r^*) = Ea(r^*)$$

  and a stationary RCE.
- We cannot ensure uniqueness.
- We lack results about stability.

Interest rate in equilibrium

- Complete markets model: $r^{CM} = \frac{1}{\beta} - 1$.
- With incomplete markets: $r^* < r^{CM}$.
- Why? Overaccumulation of capital and oversaving (because of precautionary reasons: liquidity constraints, prudence, or both).
- Policy implications.

Computation

Computation of the canonical Aiyagari model

Involves three steps:

1. Fix an $r \in (-\delta, \frac{1}{\beta} - 1)$. For a fixed r, solve household's recursive problem. This yields a value function v_r and decision rules a'_r, c_r.
2. The policy function a'_r and π induce Markov transition function Q_r. Compute the unique stationary measure Φ_r associated with this transition function.
3. Compute excess demand for capital
$$d(r) = K(r) - Ea(r)$$
If zero, stop, if not, adjust r.

Solving the household's recursive problem

- Any acceptable solution method for recursive problems is valid: value function iteration, projection, etc.
- However, speed is at a premium.
- Thus, value function iteration (at least, without further refinements) might not be fast enough.
- Standard "tricks": monotonicity and concavity.
- Be smart about initial guesses in the updates.
- Fix variable values in steady state, not parameters!
- Also, explore multigrid schemes.

How to compute the unique stationary measure, I

- Grid. Suppose $A = \{a_1, \ldots, a_M\}$.
- Then Φ is $M * N \times 1$ column vector and $Q = (q_{ij,kl})$ is $M * N \times M * N$ matrix with

$$q_{ij,kl} = \Pr\left((a', y') = (a_k, y_l) | (a, y) = (a_i, y_l)\right)$$

- Stationary measure Φ satisfies matrix equation

$$\Phi = Q^T \Phi$$

- Φ is (rescaled) eigenvector associated with eigenvalue $\lambda = 1$ of Q^T.
- Q^T is a stochastic matrix and thus has at least one unit eigenvalue. If it has more than one unit eigenvalue, continuum of stationary measures.

How to compute the unique stationary measure, II

- Variation of grid method I: allocate mass between two grid points according to relative distance.
- Variation of grid method II: uniform mass between two grid points.
- Both cases: sufficiently small grid. Otherwise, no convergence.
- Simulation.
- Parameterized cross-sectional distribution: Algan, Allais, and Den Haan (2006).

Parallelization

- We can parallelize the value function for a given interest rate.
- We can also parallelize the computation of the stationary distribution.
- You cannot (easily) parallelize the iteration over prices.

Transitional dynamics

Transitional dynamics

- Often, we are interested in the effects of the change in a parameter of the model (transitory or permanent).
- We want to compute both the new steady state and the transitional dynamics.
- Example: permanent introduction of a capital income tax at rate τ. Receipts are rebated lump-sum to households as government transfers T.

Model with a capital income tax

- State space: $Z = Y \times \mathbf{R}_+$, the set of all possible (y, a).
- Let $\mathcal{B}(Z) = \mathcal{P}(Y) \times \mathcal{B}(\mathbf{R}_+)$ and $\mathbf{M}$ be the set of all finite measures on the measurable space $(Z, \mathcal{B}(Z))$.
- Household problem:

$$v_t(a, y) = \max_{c \geq 0, a' \geq 0} u(c) + \beta \sum_{y' \in Y} \pi(y'|y) v_{t+1}(a', y')$$

$$\text{s.t. } c + a' = w_t y + (1 + (1 - \tau_t) r_t) a + T_t$$

A competitive equilibrium with taxes, I

Given initial distribution Φ_0 and fiscal legislation $\{\tau_t\}_{t=0}^{\infty}$, a competitive equilibrium is sequence of functions for the household $\{v_t, c_t, a_{t+1} : Z \to \mathbf{R}\}_{t=0}^{\infty}$, sequence of firm production plans $\{L_t, K_t\}_{t=0}^{\infty}$, factor prices $\{w_t, r_t\}_{t=0}^{\infty}$, government transfers $\{T_t\}_{t=0}^{\infty}$, and sequence of measures $\{\Phi\}_{t=1}^{\infty}$ s.t. $\forall t$,

- Given $\{w_t, r_t\}$ and $\{T_t, \tau_t\}$ the functions $\{v_t\}$ solve Bellman equation in t and $\{c_t, a_{t+1}\}$ are associated policy functions.
- Prices w_t and r_t satisfy

$$w_t = F_L(K_t, L_t)$$
$$r_t = F_K(K_t, L_t) - \delta$$

- Government Budget Constraint: for all $t \geq 0$.

$$T_t = \tau_t r_t K_t$$

A competitive equilibrium with taxes, II

- Market Clearing:

$$\begin{aligned}\int c_t(a_t, y_t) d\Phi_t + K_{t+1} &= F(K_t, L_t) + (1-\delta)K_t \\ L_t &= \int y_t d\Phi_t \\ K_{t+1} &= \int a_{t+1}(a_t, y_t) d\Phi_t\end{aligned}$$

- Aggregate Law of Motion: Define Markov transition functions $Q_t : Z \times \mathcal{B}(Z) \to [0,1]$ induced by the transition probabilities π and optimal policy $a_{t+1}(y,a)$ as

$$Q_t((a,y),(\mathcal{A},\mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \begin{cases} \pi(y'|y) \text{ if } a_{t+1}(a,y) \in \mathcal{A} \\ 0 \text{ else} \end{cases}$$

for all $(a,y) \in Z$ and all $(\mathcal{A},\mathcal{Y}) \in \mathcal{B}(Z)$. Then for all $(\mathcal{A},\mathcal{Y}) \in \mathcal{B}(Z)$

$$\Phi_{t+1}(\mathcal{A},\mathcal{Y}) = [\Gamma_t(\Phi_t)](\mathcal{A},\mathcal{Y}) = \int Q_t((a,y),(\mathcal{A},\mathcal{Y})) d\Phi_t$$

Stationary equilibrium and transitions

- A stationary equilibrium is an equilibrium such that all elements of the equilibrium that are indexed by t are constant over time.

- Transitions are likely to be asymptotic.

- However, *assume* that after T periods the transition from old to new stationary equilibrium is completed.

- Under the assumption $v_T = v_\infty$, for a given sequence of prices $\{r_t, w_t\}_{t=1}^{T}$ household problem can be solved backwards.

Computation, I

- Fix T.
- Compute stationary equilibrium $\Phi_0, v_{0,} r_0, w_0, K_0$ associated with $\tau = \tau_0 = 0$.
- Compute stationary equilibrium $\Phi_\infty, v_\infty, r_\infty, w_\infty, K_\infty$ associated with $\tau_\infty = \tau$. Assume that

$$\Phi_T, v_T, r_T, w_T, K_T = \Phi_\infty, v_\infty, r_\infty, w_\infty, K_\infty$$

- Guess sequence of capital stocks $\{\hat{K}_t\}_{t=1}^{T-1}$ The capital stock at time $t = 1$ is determined by decisions at time 0, $\hat{K}_1 = K_0$. Note that $L_t = L_0 = L$ is fixed. We also obtain

$$\begin{aligned} \hat{w}_t &= F_L(\hat{K}_t, L) \\ \hat{r}_t &= F_K(\hat{K}_t, L) - \delta \\ \hat{T}_t &= \tau_t \hat{r}_t \hat{K}_t \end{aligned}$$

Computation, II

- Since we know $v_T(a, y)$ and $\{\hat{r}_t, \hat{w}_t, \hat{T}_t\}_{t=1}^{T-1}$ we can solve for $\{\hat{v}_t, \hat{c}_t, \hat{a}_{t+1}\}_{t=1}^{T-1}$ backwards.
- With policy functions $\{\hat{a}_{t+1}\}$ define transition laws $\{\hat{\Gamma}_t\}_{t=1}^{T-1}$. We know $\Phi_0 = \Phi_1$ from the initial stationary equilibrium. Iterate the distributions forward

  $$\hat{\Phi}_{t+1} = \hat{\Gamma}_t(\hat{\Phi}_t)$$

  for $t = 1, \ldots, T-1$.
- With $\{\hat{\Phi}_t\}_{t=1}^{T}$ we can compute, for $t = 1, \ldots, T$.

  $$\hat{A}_t = \int a d\hat{\Phi}_t$$

Computation, III

- Test
$$\max_{1\leq t<T}\left|\hat{A}_t-\hat{K}_t\right|<\varepsilon$$
If yes, go to next step. If not, adjust your guesses for $\{\hat{K}_t\}_{t=1}^{T-1}$.

- Test
$$\left\|\hat{\Phi}_T-\Phi_T\right\|<\varepsilon$$
If yes, the transition converges smoothly into the new steady state and we are done and should save $\{\hat{v}_t,\hat{a}_{t+1},\hat{c}_t,\hat{\Phi}_t,\hat{r}_t,\hat{w}_t,\hat{K}_t\}$. If not, increase T.

- We can be smart with the initial guess: compute associated RA transition.

Welfare analysis

Welfare analysis

- This procedure determines aggregate variables such as r_t, w_t, Φ_t, K_t and individual decision rules c_r, a_{t+1}.
- The value functions enable us to make statements about the welfare consequences of the tax reform.
- We have value functions $\{v_t\}_{t=0}^{T}$.
- Interpretation of the value functions: $v_0(a, y)$, $v_1(a, y)$ and $v_T(a, y) = v_\infty(a, y)$.
- We can use v_0, v_1 and v_T to determine the welfare consequences from the reform.

Consumption equivalent variation, I

- Suppose that

$$U(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

- Optimal consumption allocation in initial stationary equilibrium, in sequential formulation, $\{c_s\}_{s=0}^{\infty}$:

$$v_0(a,y) = \mathbb{E}_0 \sum_{s=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

- If increase consumption in each date, in each state, in the old stationary equilibrium, by a fraction g. Then $\{(1+g)c_s\}_{s=0}^{\infty}$ and:

$$\begin{aligned} v_0(a,y;g) &= \mathbb{E}_0 \sum_{s=0}^{\infty} \beta^t \frac{[(1+g)c_t]^{1-\sigma}}{1-\sigma} = (1+g)^{1-\sigma} \mathbb{E}_0 \sum_{s=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \\ &= (1+g)^{1-\sigma} v_0(a,y) \end{aligned}$$

Consumption equivalent variation, II

- By what percent g do we have to increase consumption in the old stationary equilibrium for agent to be indifferent between old stationary equilibrium and transition induced by policy reform?
- This percent g solves

$$\begin{aligned} v_0(a, y; g) &= v_1(a, y) \\ g(a, y) &= \left[\frac{v_1(a, y)}{v_0(a, y)}\right]^{\frac{1}{1-\sigma}} - 1 \end{aligned}$$

- $g(a, y) > 0$ iff $v_1(a, y) > v_0(a, y)$. $g(a, y)$ varies by (a, y).

Steady state welfare comparisons

- Steady state welfare gain (of agent being born with (a, y) into new as opposed to old stationary equilibrium.):

$$g_{ss}(a, y) = \left[\frac{v_T(a, y)}{v_0(a, y)}\right]^{\frac{1}{1-\sigma}} - 1$$

- Define as expected steady state welfare gain

$$g_{ss} = \left[\frac{\int v_T(a, y) d\Phi_T}{\int v_0(a, y) d\Phi_0}\right]^{\frac{1}{1-\sigma}} - 1$$

- For these measures need not compute transition path. But: welfare measures based on steady state comparisons may be misleading.

An application

Motivation

- Owner-occupied housing takes account of a substantial portion in most households' portfolios and is also associated with policy goals.
- What factors affect housing decisions is an important question.
- Over the past decades, we have seen big changes in how likely people are getting married and divorced.
- Chang (2019) argues the evolving likelihood of marriage and divorce is an essential factor in accounting for some changes in housing demand.

Specific questions

- What effect does the change in likelihood of marriage and divorce have on housing decisions?
- To what extent does this channel help account for the change in housing decisions?

Key Mechanism

- What effect does the change in likelihood of marriage and divorce have on housing decisions?
    - Marriage more likely $\rightarrow$ Singles do not buy a house due to i) transaction cost to sell/resize and ii) less savings from expecting free rider problem.
    - Divorce more likely $\rightarrow$ The married invest less in housing due to cost to split.

Periods for Quantitative Analysis

- To what extent does this channel help account for the change in housing decisions?
  - **Compare 1970 vs. 1995**: $\begin{cases} \text{Similar real house prices} \\ \text{Different probabilities of marital transitions} \end{cases}$
  - **Include** the periods after 1995 with changing house prices.

Motivation

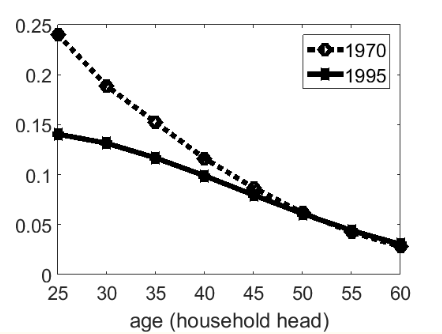
- Comparing 1995 to 1970,

Fact (1) Single households' homeownership rate increased significantly.

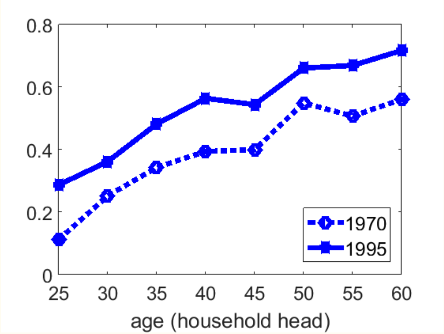
Fact (2) Young married households held a lower fraction of total asset in housing.

Fact (1) on singles

Likelihood of Marriage

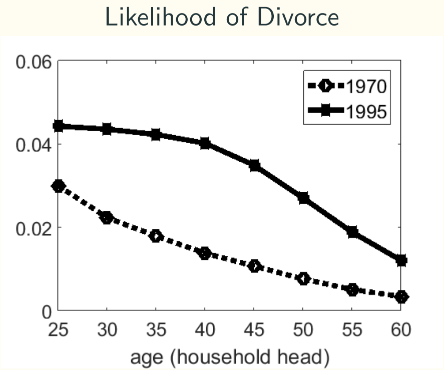


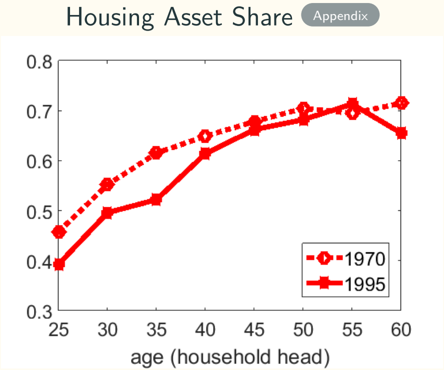
Homeownership



Single households' homeownership rate increased significantly.

Fact (2) on married households





Young married households held a lower fraction of total asset in housing.

Motivation

- Comparing 1995 to 1970,

Fact 1) Single households' homeownership rate increased significantly.

Fact 2) Young married households held a lower fraction of total asset in housing.

- **Hypothesis:** Change in housing decisions is affected by change in likelihood of marital transitions.

- **Assumption:** marriage and divorce as exogenous shocks.

What does the paper do?

- **Build** a life-cycle model of single and married households that face exogenous marital transition shocks.
- **Estimate** the parameters by matching the data moments in 1995.
- **Quantify** how much of the change in likelihood of marriage/divorce can account for the change in housing variables (1970 *vs.* 1995).
- **Assess** how the model accounts for housing choices in recent years with changing housing prices (1995 *vs.* boom in the mid 2000s).

Preview of results

- **Main channel:** Change in marriage and divorce probabilities
- **1970 vs. 1995**
  - The main channel accounts for **29%** of the change in the single's homeownership rate.
    - Other changes: Downpayment, earnings risk, spousal labor productivity
  - Without this channel, the married's housing asset share is generated to **increase**, which is **opposite** to the data's pattern falling by **11%**.
- **1995 vs. Boom in the mid 2000s**
  - The decreasing likelihood of marriage increases the single's homeownership rate by **6.8%**.
    - Other changes: house prices, beliefs, credit constraints, wage
  - This demographic force contributes to replicate partly the increase in homeownership when the house price was expensive.

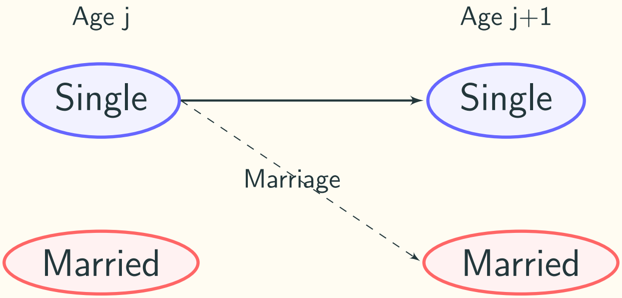
Model — Overview

- Life-cycle model of singles and married households.
- Households decide how much to consume, rent, save in non-housing and housing assets, and work (head/spouse).
- Households face $\begin{cases} \text{age-dependent marital transition shock} \\ \text{idiosyncratic labor productivity, house price shock} \end{cases}$
- A finite number of housing sizes are available to own.

  (Pros) higher service flow than renting; collateral for borrowing

  (Cons) substantial transaction cost whenever adjusted
- A status change makes the house owned prior to the change not suited.

Model — Overview



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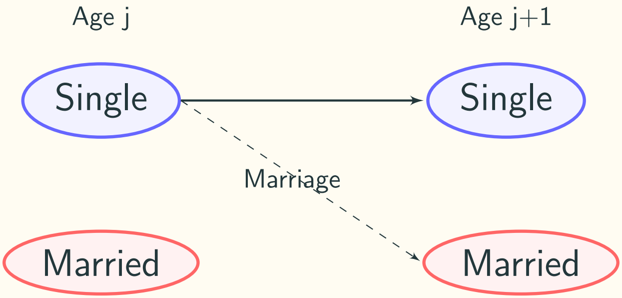
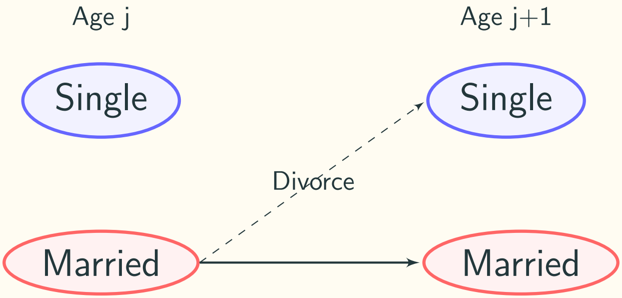


- Marriage:
  1) **Gain** economies of scale
  2) **Sell** any house owned as single
  3) **Pool** the asset with a spouse
  4) **Take on** roles (different labor productivity between spouses)

Model — Overview



- Divorce:
  1) **Lose** economies of scale
  2) **Sell** any house owned as married
  3) **Split** the asset equally
  4) **Go back** to single's labor productivity

Preferences

- **Single Agent**:

$$u(c, s, l) = \frac{(c^{\alpha} s^{1-\alpha})^{(1-\sigma)}}{1-\sigma} - B_s \frac{l^{1+\frac{1}{\gamma}}}{1+\frac{1}{\gamma}} - \phi \cdot I(l > 0) + uhp(l)$$

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where $s = m + \zeta(h) \cdot h$ (m: renting, h: owned housing) and

$$\zeta(h) = \left(\zeta_1 + \zeta_2 \cdot \frac{h - h_{min}}{h_{max} - h_{min}}\right)$$

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$uhp(l)$ reflects utility from home production

$$uhp(l) = \begin{cases} 0 & \text{if working, } l > 0 \\ \omega_{uhp} & \text{if not working, } l = 0. \end{cases}$$

Preferences

- **Married Agent**:

$$u^{\text{head}}(c,s,l,\widetilde{l}) = \varphi_j \frac{((\gamma_e c)^{\alpha}(\gamma_e s)^{1-\alpha})^{1-\sigma}}{1-\sigma} - B_m \frac{l^{1+\frac{1}{\gamma}}}{1+\frac{1}{\gamma}} - \phi \cdot I(l>0) + uhp(l,\widetilde{l})$$

$$u^{\text{spouse}}(c,s,l,\widetilde{l}) = \varphi_j \frac{\left((\gamma_e c)^{\alpha}(\gamma_e s)^{1-\alpha}\right)^{1-\sigma}}{1-\sigma} - \widetilde{B}_m \frac{\widetilde{l}^{1+\frac{1}{\gamma}}}{1+\frac{1}{\gamma}} - \widetilde{\phi} \cdot I(\widetilde{l}>0) + uhp(l,\widetilde{l})$$

Preferences

- **Married Agent**:

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c, s are "joint" consumption and housing services.

γ_e transforms them into per capita terms (e.g. $\gamma_e > 0.5$).

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Utility from home production is modeled as

$$uhp(l, \widetilde{l}) = \begin{cases} 0 & \text{if both working} \\ \frac{\omega_{uhp}}{n_{\psi}} & \text{if only one spouse working} \\ \omega_{uhp} & \text{if no one working.} \end{cases}$$

Single household's problem (Age $j < J$)

- State variables: $X_j^s := (j, a, h, y)$. Heterogeneity across age(j), total asset(a), housing asset(h), productivity(y)

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$$V^s(X_j^s) = \max_{c,m,b',h',l} \Big\{ u(c,s,l) + \beta \big[q_{ss,j} \cdot \mathbb{E}V^s(j+1, a', h', y') + q_{sm,j} \cdot \mathbb{E}V^m(j+1, (a' + \widetilde{a}') \boxed{-\kappa_s P^H (h' + \widetilde{h}')}, \boxed{0}, y', \widetilde{y}') \big] \Big\}$$

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$$\text{s.t.} \quad c + m + b' + P^H h' = wyl + a - I(h' \neq h) \cdot (\kappa_b P^H h' + \kappa_s P^H h)$$

$$c \geq 0, \quad s \geq 0, \quad b' \geq -\eta P^H h' \text{ with } 0 \leq \eta \leq 1.$$

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$$V^s(X_j^s) = \max_{c,m,b',h',l} \Big\{ u(c,s,l) \ + \beta\big[q_{ss,j} \cdot \mathbb{E}V^s(j+1, a', h', y')$$

$$+ q_{sm,j} \cdot \mathbb{E}V^m(j+1, (a' + \widetilde{a}') \boxed{-\kappa_s P^H(h' + \widetilde{h}')}, \boxed{0}, y', \widetilde{y}')\big] \Big\}$$

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$$c \geq 0, \quad s \geq 0, \quad b' \geq -\eta P^H h' \ \text{ with } \ 0 \leq \eta \leq 1.$$

$$a' = (1 + r(b'))b' + P^H(1 - \delta')h', \quad \text{where} \quad r(b') = \begin{cases} r & \text{if } b' \geq 0 \\ r^H & \text{otherwise} \end{cases}$$

Married household's problem

- State variables: $X_j^m := (j, a, h, y, \widetilde{y})$.

- Choose spousal labor supply $\widetilde{l}$. So labor income is $w(yl + \widetilde{y}\widetilde{l})$.

- A married household solves a joint problem which maximizes the average utility with equal weights.

$$V^m(X_j^m) = \max_{c,m,b',h',l,\widetilde{l}} \Big\{ u(c,s,l,\widetilde{l}) \; + \beta \Big[q_{mm,j} \cdot \mathbb{E} V^m(j+1, a', h', y', \widetilde{y}') + q_{ms,j} \cdot \mathbb{E} V^s\Big(j+1, \boxed{\frac{1}{2}(a' - \kappa_s P^H h')}, \boxed{0}, y'\Big)\Big]\Big\}$$

$$\text{where} \quad u(c,s,l,\widetilde{l}) = \frac{u^{\text{head}}(c,s,l,\widetilde{l}) + u^{\text{spouse}}(c,s,l,\widetilde{l})}{2}$$

- There is no margin of disagreement between the spouses.

Shocks

- Labor efficiency shock y is modeled to be combination of age trend $\chi(j)$ and idiosyncratic shock x of AR(1) after taken log.

$$y = \chi(j)x$$

$$\log(x') = \rho_x \log(x) + \epsilon^x$$

$$\epsilon^x \sim \mathcal{N}(0, \sigma_x^2) \quad \text{i.i.d.}$$

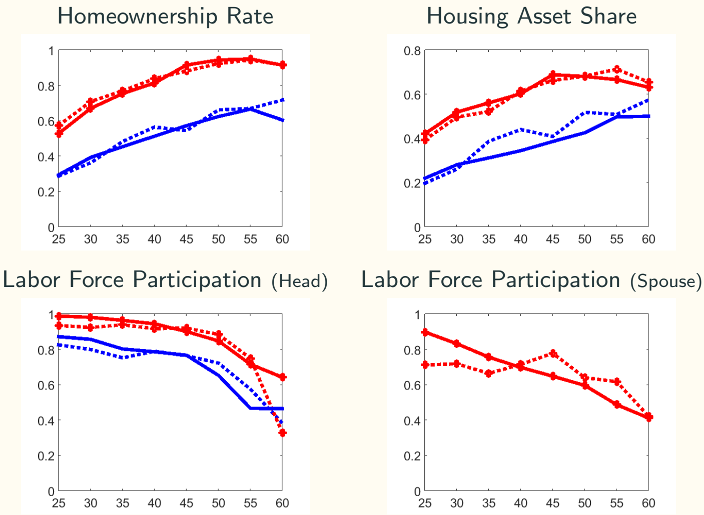
- Idiosyncratic house price shock is uniformly distributed, $\delta \sim \mathcal{U}[\underline{\delta}, \bar{\delta}]$.
- Age-dependent probabilities of marital transitions $(q_{mm,j}, q_{ms,j}, q_{sm,j}, q_{ss,j})$ are constructed from the data.
- We can use this model to analyze 1970 *vs.* 1995 as two steady states.
    - Common house price P^H is fixed.
    - This will be extended to model time-varying house price.

Estimation

- Some parameters are calibrated with external information.
  - e.g.) transaction cost of buying: 2.5%/ selling: 7% of house value
- We then estimate the other parameters by a limited information Bayesian method to match the moments from 1995's cross-section data.
  - e.g.) life-cycle profiles for homeownership, portfolio share, labor supply across marital status
- List of parameters estimated:

Parameter	Explanation	Posterior median
ζ_1	Housing preference: constant	1.36
ζ_2	Housing preference: slope	0.34
ω_{uhp}	Utility from home production	1.02
γ_e	Economies of scale	0.61
$\phi, \widetilde{\phi}, B_s, B_m, \widetilde{B}_m$	Fixed cost, disutility of labor supply	1.37, 0.83, 48.67, 17.49, 48.32

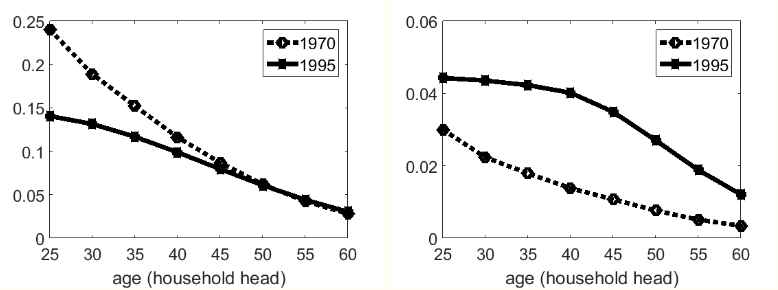
Life-cycle Profiles: Model vs. Data (1995)



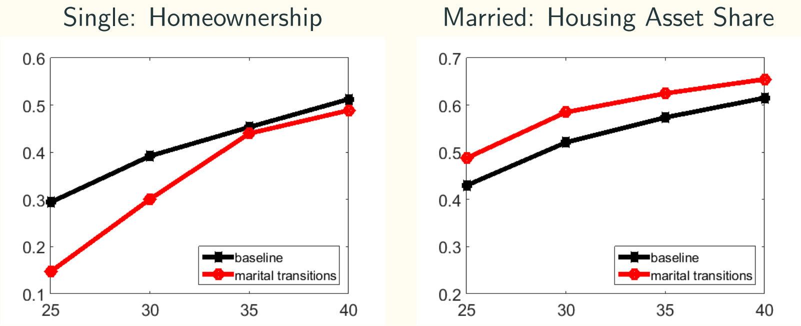
Notes: Solid - Model, Dotted - Data, Blue - Single, Red - Married.

Major changes between 1970 and 1995

(1) Marital transition probabilities changed. (Left: Marriage, Right: Divorce)



(1) Marital transition probabilities change



- High marriage probability $\rightarrow$ The single's homeownership rate $\Downarrow$
- Lower divorce probability $\rightarrow$ The married's housing asset share $\Uparrow$

Major changes between 1970 and 1995

(2) Downpayment constraint was tighter in 1970.

$$(1-\eta_{1995}) = (1-\eta_{1970}) \times 2/3$$

* Reference: Fisher and Gervais (2011), Bullard (2012)

(3) Earnings risk was lower in 1970.

$$\sigma^2_{x,1995} = \sigma^2_{x,1970} \times (1+0.4)$$

* Reference: Fisher and Gervais (2011), Santos and Weiss (2013)

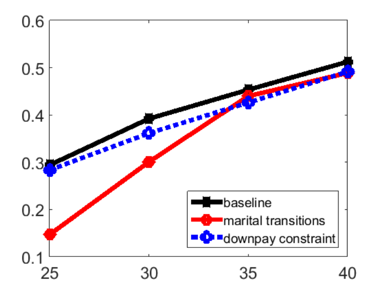
(4) Spousal labor productivity was lower in 1970.

change wage gap via $\widetilde{\chi}(j)$ and fixed cost of working $\widetilde{\phi}$

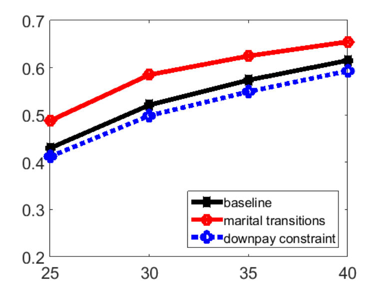
* Reference: Francis and Ramey (2009), Heathcote et al. (2010)

(2) Downpayment constraint change

Single: Homeownership



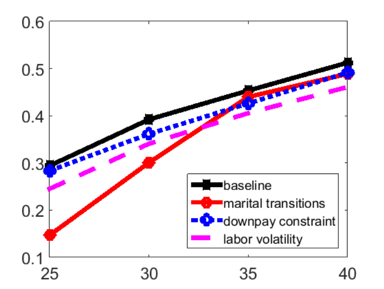
Married: Housing Asset Share



- Tighter constraint $\begin{cases} \text{The single's ownership rate } \Downarrow \\ \text{The married's housing asset share } \Downarrow \end{cases}$

(3) Earnings risk change

Single: Homeownership



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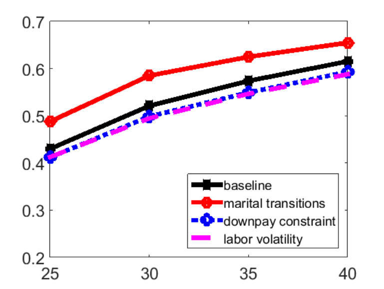
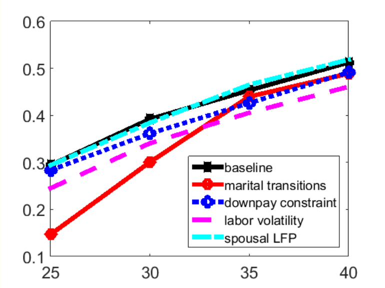


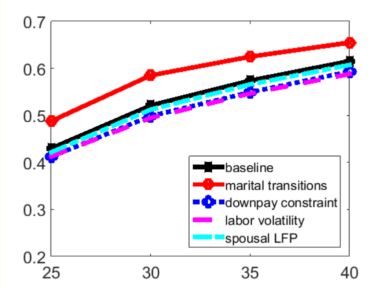
- Earnings risk $\begin{cases} \text{Precautionary savings (ownership, share } \Uparrow) \\ \text{Delay until wealthy enough (ownership, share } \Downarrow) \end{cases}$
- Lower volatility makes (ownership, share $\Downarrow$) for the single/the married.

(4) Spousal labor productivity change

Single: Homeownership



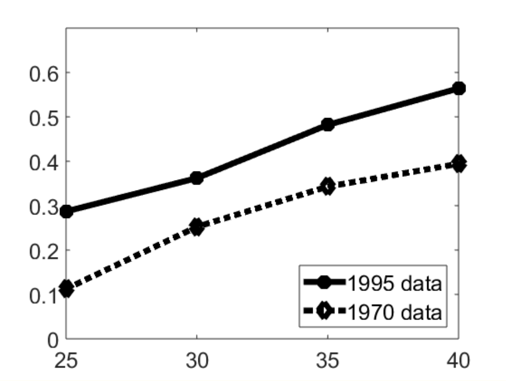
Married: Housing Asset Share



- Not much change due to labor supply adjustment between spouses.
- The married head works more as the spouse works less.

Decomposition - Single's homeownership

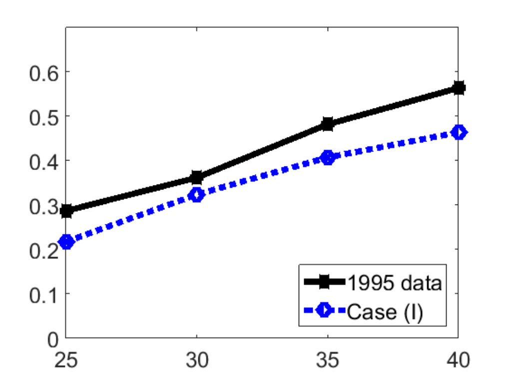
	Data	Counterfactuals	
Age	% Change	Case (I)	Case (II)
25 - 29	- 61%		
30 - 34	- 30%		
35 - 39	- 29%		
40 - 44	- 30%		
Average	- 38%		



- For counterfactuals,
  - **Case (I)**: Changes (2)+(3)+(4) applied
  - **Case (II)**: **Case (I)** + (1) Change in likelihood of marital transitions

Decomposition - Single's homeownership

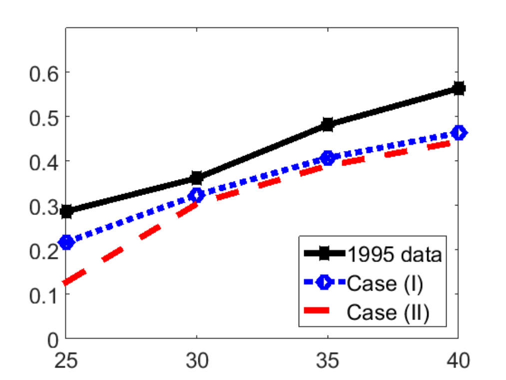
	Data	Counterfactuals	
Age	% Change	Case (I)	Case (II)
25 - 29	- 61%	- 24%	
30 - 34	- 30%	- 11%	
35 - 39	- 29%	- 16%	
40 - 44	- 30%	- 18%	
Average	- 38%	- 17%	



- For the change in single's homeownership rate,
  - 45% of the change can be accounted for by the other three factors.

Decomposition - Single's homeownership

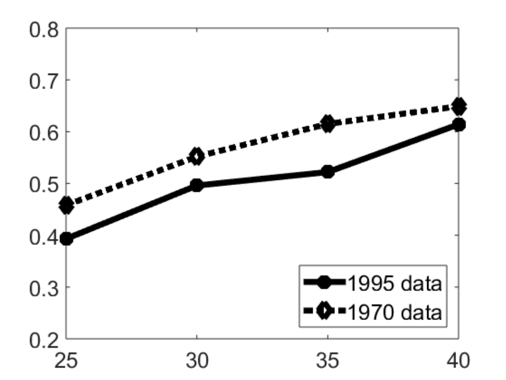
	Data	Counterfactuals	
Age	% Change	Case (I)	Case (II)
25 - 29	- 61%	- 24%	- 56%
30 - 34	- 30%	- 11%	- 16%
35 - 39	- 29%	- 16%	- 19%
40 - 44	- 30%	- 18%	- 21%
Average	- 38%	- 17%	- 28%



- For the change in single's home ownership rate,
  - 45% of the change can be accounted for by the other three factors.
  - 29% of the change can be by the likelihood of marital transitions.

Decomposition - Married's housing asset share

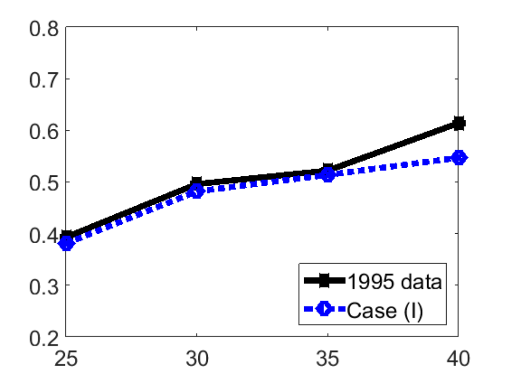
	Data	Counterfactuals	
Age	% Change	Case (I)	Case (II)
25 - 29	17%		
30 - 34	11%		
35 - 39	18%		
40 - 44	6%		
Average	13%		



- For counterfactuals,
  - **Case (I)**: Changes (2)+(3)+(4) applied.
  - **Case (II)**: **Case (I)** + (1) Change in likelihood of marital transitions.

Decomposition - Married's housing asset share

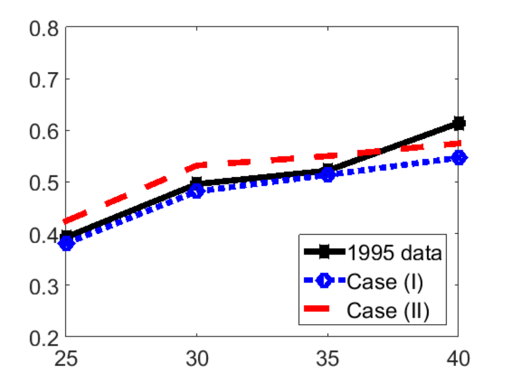
	Data	Counterfactuals	
Age	% Change	Case (I)	Case (II)
25 - 29	17%	- 3%	
30 - 34	11%	- 3%	
35 - 39	18%	- 2%	
40 - 44	6%	- 11%	
Average	13%	- 3%	



- Without the channel (1), the sign of change becomes the opposite.

Decomposition - Married's housing asset share

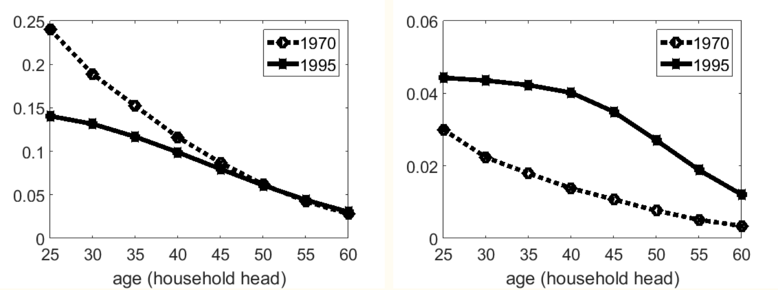
	Data	Counterfactuals	
Age	% Change	Case (I)	Case (II)
25 - 29	17%	- 3%	8%
30 - 34	11%	- 3%	7%
35 - 39	18%	- 2%	5%
40 - 44	6%	- 11%	- 6%
Average	13%	- 3%	4%



- Without the channel (1), the sign of change becomes the opposite.
- 31% of the change can be accounted for by applying all changes.

Marital transitions: 1995 vs. mid-2000s

- Marital transition is an important risk factor to understand the change in housing variables between 1970 and 1995.
- What about more recent periods? (Left: Marriage, Right: Divorce)



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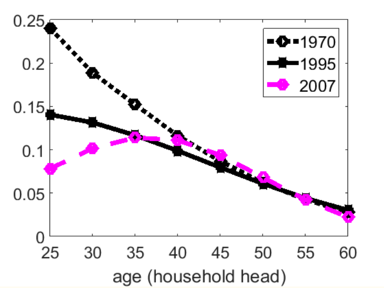

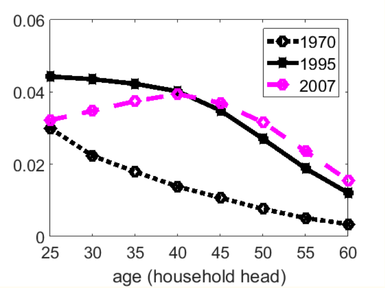

Marriage probabilities continued to fall for the young.

House price dynamics

- We need to incorporate house price boom happened over the 2000s!
- Common house price shock P_t^H is a three-point process with a Markov transition matrix similar to Corbae and Quintin (2015).
- Just with P_t^H, households do not own more when housing is expensive.
- I incorporate an additional shock o_t to capture that households expect housing price appreciation.

$$o_t \in \{0, \epsilon = 0.6\}, \quad \Pi_o \equiv \begin{bmatrix} \pi_{00} & \pi_{o\epsilon} \\ \pi_{\epsilon 0} & \pi_{\epsilon\epsilon} \end{bmatrix} = \begin{bmatrix} 0.85 & 0.15 \\ 0.5 & 0.5 \end{bmatrix}.$$

  - If $o_t = \epsilon$, households expect $P_{t+1}^H \times (1 + \epsilon)$ with probability 0.5.

Changes over time

Year	1995	2000	2005	2010
house price P_t^H	p_2^H			
belief/optimism o_t	0			
wage w	1.0			
downpayment constraint $(1-\eta)$	0.25			
borrowing interest rate r^H	0.07			
savings interest rate r	0.02			
marital transition prob.	baseline			

Notes: The wage w is normalized so that the value in 1995 is 1.0. The interest rates are annual.

Changes over time

Year	1995	2000	2005	2010
house price P_t^H	p_2^H	p_2^H	p_3^H	p_2^H
belief/optimism o_t	0	ϵ	ϵ	0
wage w	1.0			
downpayment constraint $(1-\eta)$	0.25			
borrowing interest rate r^H	0.07			
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Changes over time

Year	1995	2000	2005	2010
house price P_t^H	p_2^H	p_2^H	p_3^H	p_2^H
belief/optimism o_t	0	ϵ	ϵ	0
wage w	1.0	1.07	1.07	1.0
downpayment constraint $(1-\eta)$	0.25			
borrowing interest rate r^H	0.07			
savings interest rate r	0.02			
marital transition prob.	baseline			

Notes: The wage w is normalized so that the value in 1995 is 1.0. The interest rates are annual.

Changes over time

Year	1995	2000	2005	2010
house price P_t^H	p_2^H	p_2^H	p_3^H	p_2^H
belief/optimism o_t	0	ϵ	ϵ	0
wage w	1.0	1.07	1.07	1.0
downpayment constraint $(1-\eta)$	0.25	0.2	0.15	0.25
borrowing interest rate r^H	0.07	0.06	0.05	0.07
savings interest rate r	0.02	0.018	0.015	0.02
marital transition prob.	baseline			

Notes: The wage w is normalized so that the value in 1995 is 1.0. The interest rates are annual.

Baseline under high house price

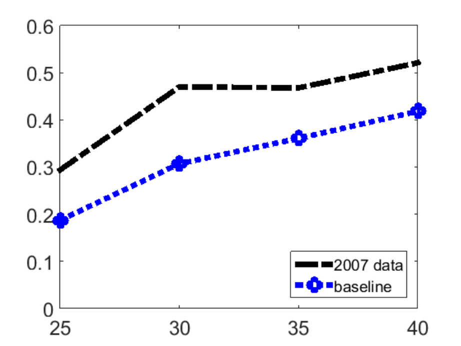
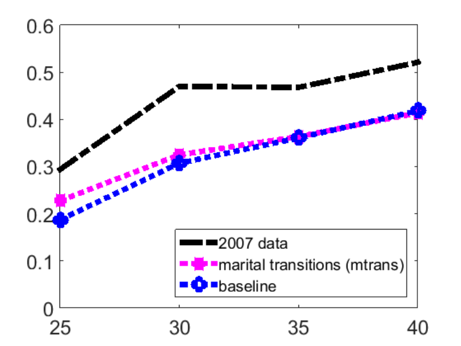


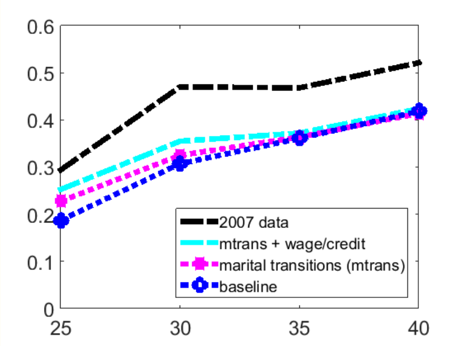
Figure 1: Homeownership Rate of Singles (Boom: Data *vs.* Model)

Marital transition probabilities change



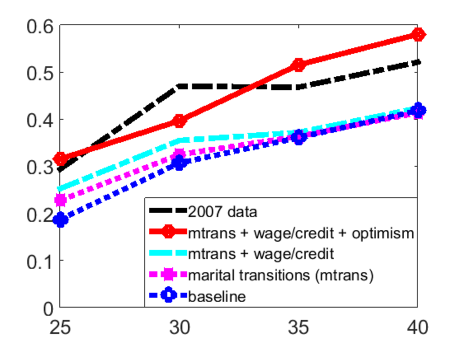
- With marriage ⇓, singles' homeownership rate ⇑

Higher wage + relaxed credit constraints



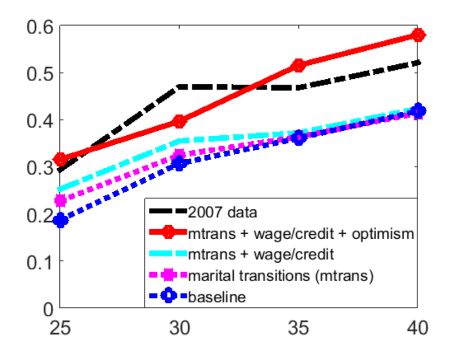
- With these changes, homeownership rate ⇑
- It is still insufficient to generate what's observed in the data.

Optimism about house price appreciation



- The data's pattern is matched by beliefs about appreciation coupled with changes in labor income and borrowing capacity.

Optimism about house price appreciation



- Change in the likelihood of marital transitions puts upward pressure on the singles' homeownership by 6.8%.