Perturbation Methods III: Change of Variables

(Lectures on Solution Methods for Economists VII)

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It is not the process of linearization that limits insight.

It is the nature of the state that we choose to linearize about.

Change of variables

- We approximated our solution in levels.
- We could have done it in logs.
- Why stop there? Why not in powers of the state variables?
- Judd (2002) has provided methods for changes of variables.
- We apply and extend ideas to the stochastic neoclassical growth model.

A general transformation

- We look at solutions of the form:

$$
\begin{aligned}
c^{\mu} - c_0^{\mu} &= a\left(k^{\zeta} - k_0^{\zeta}\right) + bz \\
k'^{\gamma} - k_0^{\gamma} &= c\left(k^{\zeta} - k_0^{\zeta}\right) + dz
\end{aligned}
$$

- Note that:

1. If γ, ζ, and μ are 1, we get the linear representation.
2. As γ, ζ, and μ tend to zero, we get the loglinear approximation.

Theory

- The first order solution can be written as

$$f\left(x\right) \simeq f\left(a\right) + \left(x - a\right) f'\left(a\right)$$

- Expand $g(y) = h\left(f\left(X\left(y\right)\right)\right)$ around $b = Y\left(a\right)$, where $X\left(y\right)$ is the inverse of $Y\left(x\right)$.
- Then:

$$g\left(y\right) = h\left(f\left(X\left(y\right)\right)\right) = g\left(b\right) + g_{\alpha}\left(b\right)\left(Y^{\alpha}\left(x\right) - b^{\alpha}\right)$$

  where $g_{\alpha} = h_A f_i^A X_{\alpha}^i$ comes from the application of the chain rule.
- From this expression it is easy to see that if we have computed the values of f_i^A, then it is straightforward to find the value of g_{α}.

Coefficients relation

- Remember that the linear solution is:

$$\begin{aligned}(k' - k_0) &= a_1 (k - k_0) + b_1 z \\ (l - l_0) &= c_1 (k - k_0) + d_1 z\end{aligned}$$

- Then we show that:

$a_3 = \frac{\gamma}{\zeta} k_0^{\gamma-\zeta} a_1$	$b_3 = \gamma k_0^{\gamma-1} b_1$
$c_3 = \frac{\mu}{\zeta} l_0^{\mu-1} k_0^{1-\zeta} c_1$	$d_3 = \mu l_0^{\mu-1} d_1$

Finding the parameters

- Minimize over a grid the Euler Error.
- Some optimal results

Euler Equation Errors

γ	ζ	μ	SEE
1	1	1	0.0856279
0.986534	0.991673	2.47856	0.0279944

Sensitivity analysis

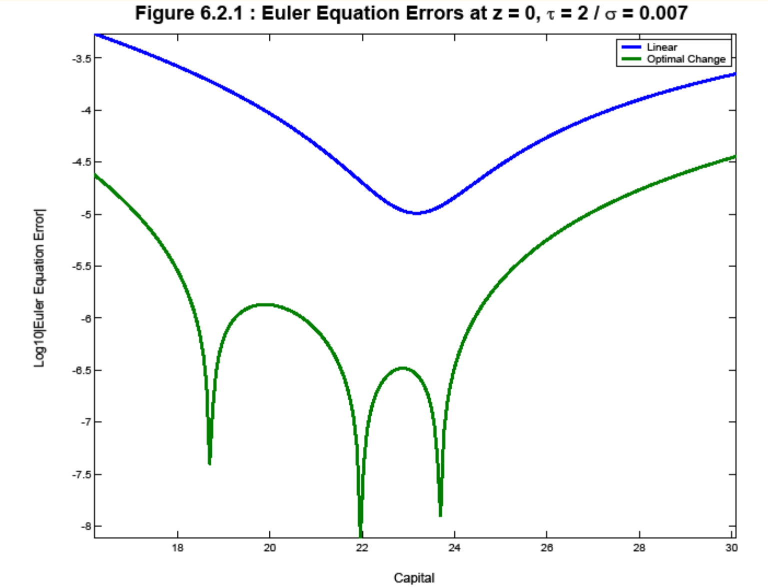
- Different parameter values.
- Most interesting finding is when we change σ:

Optimal Parameters for different σ's

σ	γ	ζ	μ
0.014	0.98140	0.98766	2.47753
0.028	1.04804	1.05265	1.73209
0.056	1.23753	1.22394	0.77869

- A first order approximation corrects for changes in variance!

Euler equation errors



A quasi-optimal approximation

- Sensitivity analysis reveals that for different parametrizations $\gamma \simeq \zeta$.

- This suggests the quasi-optimal approximation:

$$
\begin{aligned}
k'^{\gamma} - k_0^{\gamma} &= a_3 \left(k^{\gamma} - k_0^{\gamma}\right) + b_3 z \\
l^{\mu} - l_0^{\mu} &= c_3 \left(k^{\gamma} - k_0^{\gamma}\right) + d_3 z
\end{aligned}
$$

- Note that if define $\widehat{k} = k^{\gamma} - k_0^{\gamma}$ and $\widehat{l} = l^{\mu} - l_0^{\mu}$ we get:

$$
\begin{aligned}
\widehat{k}' &= a_3 \widehat{k} + b_3 z \\
\widehat{l} &= c_3 \widehat{k} + d_3 z
\end{aligned}
$$

- Linear system:

  1. Use for analytical study.
  2. Use for estimation with a Kalman Filter.