Perturbation Methods II: General Case

(Lectures on Solution Methods for Economists VI)

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Most of arguments in the previous set of lecture notes are easy to generalize.

The set of equilibrium conditions of many DSGE models can be written using recursive notation as:

$$\mathbb{E}_t \mathcal{H}(y, y', x, x') = 0,$$

where $y_t$ is a $n_y \times 1$ vector of controls and $x_t$ is a $n_x \times 1$ vector of states.

$n = n_x + n_y$.

$\mathcal{H}$ maps $R^{n_y} \times R^{n_y} \times R^{n_x} \times R^{n_x}$ into $R^n$. 
The state vector $x_t$ can be partitioned as $x = [x_1; x_2]^t$.

- $x_1$ is a $(n_x - n_\epsilon) \times 1$ vector of endogenous state variables.
- $x_2$ is a $n_\epsilon \times 1$ vector of exogenous state variables.

Why do we want to partition the state vector?
Exogenous stochastic process 1

\[ x_2' = Ax_2 + \lambda \eta \epsilon' \]

- Process with 3 parts:
  1. The deterministic component \( Ax_2 \), where \( A \) is a \( n_e \times n_e \) matrix, with all eigenvalues with modulus less than one.
  2. The scaled innovation \( \eta \epsilon' \), where:
     2.1 \( \eta \) is a known \( n_e \times n_e \) matrix.
     2.2 \( \epsilon \) is a \( n_e \times 1 \) i.i.d. innovation with bounded support, zero mean, and variance/covariance matrix \( I \).
  3. The perturbation parameter \( \lambda \).
• We can accommodate very general structures of $x_2$ through changes in the definition of the state space: i.e., stochastic volatility.

• More general structure:

$$x'_2 = \Gamma(x_2) + \lambda \eta \epsilon'$$

where $\Gamma$ is a non-linear function satisfying that all eigenvalues of its first derivative evaluated at the non-stochastic steady state lie within the unit circle.

• Note we do not impose Gaussianity.
The perturbation parameter

- The scalar $\lambda \geq 0$ is the perturbation parameter.
- If we set $\lambda = 0$, we have a deterministic model.
- Important: there is only ONE perturbation parameter. The matrix $\eta_e$ takes account of relative sizes of different shocks.
Solution of the model

• The solution to the model is of the form:

\[ y = g(x; \lambda) \]
\[ x' = h(x; \lambda) + \lambda \eta \epsilon' \]

where \( g \) maps \( \mathbb{R}^{n_x} \times \mathbb{R}^+ \) into \( \mathbb{R}^{ny} \) and \( h \) maps \( \mathbb{R}^{n_x} \times \mathbb{R}^+ \) into \( \mathbb{R}^{nx} \).

• The matrix \( \eta \) is of order \( n_x \times n_{\epsilon} \) and is given by:

\[ \eta = \begin{bmatrix} \emptyset \\ \eta_{\epsilon} \end{bmatrix} \]
• We wish to find a perturbation approximation of the functions $g$ and $h$ around the non-stochastic steady state, $x_t = \bar{x}$ and $\lambda = 0$.

• We define the non-stochastic steady state as vectors $(\bar{x}, \bar{y})$ such that:

$$\mathcal{H}(\bar{y}, \bar{y}, \bar{x}, \bar{x}) = 0.$$ 

• Note that $\bar{y} = g(\bar{x}; 0)$ and $\bar{x} = h(\bar{x}; 0)$.

• This is because, if $\lambda = 0$, $E_t \mathcal{H} = \mathcal{H}$. 
Plugging-in the proposed solution

- Substituting the proposed solution, we define:

\[ F(x; \lambda) \equiv \mathbb{E}_t \mathcal{H}(g(x; \lambda), g(h(x; \lambda) + \eta \lambda \epsilon', \lambda), x, h(x; \lambda) + \eta \lambda \epsilon') = 0 \]

- Since \( F(x; \lambda) = 0 \) for any values of \( x \) and \( \lambda \), the derivatives of any order of \( F \) must also be equal to zero.

- Formally:

\[ F_{x^k \lambda^j}(x; \lambda) = 0 \quad \forall x, \lambda, j, k, \]

where \( F_{x^k \lambda^j}(x, \lambda) \) denotes the derivative of \( F \) with respect to \( x \) taken \( k \) times and with respect to \( \lambda \) taken \( j \) times.
First-order approximation

• We are looking for approximations to $g$ and $h$ around $(x, \lambda) = (\bar{x}, 0)$ of the form:

\[
\begin{align*}
g(x; \lambda) &= g(\bar{x}; 0) + g_x(\bar{x}; 0)(x - \bar{x}) + g_\lambda(\bar{x}; 0)\lambda \\
h(x; \lambda) &= h(\bar{x}; 0) + h_x(\bar{x}; 0)(x - \bar{x}) + h_\lambda(\bar{x}; 0)\lambda
\end{align*}
\]

• As explained earlier, $g(\bar{x}; 0) = \bar{y}$ and $h(\bar{x}; 0) = \bar{x}$.

• The remaining four unknown coefficients of the first-order approximation to $g$ and $h$ are found by using the fact that:

\[
F_x(\bar{x}; 0) = 0
\]

and

\[
F_\lambda(\bar{x}; 0) = 0
\]

• Before doing so, we need to introduce the tensor notation.
Tensors

- General trick from physics.

- An \(n^{th}\)-rank tensor in a \(m\)-dimensional space is an operator that has \(n\) indices and \(m^n\) components and obeys certain transformation rules.

- \([H_y]_i^\alpha\) is the \((i, \alpha)\) element of the derivative of \(H\) with respect to \(y\):
  1. The derivative of \(H\) with respect to \(y\) is an \(n \times n_y\) matrix.
  2. Thus, \([H_y]_i^\alpha\) is the element of this matrix located at the intersection of the \(i\)-th row and \(\alpha\)-th column.
  3. Thus, \([H_y]_i^\alpha[g_x]^\alpha_\beta[h_x]^\beta_j = \sum_{\alpha=1}^{n_y} \sum_{\beta=1}^{n_x} \frac{\partial H^i}{\partial y^\alpha} \frac{\partial g^\alpha}{\partial x^\beta} \frac{\partial h^\beta}{\partial x^j} \). 

- \([H_{y'y'}]_i^\alpha_\gamma\):
  1. \(H_{y'y'}\) is a three dimensional array with \(n\) rows, \(n_y\) columns, and \(n_y\) pages.
  2. Then \([H_{y'y'}]_i^\alpha_\gamma\) denotes the element of \(H_{y'y'}\) located at the intersection of row \(i\), column \(\alpha\) and page \(\gamma\).
Solving the system I

- $g_x$ and $h_x$ can be found as the solution to the system:

\[
[F_x(\bar{x}; 0)]_j^i = [\mathcal{H}_y]_\alpha^i [g_x]_\beta^\alpha [h_x]_j^\beta + [\mathcal{H}_y]_\alpha^i [g_x]_j^\alpha + [\mathcal{H}_x']_\beta^i [h_x]_j^\beta + [\mathcal{H}_x]_j^i = 0;
\]

\[i = 1, \ldots, n; \quad j, \beta = 1, \ldots, n_x; \quad \alpha = 1, \ldots, n_y\]

- Note that the derivatives of $\mathcal{H}$ evaluated at $(y, y', x, x') = (\bar{y}, \bar{y}, \bar{x}, \bar{x})$ are known.

- Then, we have a system of $n \times n_x$ quadratic equations in the $n \times n_x$ unknowns given by the elements of $g_x$ and $h_x$.

- We can solve with a standard quadratic matrix equation solver.
Solving the system II

- \( g_\lambda \) and \( h_\lambda \) are the solution to the \( n \) equations:

\[
[F_\lambda(\bar{x}; 0)]^i = \mathbb{E}_t\{[\mathcal{H}_y]^i\alpha [g_x]^\alpha [h_\lambda]^\beta + [\mathcal{H}_y]^i\alpha [g_x]^\alpha [\eta]^\beta [\epsilon']^\phi + [\mathcal{H}_y]^i\alpha [g_\lambda]^\alpha + [\mathcal{H}_y]^i\alpha [g_\lambda]^\alpha + [\mathcal{H}_x]^i\beta [h_\lambda]^\beta + [\mathcal{H}_x]^i\beta [\eta]^\beta [\epsilon']^\phi\}
\]

\[ i = 1, \ldots, n; \quad \alpha = 1, \ldots, n_y; \quad \beta = 1, \ldots, n_x; \quad \phi = 1, \ldots, n_\epsilon. \]

- Then:

\[
[F_\lambda(\bar{x}; 0)]^i = [\mathcal{H}_y]^i\alpha [g_x]^\alpha [h_\lambda]^\beta + [\mathcal{H}_y]^i\alpha [g_\lambda]^\alpha + [\mathcal{H}_y]^i\alpha [g_\lambda]^\alpha + [f_x]^i\beta [h_\lambda]^\beta = 0;
\]

\[ i = 1, \ldots, n; \quad \alpha = 1, \ldots, n_y; \quad \beta = 1, \ldots, n_x; \quad \phi = 1, \ldots, n_\epsilon. \]

- Certainty equivalence: linear and homogeneous equation in \( g_\lambda \) and \( h_\lambda \). Thus, if a unique solution exists, it satisfies:

\[
\begin{align*}
h_\lambda &= 0 \\
g_\lambda &= 0
\end{align*}
\]
Second-order approximation I

The second-order approximations to $g$ around $(x; \lambda) = (\bar{x}; 0)$ is

$$[g(x; \lambda)]^i = [g(\bar{x}; 0)]^i + [g_x(\bar{x}; 0)]^i_a[(x - \bar{x})]_a + [g_\lambda(\bar{x}; 0)]^i[\lambda]$$
$$+ \frac{1}{2} [g_{xx}(\bar{x}; 0)]^i_{ab}[(x - \bar{x})]_a[(x - \bar{x})]_b$$
$$+ \frac{1}{2} [g_{x\lambda}(\bar{x}; 0)]^i_a[(x - \bar{x})]_a[\lambda]$$
$$+ \frac{1}{2} [g_{\lambda x}(\bar{x}; 0)]^i_a[(x - \bar{x})]_a[\lambda]$$
$$+ \frac{1}{2} [g_{\lambda\lambda}(\bar{x}; 0)]^i[\lambda][\lambda]$$

where $i = 1, \ldots, n_y$, $a, b = 1, \ldots, n_x$, and $j = 1, \ldots, n_x$. 
The second-order approximations to $h$ around $(x; \lambda) = (\bar{x}; 0)$ is

$$[h(x; \lambda)]^i = [h(\bar{x}; 0)]^i + [h_x(\bar{x}; 0)]^i_a[(x - \bar{x})]_a + [h_\lambda(\bar{x}; 0)]^i[\lambda]$$

$$+ \frac{1}{2} [h_{xx}(\bar{x}; 0)]^i_{ab}[(x - \bar{x})]_a[(x - \bar{x})]_b$$

$$+ \frac{1}{2} [h_{x\lambda}(\bar{x}; 0)]^i_a[(x - \bar{x})]_a[\lambda]$$

$$+ \frac{1}{2} [h_{\lambda x}(\bar{x}; 0)]^i_a[(x - \bar{x})]_a[\lambda]$$

$$+ \frac{1}{2} [h_{\lambda\lambda}(\bar{x}; 0)]^i[\lambda][\lambda],$$

where $i = 1, \ldots, n_y$, $a, b = 1, \ldots, n_x$, and $j = 1, \ldots, n_x$. 
The unknowns of these expansions are 

\[
[g_{xx}]^i_{ab}, [g_{x\lambda}]^i_a, [g_{\lambda x}]^i_a, [g_{\lambda\lambda}]^i, [h_{xx}]^j_{ab}, [h_{x\lambda}]^j_a, [h_{\lambda x}]^j_a, [h_{\lambda\lambda}]^j.
\]

These coefficients can be identified by taking the derivative of \( F(x; \lambda) \) with respect to \( x \) and \( \lambda \) twice and evaluating them at \( (x; \lambda) = (\bar{x}; 0) \).

By the arguments provided earlier, these derivatives must be zero.
We use $F_{xx}(\bar{x}; 0)$ to identify $g_{xx}(\bar{x}; 0)$ and $h_{xx}(\bar{x}; 0)$:

$$[F_{xx}(\bar{x}; 0)]^i_{jk} =$$

$$( [H_{y'y'}]^i_{\alpha\gamma}[g_x]^\gamma_k[h_x]^\delta_k + [H_{y'y}]^i_{\alpha\gamma}[g_x]^\gamma_k + [H_{y'x}^i_{\alpha\delta}[h_x]^\delta_k + [H_{y'x}^i_{\alpha k}) [g_x]^\alpha_{\beta} [h_x]^\beta_j$$

$$+ [H_{y'}^i_{\alpha}[g_{xx}]^\alpha_{\beta\delta}[h_x]^\delta_k [h_x]^\beta_j + [H_{y'}^i_{\alpha}[g_x]^\alpha_{\beta} [h_{xx}]^\beta_{jk}$$

$$+ ( [H_{yy'}^i_{\alpha\gamma}[g_x]^\gamma_k[h_x]^\delta_k + [H_{yy}^i_{\alpha\gamma}[g_x]^\gamma_k + [H_{yx}^i_{\alpha\delta}[h_x]^\delta_k + [H_{yx}^i_{\alpha k}) [g_x]^\alpha_{j}$$

$$+ [H_{y}^i_{\alpha}[g_{xx}]^\alpha_{jk}$$

$$+ ( [H_{x'y'}^i_{\beta\gamma}[g_x]^\gamma_k[h_x]^\delta_k + [H_{x'y}^i_{\beta\gamma}[g_x]^\gamma_k + [H_{x'x}^i_{\beta\delta}[h_x]^\delta_k + [H_{x'x}^i_{\beta k}) [h_x]^\beta_j$$

$$+ [H_{x'}^i_{\beta}[h_{xx}]^\beta_{jk}$$

$$+ [H_{xy}^i_{j\gamma}[g_x]^\gamma_k[h_x]^\delta_k + [H_{xy}^i_{j\gamma}[g_x]^\gamma_k + [H_{xx}^i_{j\delta}[h_x]^\delta_k + [H_{xx}^i_{j k}] = 0;$$

$$i = 1, \ldots n, \quad j, k, \beta, \delta = 1, \ldots n_x; \quad \alpha, \gamma = 1, \ldots n_y.$$
Solving the system II

- We know the derivatives of $\mathcal{H}$.
- We also know the first derivatives of $g$ and $h$ evaluated at $(y, y', x, x') = (\bar{y}, \bar{y}, \bar{x}, \bar{x})$.
- Hence, the above expression represents a system of $n \times n_x \times n_x$ linear equations in then $n \times n_x \times n_x$ unknowns elements of $g_{xx}$ and $h_{xx}$. 

Similarly, $g_{\lambda\lambda}$ and $h_{\lambda\lambda}$ can be obtained by solving:

$$[F_{\lambda\lambda}(\bar{x}; 0)]^i = [H_y']^i_\alpha [g_{\lambda\lambda}]^\alpha_\beta [h_{\lambda\lambda}]^\beta + [H_y'y']^i_\alpha\gamma [g_{\lambda\lambda}]^\alpha_\delta [\eta]\delta_\xi [g_{\lambda\lambda}]^\beta_\phi [l]^\phi \xi$$

$$+[H_y'y']^i_\alpha\delta [\eta]\xi [g_{\lambda\lambda}]^\alpha_\beta [\eta]\phi [l]^\phi \xi$$

$$+[H_y'y']^i_\alpha [g_{\lambda\lambda}]^\alpha_\gamma [\eta]\delta_\xi [\eta]\phi [l]^\phi \xi + [H_y']^i_\alpha [g_{\lambda\lambda}]^\alpha$$

$$+[H_{y'y'}]_\alpha [g_{\lambda\lambda}]^\alpha + [H_{x'y'}]_\beta [h_{\lambda\lambda}]^\beta$$

$$+[H_{x'y'}]_\beta [g_{\lambda\lambda}]^\gamma [\eta]\delta_\xi [\eta]\phi [l]^\phi \xi$$

$$+[H_{x'y'}]_\beta [\eta]\delta_\xi [\eta]\phi [l]^\phi \xi = 0;$$

$$i = 1, \ldots, n; \alpha, \gamma = 1, \ldots, n_y; \beta, \delta = 1, \ldots, n_x; \phi, \xi = 1, \ldots, n_\epsilon$$

a system of $n$ linear equations in the $n$ unknowns given by the elements of $g_{\lambda\lambda}$ and $h_{\lambda\lambda}$.
Cross-derivatives

- The cross derivatives $g_{x\lambda}$ and $h_{x\lambda}$ are zero when evaluated at $(\bar{x}, 0)$.

- Why? Write the system $F_{\lambda x}(\bar{x}; 0) = 0$ taking into account that all terms containing either $g_{\lambda}$ or $h_{\lambda}$ are zero at $(\bar{x}, 0)$.

- Then:

$$[F_{\lambda x}(\bar{x}; 0)]^i_j = [H_y']^i_\alpha [g_x]^{\alpha}_\beta [h_{\lambda x}]^\beta_j + [H_y']^i_\alpha [g_{\lambda x}]^\alpha_\gamma [h_x]^{\gamma}_j + [H_y]^{i}_\alpha [g_{\lambda x}]^\alpha_j + [H_x']^{i}_\beta [h_{\lambda x}]^{\beta}_j = 0;$$

$$i = 1, \ldots, n; \quad \alpha = 1, \ldots, n_y; \quad \beta, \gamma, j = 1, \ldots, n_x.$$

- This is a system of $n \times n_x$ equations in the $n \times n_x$ unknowns given by the elements of $g_{\lambda x}$ and $h_{\lambda x}$.

- The system is homogeneous in the unknowns.

- Thus, if a unique solution exists, it is given by:

$$g_{\lambda x} = 0$$
$$h_{\lambda x} = 0$$
Structure of the solution

• The perturbation solution of the model satisfies:

\[ g_{\lambda}(\bar{x}; 0) = 0 \]
\[ h_{\lambda}(\bar{x}; 0) = 0 \]
\[ g_{x\lambda}(\bar{x}; 0) = 0 \]
\[ h_{x\lambda}(\bar{x}; 0) = 0 \]

• Standard deviation only appears in:

  1. A constant term given by \( \frac{1}{2} g_{\lambda\lambda} \lambda^2 \) for the control vector \( y_t \).

  2. The first \( n_x - n_\epsilon \) elements of \( \frac{1}{2} h_{\lambda\lambda} \lambda^2 \).

• Correction for risk.

• Quadratic terms in endogenous state vector \( x_1 \).

• Those terms capture non-linear behavior.
Higher-order approximations

• We can iterate this procedure as many times as we want.

• We can obtain $n$-th order approximations.

• Problems:
  1. Existence of higher order derivatives (Santos, 1992).
  3. Computational costs.