

Perturbation Methods V: Pruning

(Lectures on Solution Methods for Economists IX)

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Introduction

- Advantages:
 1. Intuitive.
 2. Straightforward to compute.
 3. Fast.
 4. Accurate.
- Problem with simulations.

First-order approximation

- First-order approximation of a canonical RBC model without persistence in productivity shocks:

$$\widehat{k}_{t+1} = a_1 \widehat{k}_t + a_2 \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, 1)$$

- Then:

$$\begin{aligned}\widehat{k}_{t+1} &= a_1 \left(a_1 \widehat{k}_{t-1} + a_2 \varepsilon_{t-1} \right) + a_2 \varepsilon_t \\ &= a_1^2 \widehat{k}_{t-1} + a_1 a_2 \varepsilon_{t-1} + a_2 \varepsilon_t\end{aligned}$$

- Since $a_1 < 1$ and assuming $\widehat{k}_0 = 0$

$$\widehat{k}_{t+1} = a_2 \sum_{j=0}^t a_1^j \varepsilon_{t-j}$$

which is a well-understood system.

Higher-order approximations

- Second-order approximation:

$$\widehat{k}_{t+1} = a_0 + a_1 \widehat{k}_t + a_2 \varepsilon_t + a_3 \widehat{k}_t^2 + a_4 \varepsilon_t^2 + a_5 \widehat{k}_t \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, 1)$$

- Then:

$$\begin{aligned} \widehat{k}_{t+1} &= a_0 + a_1 \left(a_0 + a_1 \widehat{k}_t + a_2 \varepsilon_t + a_3 \widehat{k}_t^2 + a_4 \varepsilon_t^2 + a_5 \widehat{k}_t \varepsilon_t \right) + a_2 \varepsilon_t \\ &\quad + a_3 \left(a_0 + a_1 \widehat{k}_t + a_2 \varepsilon_t + a_3 \widehat{k}_t^2 + a_4 \varepsilon_t^2 + a_5 \widehat{k}_t \varepsilon_t \right)^2 + a_4 \varepsilon_t^2 \\ &\quad + a_5 \left(a_0 + a_1 \widehat{k}_t + a_2 \varepsilon_t + a_3 \widehat{k}_t^2 + a_4 \varepsilon_t^2 + a_5 \widehat{k}_t \varepsilon_t \right) \varepsilon_t \end{aligned}$$

- We have terms in \widehat{k}_t^3 and \widehat{k}_t^4 .

Problem

- For a large realization of ε_t , the terms in \hat{k}_t^3 and \hat{k}_t^4 make the system explode.
- This will happen as soon as we have a large simulation \Rightarrow no unconditional moments would exist based on this approximation.
- This is true even when the corresponding linear approximation is stable.
- Then:
 1. How do you calibrate? (translation, spread, and deformation).
 2. How do you GMM or SMM?
 3. Asymptotics?

- For second-order approximations, [Kim et al. \(2008\)](#): pruning.
- Idea:

$$\begin{aligned}\widehat{k}_{t+1} &= a_0 + a_1 \left(a_0 + a_1 \widehat{k}_t + a_2 \varepsilon_t + a_3 \widehat{k}_t^2 + a_4 \varepsilon_t^2 + a_5 \widehat{k}_t \varepsilon_t \right) + a_2 \varepsilon_t \\ &\quad + a_3 \left(a_0 + a_1 \widehat{k}_t + a_2 \varepsilon_t + a_3 \widehat{k}_t^2 + a_4 \varepsilon_t^2 + a_5 \widehat{k}_t \varepsilon_t \right)^2 + a_4 \varepsilon_t^2 \\ &\quad + a_5 \left(a_0 + a_1 \widehat{k}_t + a_2 \varepsilon_t + a_3 \widehat{k}_t^2 + a_4 \varepsilon_t^2 + a_5 \widehat{k}_t \varepsilon_t \right) \varepsilon_t\end{aligned}$$

- We omit terms raised to powers higher than 2.
- Pruned approximation does not explode.

What do we do?

- Build a pruned state-space system.
- Apply pruning to an approximation of any arbitrary order.
- Prove that first and second unconditional moments exist.
- Closed-form expressions for first and second unconditional moments and IRFs.
- Conditions for the existence of some higher unconditional moments, such as skewness and kurtosis.
- Apply to a New Keynesian model with EZ preferences.
- Software available online.

Consequences

1. GMM and IRF-matching can be implemented without simulation.
2. First and second unconditional moments or IRFs can be computed in a trivial amount of time for medium-sized DSGE models approximated up to third-order.
3. Use the unconditional moment conditions in optimal GMM estimation to build a limited information likelihood function for Bayesian inference ([Kim, 2002](#)).
4. Foundation for indirect inference as in [Smith \(1993\)](#) and SMM as in [Duffie and Singleton \(1993\)](#).
5. Calibration.

State-Space Representations

- Dynamic model:

$$\begin{aligned}\mathbf{x}_{t+1} &= \mathbf{h}(\mathbf{x}_t, \sigma) + \sigma\eta\epsilon_{t+1}, \epsilon_{t+1} \sim IID(\mathbf{0}, \mathbf{I}) \\ \mathbf{y}_t &= \mathbf{g}(\mathbf{x}_t, \sigma)\end{aligned}$$

- What is behind this system?
- General structure (use of augmented state vector).

The state-space system I

- Perturbation methods approximate $\mathbf{h}(\mathbf{x}_t, \sigma)$ and $\mathbf{g}(\mathbf{x}_t, \sigma)$ with Taylor-series expansions around $\mathbf{x}_{ss} = \sigma = 0$.
- A first-order approximated state-space system replaces $\mathbf{g}(\mathbf{x}_t, \sigma)$ and $\mathbf{h}(\mathbf{x}_t, \sigma)$ with $\mathbf{g}_x \mathbf{x}_t$ and $\mathbf{h}_x \mathbf{x}_t$.
- If $\forall \text{mod}(\text{eig}(\mathbf{h}_x)) < 1$, the approximation fluctuates around the steady state (also its mean value).
- Thus, easy to calibrate the model based on first and second moments or to estimate it using Bayesian methods, MLE, GMM, SMM, etc.

The state-space system II

- We can replace $\mathbf{g}(\mathbf{x}_t, \sigma)$ and $\mathbf{h}(\mathbf{x}_t, \sigma)$ with their higher-order Taylor-series expansions.
- However, the approximated state-space system cannot, in general, be shown to have any finite moments.
- Also, it often displays explosive dynamics.
- This occurs even with simple versions of the New Keynesian model.
- Hence, it is difficult to use the approximated state-space system to calibrate or to estimate the parameters of the model.

The pruning method: second-order approximation I

- Partition states:

$$\begin{bmatrix} (\mathbf{x}_t^f)' & (\mathbf{x}_t^s)' \end{bmatrix}$$

- Original state-space representation:

$$\begin{aligned} \mathbf{x}_{t+1}^{(2)} &= \mathbf{h}_x (\mathbf{x}_t^f + \mathbf{x}_t^s) + \frac{1}{2} \mathbf{H}_{xx} ((\mathbf{x}_t^f + \mathbf{x}_t^s) \otimes (\mathbf{x}_t^f + \mathbf{x}_t^s)) + \frac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 + \sigma \eta \epsilon_{t+1} \\ \mathbf{y}_t^{(2)} &= \mathbf{g}_x \mathbf{x}_t^{(2)} + \frac{1}{2} \mathbf{G}_{xx} (\mathbf{x}_t^{(2)} \otimes \mathbf{x}_t^{(2)}) + \frac{1}{2} \mathbf{g}_{\sigma\sigma} \sigma^2 \end{aligned}$$

The pruning method: second-order approximation II

- New state-space representation:

$$\begin{aligned}\mathbf{x}_{t+1}^f &= \mathbf{h}_x \mathbf{x}_t^f + \sigma \eta \epsilon_{t+1} \\ \mathbf{x}_{t+1}^s &= \mathbf{h}_x \mathbf{x}_t^s + \frac{1}{2} \mathbf{H}_{xx} (\mathbf{x}_t^f \otimes \mathbf{x}_t^f) + \frac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 \\ \mathbf{y}_t^f &= \mathbf{g}_x \mathbf{x}_t^f \\ \mathbf{y}_t^s &= \mathbf{g}_x (\mathbf{x}_t^f + \mathbf{x}_t^s) + \frac{1}{2} \mathbf{G}_{xx} (\mathbf{x}_t^f \otimes \mathbf{x}_t^f) + \frac{1}{2} \mathbf{g}_{\sigma\sigma} \sigma^2\end{aligned}$$

- All variables are second-order polynomials of the innovations.

The pruning method: third-order approximation I

- Partition states:

$$\begin{bmatrix} (\mathbf{x}_t^f)' & (\mathbf{x}_t^s)' & (\mathbf{x}_t^{rd})' \end{bmatrix}$$

- Original state-space representation:

$$\begin{aligned} \mathbf{x}_{t+1}^{(3)} &= \mathbf{h}_x \mathbf{x}_t^{(3)} + \frac{1}{2} \mathbf{H}_{xx} \left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \right) + \frac{1}{6} \mathbf{H}_{xxx} \left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \right) \\ &\quad + \frac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 + \frac{3}{6} \mathbf{h}_{\sigma\sigma x} \sigma^2 \mathbf{x}_t^{(3)} + \frac{1}{6} \mathbf{h}_{\sigma\sigma\sigma} \sigma^3 + \sigma \eta \epsilon_{t+1} \\ \mathbf{y}_t^{(3)} &= \mathbf{g}_x \mathbf{x}_t^{(3)} + \frac{1}{2} \mathbf{G}_{xx} \left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \right) + \frac{1}{6} \mathbf{G}_{xxx} \left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \right) \\ &\quad + \frac{1}{2} \mathbf{g}_{\sigma\sigma} \sigma^2 + \frac{3}{6} \mathbf{g}_{\sigma\sigma x} \sigma^2 \mathbf{x}_t^{(3)} + \frac{1}{6} \mathbf{g}_{\sigma\sigma\sigma} \sigma^3 \end{aligned}$$

The pruning method: third-order approximation II

- New state-space representation:

Second-order pruned state-space representation+

$$\begin{aligned}\mathbf{x}_{t+1}^{rd} &= \mathbf{h}_x \mathbf{x}_t^{rd} + \mathbf{H}_{xx} (\mathbf{x}_t^f \otimes \mathbf{x}_t^s) + \frac{1}{6} \mathbf{H}_{xxx} (\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f) \\ &\quad + \frac{3}{6} \mathbf{h}_{\sigma\sigma x} \sigma^2 \mathbf{x}_t^f + \frac{1}{6} \mathbf{h}_{\sigma\sigma\sigma} \sigma^3 \\ \mathbf{y}_t^{rd} &= \mathbf{g}_x (\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}) + \frac{1}{2} \mathbf{G}_{xx} ((\mathbf{x}_t^f \otimes \mathbf{x}_t^f) + 2 (\mathbf{x}_t^f \otimes \mathbf{x}_t^s)) \\ &\quad + \frac{1}{6} \mathbf{G}_{xxx} (\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f) + \frac{1}{2} \mathbf{g}_{\sigma\sigma} \sigma^2 + \frac{3}{6} \mathbf{g}_{\sigma\sigma x} \sigma^2 \mathbf{x}_t^f + \frac{1}{6} \mathbf{g}_{\sigma\sigma\sigma} \sigma^3\end{aligned}$$

- All variables are third-order polynomials of the innovations.

Higher-order approximations

- We can generalize previous steps:
 1. Decompose the state variables into first-, second-, ... , and k th-order effects.
 2. Set up laws of motions for the state variables capturing only first-, second-, ... , and k th-order effects.
 3. Construct the expression for control variables by preserving only effects up to k th-order.

Statistical properties: second-order approximation I

Theorem

If $\forall \text{mod}(\text{eig}(\mathbf{h}_x)) < 1$ and ϵ_{t+1} has finite fourth moments, the pruned state-space system has finite first and second moments.

Theorem

If $\forall \text{mod}(\text{eig}(\mathbf{h}_x)) < 1$ and ϵ_{t+1} has finite sixth and eighth moments, the pruned state-space system has finite third and fourth moments.

Statistical properties: second-order approximation I

- We introduce the vectors

$$\mathbf{z}_t^{(2)} \equiv \begin{bmatrix} (\mathbf{x}_t^f)' & (\mathbf{x}_t^s)' & (\mathbf{x}_t^f \otimes \mathbf{x}_t^f)' \end{bmatrix}'$$
$$\xi_{t+1}^{(2)} \equiv \begin{bmatrix} \epsilon_{t+1} \\ \epsilon_{t+1} \otimes \epsilon_{t+1} - \text{vec}(\mathbf{I}_{n_e}) \\ \epsilon_{t+1} \otimes \mathbf{x}_t^f \\ \mathbf{x}_t^f \otimes \epsilon_{t+1} \end{bmatrix}$$

- First moment:

$$\mathbb{E}[\mathbf{x}_t^{(2)}] = \underbrace{\mathbb{E}[\mathbf{x}_t^f]}_{=0} + \underbrace{\mathbb{E}[\mathbf{x}_t^s]}_{\neq 0}$$
$$\mathbb{E}[\mathbf{x}_t^s] = (\mathbf{I} - \mathbf{h}_x)^{-1} \left(\frac{1}{2} \mathbf{H}_{xx} (\mathbf{I} - \mathbf{h}_x \otimes \mathbf{h}_x)^{-1} (\sigma\eta \otimes \sigma\eta) \text{vec}(\mathbf{I}_{n_e}) + \frac{1}{2} \mathbf{h}_{\sigma\sigma} \sigma^2 \right)$$
$$\mathbb{E}[\mathbf{y}_t^s] = \mathbf{C}^{(2)} \mathbb{E}[\mathbf{z}_t^{(2)}] + \mathbf{d}^{(2)}$$

Statistical properties: second-order approximation III

- Second moment:

$$\mathbb{V} \left(\mathbf{z}_t^{(2)} \right) = \mathbf{A}^{(2)} \mathbb{V} \left(\mathbf{z}_t^{(2)} \right) \left(\mathbf{A}^{(2)} \right)' + \mathbf{B}^{(2)} \mathbb{V} \left(\xi_t^{(2)} \right) \left(\mathbf{B}^{(2)} \right)'$$

$$\text{Cov} \left(\mathbf{z}_{t+l}^{(2)}, \mathbf{z}_t^{(2)} \right) = \left(\mathbf{A}^{(2)} \right)^l \mathbb{V} \left(\mathbf{z}_t^{(2)} \right) \quad \text{for } l = 1, 2, 3, \dots$$

$$\mathbb{V} \left[\mathbf{x}_t^{(2)} \right] = \mathbb{V} \left(\mathbf{x}_t^f \right) + \mathbb{V} \left(\mathbf{x}_t^s \right) + \text{Cov} \left(\mathbf{x}_t^f, \mathbf{x}_t^s \right) + \text{Cov} \left(\mathbf{x}_t^s, \mathbf{x}_t^f \right)$$

$$\mathbb{V} \left[\mathbf{y}_t^s \right] = \mathbf{C}^{(2)} \mathbb{V} \left[\mathbf{z}_t \right] \left(\mathbf{C}^{(2)} \right)'$$

$$\text{Cov} \left(\mathbf{y}_t^s, \mathbf{y}_{t+l}^s \right) = \mathbf{C}^{(2)} \text{Cov} \left(\mathbf{z}_{t+l}^{(2)}, \mathbf{z}_t^{(2)} \right) \left(\mathbf{C}^{(2)} \right)' \quad \text{for } l = 1, 2, 3, \dots$$

where we solve for $\mathbb{V} \left(\mathbf{z}_t^{(2)} \right)$ by standard methods for discrete Lyapunov equations.

- Generalized impulse response function (GIRF): [Koop et al. \(1996\)](#)

$$GIRF_{\text{var}}(l, \nu, \mathbf{w}_t) = \mathbb{E}[\mathbf{var}_{t+l} | \mathbf{w}_t, \epsilon_{t+1} = \nu] - \mathbb{E}[\mathbf{var}_{t+l} | \mathbf{w}_t]$$

- Importance in models with volatility shocks.

Statistical properties: third-order approximation I

Theorem

If $\forall \text{mod}(\text{eig}(\mathbf{h}_x)) < 1$ and ϵ_{t+1} has finite sixth moments, the pruned state-space system has finite first and second moments.

Theorem

If $\forall \text{mod}(\text{eig}(\mathbf{h}_x)) < 1$ and ϵ_{t+1} has finite ninth and twelfth moments, the pruned state-space system has finite third and fourth moments.

- Similar (but long!!!!) formulae for first and second moments and IRFs.

Application

- A middle-scale New Keynesian model with habit formation and EZ preferences.
- Why?
 1. Standard model for policy analysis.
 2. Sizable higher-order terms.
 3. The model should not generate explosive sample paths when simulated with the unpruned state-space system.

- What will we do?
 1. Check the accuracy of pruned state-space representations.
 2. Estimate the model with GMM and SMM.
 3. Explore its properties.

- Preferences:

$$V_t \equiv \begin{cases} u_t + \beta \left(\mathbb{E}_t \left[V_{t+1}^{1-\phi_3} \right] \right)^{\frac{1}{1-\phi_3}} & \text{if } u_t > 0 \text{ for all } t \\ u_t - \beta \left(\mathbb{E}_t \left[(-V_{t+1})^{1-\phi_3} \right] \right)^{\frac{1}{1-\phi_3}} & \text{if } u_t < 0 \text{ for all } t \end{cases}$$

where

$$u_t \equiv d_t \frac{(c_t - bc_{t-1})^{1-\phi_2}}{1-\phi_2} + (z_t^*)^{(1-\phi_2)} \phi_0 \frac{(1-h_t)^{1-\phi_1}}{1-\phi_1}$$

and

$$\log d_{t+1} = \rho_d \log d_t + \epsilon_{d,t+1}, \quad \epsilon_{d,t} \sim IID(0, \sigma_d^2)$$

- The budget constraint:

$$c_t + \frac{i_t}{\Upsilon_t} + \int D_{t,t+1} x_{t+1} d\omega_{t,t+1} = w_t h_t + r_t^k k_t + \frac{x_t}{\pi_t} + div_t$$

- Capital:

$$k_{t+1} = (1 - \delta) k_t + i_t - \frac{\kappa}{2} \left(\frac{i_t}{k_t} - \psi \right)^2 k_t$$

- Final firm:

$$y_t = \left(\int_0^1 y_{i,t}^{(\eta-1)/\eta} di \right)^{\eta/(\eta-1)}$$

- Intermediate goods producers:

$$y_{i,t} = a_t k_{i,t}^\theta (z_t h_{i,t})^{1-\theta}$$

$$\log z_{t+1} = \log z_t + \log \mu_{z,ss}$$

$$z_t^* \equiv \Upsilon_t^{\frac{\theta}{1-\theta}} z_t$$

$$\log a_{t+1} = \rho_a \log a_t + \epsilon_{a,t+1}, \quad \epsilon_{a,t} \sim IID(0, \sigma_a^2)$$

- Two versions: Calvo and quadratic price adjustment costs $\xi_p \geq 0$ w.r.t. π_{ss} .

- Taylor rule for the monetary authority:

$$r_{t,1} = (1 - \rho_r) r_{ss} + \rho_r r_{t-1,1} + \beta_\pi \log \left(\frac{\pi_t}{\pi_{ss}} \right) + \beta_y \log \left(\frac{y_t}{z_t^* Y_{ss}} \right)$$

- 13 state variables.
- We detrend variables.
- Second- and third-order approximation.
- We check the Euler equation errors to compare the accuracy of pruned and unpruned state-space systems with a standard calibration.
- Results are even better for large innovations.

	First-order	Second-order	
	RMSE	RMSE ^P	RMSE
Benchmark			
Household's value function	1.0767	0.9382	0.5998
Household's FOC for consumption	49.8994	1.8884	1.8069
Household's FOC for capital	5.5079	0.2006	0.1976
Household's FOC for labor	0.4605	0.1769	0.5305
Household's FOC for investment	0.0570	0.0092	0.1023
Euler-eq. for one-period interest rate	4.9705	0.1930	0.1922
Firm's FOC for prices	3.8978	0.2026	0.1709
Income identity	0.1028	0.1004	0.1840
Law of motion for capital	0.1295	0.0239	0.2499
Average error	7.3447	0.4148	0.4482

	First-order	Third-order	
	RMSE	RMSE ^P	RMSE
Benchmark			
Household's value function	1.0767	0.4801	0.3269
Household's FOC for consumption	49.8994	0.7840	1.1971
Household's FOC for capital	5.5079	0.0907	0.1473
Household's FOC for labor	0.4605	0.0406	0.1831
Household's FOC for investment	0.0570	0.0023	0.0553
Euler-eq. for one-period interest rate	4.9705	0.0770	0.1320
Firm's FOC for prices	3.8978	0.0903	0.0845
Income identity	0.1028	0.0405	0.1333
Law of motion for capital	0.1295	0.0056	0.1327
Average error	7.3447	0.1790	0.2658

Estimation

- Version with Calvo pricing.
- US Macro and financial data from 1961Q3 to 2007Q4:
 1. consumption growth Δc_t
 2. investment growth Δi_t
 3. inflation π_t
 4. 1-quarter nominal interest rate $r_{t,1}$
 5. 10-year nominal interest rate $r_{t,40}$
 6. 10-year ex post excess holding period return $xhr_{t,40} \equiv \log(P_{t,39}/P_{t-1,40}) - r_{t-1,1}$
 7. log of hours $\log h_t$.
- Use GMM and SMM.
- Computation: 0.03 seconds in second-order, 0.8 seconds in third-order, 1.4 for GIRFs (all in Matlab)

	GMM ^{2nd}	GMM ^{3rd}	SMM ^{3rd}
β	0.9925 (0.0021)	0.9926 (0.0002)	0.9926 (0.0023)
b	0.6889 (0.0194)	0.7137 (0.0004)	0.7332 (0.0085)
h_{ss}	0.3402 (0.0010)	0.3401 (0.0004)	0.3409 (0.0065)
ϕ_1	6.1405 (1.2583)	6.1252 (0.0002)	6.1169 (0.0040)
ϕ_2	1.5730 (0.1400)	1.5339 (0.0008)	1.5940 (0.0009)
ϕ_3	-196.31 (51.90)	-197.36 (0.01)	-194.22 (0.01)
κ	4.1088 (0.7213)	3.5910 (0.0160)	3.5629 (0.1085)
α	0.9269 (0.0044)	0.9189 (0.0026)	0.9195 (0.0024)
ρ_r	0.6769 (0.6086)	0.6759 (0.0723)	0.6635 (0.1464)
β_π	3.9856 (8.2779)	3.6974 (0.7892)	3.6216 (1.8555)
β_y	0.5553 (1.5452)	0.50691 (0.1465)	0.5027 (0.3685)

	GMM ^{2nd}	GMM ^{3rd}	SMM ^{3rd}
$\mu_{\gamma,ss}$	1.0018 (0.0012)	1.0017 (0.0007)	1.0016 (0.0006)
$\mu_{z,ss}$	1.0050 (0.0005)	1.0051 (0.0004)	1.0052 (0.0003)
ρ_a	0.9192 (0.0081)	0.9165 (0.0030)	0.9139 (0.0036)
ρ_d	0.9915 (0.0023)	0.9914 (0.0005)	0.9911 (0.0019)
π_{ss}	1.0407 (0.0134)	1.0419 (0.0022)	1.0432 (0.0057)
σ_α	0.0171 (0.0006)	0.0183 (0.0005)	0.0183 (0.0003)
σ_d	0.0144 (0.0017)	0.0144 (0.0005)	0.0143 (0.0018)
skew _a	—	—	0.2296 (0.0298)
tail _a	—	—	1.2526 (0.0437)
skew _d	—	—	0.0693 (0.4530)
tail _d	—	—	1.1329 (3.4724)

	Data	GMM ^{2nd}	GMM ^{3rd}	SMM ^{3rd}
Means				
$\Delta c_t \times 100$	2.439	2.399	2.429	2.435
$\Delta i_t \times 100$	3.105	3.111	3.099	3.088
$\pi_t \times 100$	3.757	3.681	3.724	3.738
$r_{t,1} \times 100$	5.605	5.565	5.548	5.582
$r_{t,40} \times 100$	6.993	6.925	6.955	6.977
$xhr_{t,40} \times 100$	1.724	1.689	1.730	1.717
$\log h_t$	-1.084	-1.083	-1.083	-1.083

	Data	GMM ^{2nd}	GMM ^{3rd}	SMM ^{3rd}
Standard deviations (in pct)				
Δc_t	2.685	1.362	1.191	1.127
Δi_t	8.914	8.888	8.878	8.944
π_t	2.481	3.744	3.918	3.897
$r_{t,1}$	2.701	4.020	4.061	4.060
$r_{t,40}$	2.401	2.325	2.326	2.308
$xhr_{t,40}$	22.978	22.646	22.883	22.949
$\log h_t$	1.676	3.659	3.740	3.721

	Data	GMM ^{2nd}	GMM ^{3rd}	SMM ^{3rd}
Auto-correlations: 1 lag				
$\text{corr}(\Delta c_t, \Delta c_{t-1})$	0.254	0.702	0.726	0.7407
$\text{corr}(\Delta i_t, \Delta i_{t-1})$	0.506	0.493	0.480	0.4817
$\text{corr}(\pi_t, \pi_{t-1})$	0.859	0.988	0.986	0.9861
$\text{corr}(r_{t,1}, r_{t-1,1})$	0.942	0.989	0.987	0.987
$\text{corr}(r_{t,40}, r_{t-1,40})$	0.963	0.969	0.969	0.968
$\text{corr}(xhr_{t,40}, xhr_{t-1,40})$	-0.024	0.000	-0.003	-0.003
$\text{corr}(\log h_t, \log h_{t-1})$	0.792	0.726	0.678	0.6706

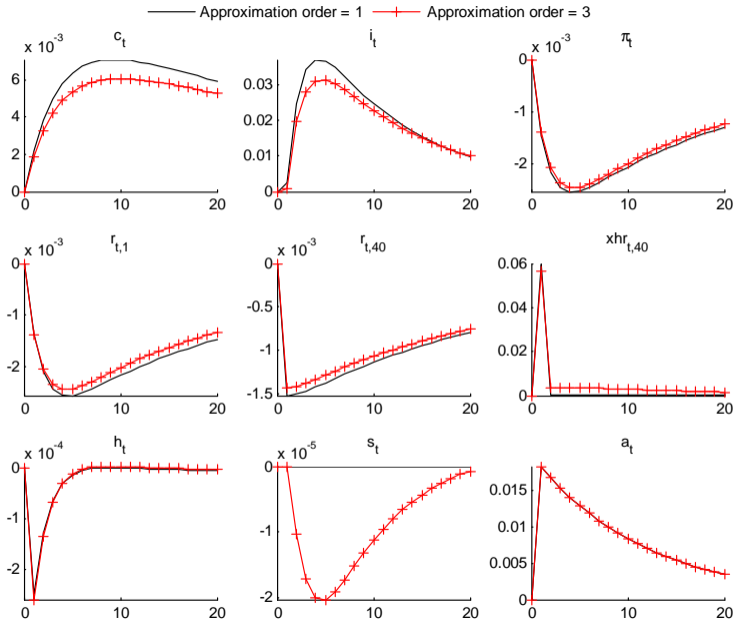
	Data	GMM ^{2nd}	GMM ^{3rd}	SMM ^{3rd}
Skewness				
Δc_t	-0.679	0.024	0.034	0.193
Δi_t	-0.762	-0.191	-0.254	-0.122
π_t	1.213	0.013	0.014	-0.054
r_t	1.053	0.012	0.011	-0.051
$r_{t,40}$	0.967	0.014	0.017	-0.043
$xhr_{t,40}$	0.364	-0.026	-0.028	0.368

	Data	GMM ^{2nd}	GMM ^{3rd}	SMM ^{3rd}
Kurtosis				
Δc_t	5.766	3.011	3.015	3.547
Δi_t	5.223	3.157	3.279	4.425
π_t	4.232	2.987	2.985	3.040
r_t	4.594	2.968	2.975	3.033
$r_{t,40}$	3.602	2.987	2.979	3.028
$xhr_{t,40}$	5.121	3.003	3.006	5.167

	Data	GMM ^{2nd}	GMM ^{3rd}	SMM ^{3rd}
$corr(\Delta c_t, \Delta i_t)$	0.594	0.590	0.579	0.582
$corr(\Delta c_t, \pi_t)$	-0.362	-0.238	-0.296	-0.310
$corr(\Delta c_t, r_{t,1})$	-0.278	-0.210	-0.274	-0.290
$corr(\Delta c_t, r_{t,40})$	-0.178	-0.3337	-0.355	-0.366
$corr(\Delta c_t, xhr_{t,40})$	0.271	0.691	0.655	0.641
$corr(\Delta c_t, \log h_t)$	0.065	-0.677	-0.670	-0.674
$corr(\Delta i_t, \pi_t)$	-0.242	-0.075	-0.098	-0.098
$corr(\Delta i_t, r_{t,1})$	-0.265	-0.058	-0.084	-0.088
$corr(\Delta i_t, r_{t,40})$	-0.153	-0.130	-0.133	-0.135
$corr(\Delta i_t, xhr_{t,40})$	0.021	0.015	0.024	0.027
$corr(\Delta i_t, \log h_t)$	0.232	-0.398	-0.406	-0.418

	Data	GMM ^{2nd}	GMM ^{3rd}	SMM ^{3rd}
$corr(\pi_t, r_{t,1})$	0.628	0.994	0.997	0.997
$corr(\pi_t, r_{t,40})$	0.479	0.990	0.988	0.987
$corr(\pi_t, xhr_{t,40})$	-0.249	-0.130	-0.142	-0.141
$corr(\pi_t, \log h_t)$	-0.467	0.132	0.128	0.154
$corr(r_{t,1}, r_{t,40})$	0.861	0.986	0.991	0.991
$corr(r_{t,1}, xhr_{t,40})$	-0.233	-0.122	-0.137	-0.138
$corr(r_{t,1}, \log h_t)$	-0.369	0.177	0.153	0.180
$corr(r_{t,40}, xhr_{t,40})$	-0.121	-0.247	-0.248	-0.249
$corr(r_{t,40}, \log h_t)$	-0.409	0.229	0.238	0.268
$corr(xhr_{t,40}, \log h_t)$	-0.132	-0.644	-0.680	-0.690

	GMM ^{2nd}	GMM ^{3rd}	SMM ^{3rd}
Objective function: Q	0.0920	0.1055	0.0958
Number of moments	42	42	54
Number of parameters	18	18	22
P-value	0.8437	0.7183	0.9797



— Approximation order = 1 —+— Approximation order = 3

