The neoclassical growth model

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Introduction
Neoclassical growth model

- Original contribution of Ramsey (1928). That is why sometimes it is known as the Ramsey model.

- Completed by David Cass (1965) and Tjalling Koopmans (1965). That is why sometimes it is known as the Cass-Koopmans model.

- William Brock and Leonard Mirman (1972) introduced uncertainty.

- Finn Kydland and Edward Prescott (1982) used it to create the real business cycle research agenda.
Environment
• Representative household with a utility function:

\[ u(c(t)) \]

**Definition**

\( u(c) \) is strictly increasing, concave, twice continuously differentiable with derivatives \( u' \) and \( u'' \), and satisfies Inada conditions:

\[
\begin{align*}
\lim_{c \to 0} u'(c) &= \infty \\
\lim_{c \to \infty} u'(c) &= 0
\end{align*}
\]
Dynastic structure

- Population evolves:
  \[ L(t) = \exp(nt) \]

  with \( L_0 = 1 \).

- Intergenerational altruism.

- Intertemporal utility function:
  \[ U(0) = \int_0^\infty e^{-(\rho-n)t} u(c(t)) \, dt \]

- \( \rho \): subjective discount rate, such that \( \rho > n \).

- \( \rho - n \): “effective” discount rate.
Budget constraint

- Asset evolution:

\[ \dot{a} = (r - \delta - n)a + w - c \]

- Who owns the capital in the economy? Role of complete markets.

- Modigliani-Miller theorems.

- Arrow securities.
No-Ponzi-game condition

- No-Ponzi games.
- Historical examples.
- Condition:

\[
\lim_{t \to \infty} a(t) \exp \left( - \int_0^t (r - \delta - n) \, ds \right) = 0
\]
Production side

- Cobb-Douglas aggregate production function:
  \[ Y = K^\alpha L^{1-\alpha} \]

- Per capita terms:
  \[ y = k^\alpha \]

- From the first order condition of firm with respect to capital \( k \):
  \[ r = \alpha k^{\alpha-1} \]
  \[ w = k^\alpha - k\alpha k^{\alpha-1} = (1 - \alpha) k^\alpha \]

- Interest rate:
  \[ r - \delta \]
Aggregate consistency conditions

- Asset market clearing:
  \[ a = k \]

- Implicitly, labor market clearing.

- Resource constraint:
  \[ \dot{k} = k^\alpha - c - (n + \delta) k \]
Competitive equilibrium
A competitive equilibrium is a sequence of per capita allocations \( \{c(t), k(t)\}_{t=0}^{\infty} \) and input prices \( \{r(t), w(t)\}_{t=0}^{\infty} \) such that:

- Given input prices, \( \{r(t), w(t)\}_{t=0}^{\infty} \), the representative household maximizes its utility:

\[
\max_{\{c(t), a(t)\}_{t=0}^{\infty}} \int_{0}^{\infty} e^{-(\rho-n)t} u(c(t)) \, dt
\]

\[
\text{s.t. } \dot{a} = (r - \delta - n) a + w - c
\]

\[
\lim_{t \to \infty} a(t) \exp \left( - \int_{0}^{t} (r - \delta - n) \, ds \right) = 0
\]

\[
a_0 = k_0
\]
Input prices, \( \{r(t), w(t)\}_{t=0}^{\infty} \), are equal to the marginal productivities:

\[
\begin{align*}
  r(t) &= \alpha k(t)^{\alpha-1} \\
  w(t) &= (1 - \alpha) k(t)^{\alpha}
\end{align*}
\]

Markets clear:

\[
\begin{align*}
  a(t) &= k(t) \\
  \dot{k} &= k(t)^{\alpha} - c(t) - (n + \delta) k(t)
\end{align*}
\]
Solving the model
Household maximization

- We can come back now to the problem of the household.

- We build the Hamiltonian:

\[ H(a, c, \mu) = u(c(t)) + \mu(t)((r(t) - n - \delta) a(t) - w(t) - c(t)) \]

where:

1. \( a(t) \) is the state variable.
2. \( c(t) \) is the control variable.
3. \( \mu(t) \) is the current-value co-state variable.
Necessary conditions

1. Partial derivative of the Hamiltonian with respect to controls is equal to zero:
   \[ H_c(a, c, \mu) = u'(c(t)) - \mu(t) = 0 \]

2. Partial derivative of the Hamiltonian with respect to states is:
   \[ H_a(a, c, \mu) = \mu(t)(r(t) - n - \delta) = (\rho - n)\mu(t) - \dot{\mu}(t) \]

3. Partial derivative of the Hamiltonian with respect to co-states is:
   \[ H_\mu(a, c, \mu) = (r(t) - n - \delta)a(t) - c(t) = \dot{a}(t) \]

4. Transversality condition:
   \[ \lim_{t \to \infty} e^{-\rho t} \mu(t) \dot{a}(t) = 0 \]
Working with the necessary conditions I

- From the second condition:

\[ \mu (r - n - \delta) = (\rho - n) \mu - \dot{\mu} \Rightarrow \]
\[ (r - n - \delta) = (\rho - n) - \frac{\dot{\mu}}{\mu} \Rightarrow \]
\[ \frac{\dot{\mu}}{\mu} = -(r - \delta - \rho) \]

- From the first condition:

\[ u'(c) = \mu \]

and taking derivatives with respect to time:

\[ u''(c) \dot{c} = \dot{\mu} \Rightarrow \]
\[ \frac{u''(c)}{u'(c)} \dot{c} = \frac{\dot{\mu}}{\mu} = -(r - \delta - \rho) \]
Now, we can combine both expression:

\[-\sigma \frac{\dot{c}}{c} = -(r - \delta - \rho)\]

where

\[\sigma = -\frac{u''(c)}{u'(c)} = \frac{d \log (c(s)/c(t))}{d \log (u'(c(s))/u'(c(t)))}\]

is the (inverse of) elasticity of intertemporal substitution (EIS).

Thus:

\[\frac{\dot{c}}{c} = \frac{1}{\sigma} (r - \delta - \rho)\]

This expression is known as the consumer Euler equation.
• In the previous equation, we have implicitly assumed that $\sigma$ is a constant.

• This will be only true of a class of utility functions.

• Constant Relative Risk Aversion (CRRA):

\[
\begin{align*}
\frac{c^{1-\sigma} - 1}{1 - \sigma} & \quad \text{for } \sigma \neq 1 \\
\log c & \quad \text{for } \sigma = 1
\end{align*}
\]

(you need to take limits and apply L’Hôpital’s rule).

• Why is it called CRRA?
Applying equilibrium conditions

- First, note that $r = \alpha k^{\alpha - 1}$. Then:

$$\frac{\dot{c}}{c} = \frac{1}{\sigma} (\alpha k^{\alpha - 1} - \delta - \rho)$$

- Second, $k = a$. Then:

$$\dot{a} = (r - \delta - n) a + w - c \Rightarrow$$

$$\dot{k} = (\alpha k^{\alpha - 1} - \delta - n) k + w - c \Rightarrow$$

$$\dot{k} = k^\alpha - c - (n + \delta) k$$

where in the last step we use the fact that $k^\alpha = \alpha k^{\alpha - 1} k + w$. 
System of differential equations
We have two differential equations:

\[
\frac{\dot{c}}{c} = \frac{1}{\sigma} \left( \alpha k^{\alpha - 1} - \delta - \rho \right)
\]

\[
\dot{k} = k^\alpha - c - (n + \delta) k
\]

on two variables, \(k\) and \(c\), plus the transversality condition:

\[
\lim_{t \to \infty} e^{-\rho t} \mu \hat{a} = \lim_{t \to \infty} e^{-\rho t} \mu \hat{k} = 0
\]

How do we solve it?
Steady state

• We search for a steady state where $\dot{c} = \dot{k} = 0$.

• Then:

$$\frac{1}{\sigma} \left( \alpha (k^*)^{\alpha - 1} - \delta - \rho \right) = 0$$
$$\left( k^* \right)^\alpha - c^* - (n + \delta) k^* = 0$$

• System of two equations on two unknowns $k^*$ and $c^*$ with solution:

$$k^* = \left( \frac{\alpha}{\rho + \delta} \right)^{\frac{1}{1-\alpha}}$$
$$c^* = \left( k^* \right)^\alpha - (n + \delta) k^*$$

• Note that EIS does not enter into the steady state. In fact, the form of the utility function is irrelevant!
Transitional dynamics

- The neoclassical growth model does not have a closed-form solution.

- We can do three things:
  1. Use a phase diagram.
  2. Solve an approximated version of the model where we linearize the equations.
  3. Use the computer to approximate the solution numerically.
Phase diagram

\[ c(t) \]

- \( c^* \)
- \( c^{**} \)
- \( c(t)=0 \)
- \( k(t) \)
- \( k^* \)
- \( k^{**} \)
- \( k_{gold} \)
- \( k_l \)

\[ \dot{k}(t)=0 \]
Comparative statics
• We can linearize the system:

\[
\begin{align*}
\dot{c} &= \frac{1}{\sigma} \left( \alpha k^{\alpha-1} - \delta - \rho \right) \\
\dot{k} &= k^\alpha - c - (n + \delta) k
\end{align*}
\]

• We get:

\[
\begin{align*}
\dot{c} &\approx \frac{c^* \alpha (\alpha - 1) (k^*)^{\alpha-2}}{\sigma} (k - k^*) + \frac{\alpha (k^*)^{\alpha-1} - \delta - \rho}{\sigma} (c - c^*) \\
&= \frac{c^*}{\sigma} \left( \alpha (\alpha - 1) (k^*)^{\alpha-2} \right) (k - k^*) \\
\end{align*}
\]

and:

\[
\begin{align*}
\dot{k} &\approx \left( \alpha (k^*)^{\alpha-1} - n - \delta \right) (k - k^*) - (c - c^*) \\
&= (\rho - n) (k - k^*) - (c - c^*)
\end{align*}
\]
Linearization II

- The behavior of the linearized system is given by the roots (eigenvalues) $\xi$ of:

$$\det \begin{pmatrix}
\rho - n - \xi & 1 \\
\frac{c^*}{\sigma} (\alpha (\alpha - 1) (k^*)^{\alpha - 2}) & -\xi
\end{pmatrix}$$

- Solving:

$$-\xi (\rho - n - \xi) + \frac{c^*}{\sigma} (\alpha (\alpha - 1) (k^*)^{\alpha - 2}) = 0 \Rightarrow$$

$$\xi^2 - \xi (\rho - n) + \frac{c^*}{\sigma} (\alpha (\alpha - 1) (k^*)^{\alpha - 2}) = 0$$

- Thus:

$$\xi = \frac{(\rho - n) \pm \sqrt{1 - 4 \left(\alpha (\alpha - 1) (k^*)^{\alpha - 2}\right)}}{2}$$

and since $\alpha (\alpha - 1) < 1$, we have one positive and one negative eigenvalue $\Rightarrow$ one stable manifold.
• We will call $\xi_1$ the positive eigenvalue and $\xi_2$ the negative one.

• With some results in differential equations, we can show:

$$k = k^* + \eta_1 e^{\xi_1 t} + \eta_2 e^{\xi_2 t} \Rightarrow$$

$$k - k^* = \eta_1 e^{\xi_1 t} + \eta_2 e^{\xi_2 t}$$

where $\eta_1$ and $\eta_2$ are arbitrary constants of integration.

• It must be that $\eta_1 = 0$. If $\eta_1 > 0$, we will violate the transversality condition and $\eta_1 < 0$ will take $k_t$ to 0.

• Then, $\eta_2$ is determined by:

$$\eta_2 = k_0 - k^*$$

• Hence:

$$k = (1 - e^{\xi_2 t}) k^* + e^{\xi_2 t} k_0 \Rightarrow$$

$$k - k^* = \eta_2 e^{\xi_2 t} = (k_0 - k^*) e^{\xi_2 t}$$
• Also:

\[ \dot{c} = \frac{c^*}{\sigma} \left( \alpha (\alpha - 1) (k^*)^{\alpha-2} \right) (k - k^*) \]

or

\[ c = \frac{c^*}{\sigma} \left( \alpha (\alpha - 1) (k^*)^{\alpha-2} \right) \frac{\eta_2}{\xi_2} e^{\xi_2 t} + c^* \]

where the constant \( c^* \) ensures that we converge to the steady state.

• Since \( y = k^\alpha \), we get:

\[ \log y = \alpha \log \left( k^* + (k_0 - k^*) e^{\xi_2 t} \right) \]
• Taking time derivatives and making \( y = y_0 \):

\[
\frac{\dot{y}}{y_0} = \frac{\alpha}{k^* + (k_0 - k^*) e^{\xi_2 t}} \left( (k_0 - k^*) \xi_2 e^{\xi_2 t} \right) \\
= \alpha \xi_2 - \alpha \xi_2 \frac{k^*}{k_0} \\
= \alpha \xi_2 - \alpha \xi_2 \left( \frac{y^*}{y_0} \right)^{\frac{1}{\alpha}}
\]

• This suggests going to the data and running convergence regressions of the form:

\[
g_{i,t,t-1} = b^0 + b^1 \log y_{i,t-1} + \varepsilon_{i,t}
\]

• We need to be careful about interpreting the coefficient \( \hat{b}^1 \).

• Where does the error come from?
Selecting parameter values

- In general, computers cannot approximate the solution for arbitrary parameter values.

- How do we determine the parameter values?

- Two main approaches:
  1. Calibration.
  2. Statistical methods: Methods of Moments, ML, Bayesian.

- Advantages and disadvantages.
Calibration as an empirical methodology


- Two sources of information:
  1. Well accepted microeconomic estimates.

- Problems of 1 and 2.

- References:
Calibration of the standard model

- Parameters: $n$, $\alpha$, $\delta$, $\rho$, and $\sigma$.
- $n$: population growth in the data.
- $\alpha$: capital income. Proprietor’s income?
- $\delta$: in steady state

$$\delta k^* = x^* \Rightarrow \delta = \frac{x^*}{k^*}$$

- $\rho$: in steady state

$$r^* = \alpha \left( \frac{\alpha}{\rho + \delta} \right)^{\frac{\alpha}{1 - \alpha} - 1} - \delta$$

Then, we take $r^*$ from the data and given $\alpha$ and $\delta$, we find $\rho$.
- $\sigma$: from microeconomic evidence.
Running the model in the computer

• We have the system:

\[
\begin{align*}
\frac{\dot{c}}{c} &= \frac{1}{\sigma} (\alpha k^{\alpha-1} - \delta - \rho) \\
\dot{k} &= k^\alpha - c - (n + \delta) k
\end{align*}
\]

• Many methods to solve it.

• A simple one is a shooting algorithm.

• A popular alternative: Runge-Kutta methods.
A shooting algorithm

- Approximate the system by:
  \[
  \frac{c(t+\Delta t) - c(t)}{\Delta t} = \frac{1}{\sigma} \left( \alpha k(t)^{\alpha-1} - \delta - \rho \right)
  \]
  \[
  \frac{k(t+\Delta t) - k(t)}{\Delta t} = k(t)^{\alpha} - c(t) - (n + \delta) k(t)
  \]
  for a small \( \Delta t \).

- Steps:
  1. Given \( k(0) \), guess \( c(0) \).
  2. Trace dynamic system for a long \( t \).
  3. Is \( k(t) \to k^* \)? If yes, we got the right \( c(0) \). If \( k(t) \to \infty \), raise \( c(0) \), if \( k(t) \to 0 \), lower \( c(0) \).

- Intuition: phase diagram.
FIGURE 4. IMPLICATIONS OF INTERTEMPORAL PREFERENCES FOR TRANSITIONAL DYNAMICS

Key: Solid line, o- = 1; dashed line, or = 10; dash-dot line, Stone Geary. For output, x denotes the quarter life; 0, the half life; and *, the three-quarter life.
Savings rate
• We can actually work on our system of differential equations a bit more to show a more intimate relation between the Solow and the neoclassical growth model.

• The savings rate is defined as:

\[ s(t) = 1 - \frac{c(t)}{y(t)} \]

• Now

\[
\frac{d \left( \frac{c(t)}{y(t)} \right)}{dt} = \frac{\dot{c}}{c} - \frac{\dot{y}}{y} = \frac{\dot{c}}{c} - \frac{\alpha}{k} \frac{\dot{k}}{k}
\]
• If we substitute in the differential equations for $\frac{\dot{c}}{c}$ and $\dot{k}$:

$$\frac{d \left( \frac{c(t)}{y(t)} \right)}{dt} = \frac{1}{c(t)/y(t)}$$

$$= \frac{1}{\sigma} \left( \alpha k^{\alpha-1} - \delta - \rho \right) - \alpha \left( k^{\alpha-1} - \frac{c}{k} - n - \delta \right)$$

$$= \frac{1}{\sigma} \left( \alpha k^{\alpha-1} - \delta - \rho \right) - \alpha \left( k^{\alpha-1} - \frac{c}{k} k^{\alpha-1} - n - \delta \right)$$

$$= -\frac{1}{\sigma} (\delta + \rho) + \alpha (n + \delta) + \left( \frac{1}{\sigma} - 1 + \frac{c}{y} \right) \alpha k^{\alpha-1}$$
Then:

\[
\frac{d (c(t)/y(t))}{dt} \frac{1}{c(t)/y(t)} = -\frac{1}{\sigma} (\delta + \rho) + \alpha (n + \delta) + \left(\frac{1}{\sigma} - 1 + \frac{c}{y}\right) \alpha k^{\alpha-1}
\]

\[\dot{k} = k^\alpha - c - (n + \delta) k\]

is another system of differential equations.

This system implies that the saving rate is monotone (always increasing, always decreasing, or constant).
• We find the locus \( \frac{d(c(t)/y(t))}{dt} = 0 \):

\[
\left( \frac{1}{\sigma} - 1 + \frac{c}{y} \right) \alpha k^{\alpha-1} = \frac{1}{\sigma} (\delta + \rho) - \alpha (n + \delta) \Rightarrow
\]

\[
\frac{c}{y} = 1 - \frac{1}{\sigma} + \left( \frac{1}{\sigma} (\delta + \rho) - \alpha (n + \delta) \right) \frac{1}{\alpha} k^{1-\alpha}
\]

• Hence, if

\[
\frac{1}{\sigma} (\delta + \rho) = \alpha (n + \delta)
\]

the savings rate is constant, and we are back into the basic Solow model!
Optimal growth
The social planner’s problem

- The Social planner’s problem can be written as:

\[
\max_{\{c(t), k(t)\}_{t=0}^{\infty}} \int_{0}^{\infty} e^{-(\rho-n)t} u(c(t)) \, dt \\
\text{s.t. } \dot{k} = k(t)^{\alpha} - c(t) - (n + \delta) k(t) \\
\lim_{t \to \infty} k(t) \exp \left( - \int_{0}^{t} (r - \delta - n) \, ds \right) = 0 \\
k_0 \text{ given}
\]

- This problem is very similar to the household’s problem.

- We can also apply the optimality principle to the Hamiltonian:

\[
\mathcal{H}(a, c, \mu) = u(c(t)) + \mu(t) (k(t)^{\alpha} - c(t) - (n + \delta) k(t))
\]
1. Partial derivative of the Hamiltonian with respect to controls is equal to zero:

\[ H_c(a, c, \mu) = u'(c(t)) - \mu(t) = 0 \]

2. Partial derivative of the Hamiltonian with respect to states is:

\[ H_a(a, c, \mu) = \mu(t) \left( \alpha k(t)^{\alpha - 1} - n - \delta \right) = (\rho - n) \mu(t) - \dot{\mu}(t) \]

3. Partial derivative of the Hamiltonian with respect to co-states is:

\[ H_\mu(a, c, \mu) = k(t)^\alpha - c(t) - (n + \delta) k(t) = \dot{k}(t) \]

4. Transversality condition:

\[ \lim_{t \to \infty} e^{-\rho t} \mu(t) \dot{k}(t) = 0 \]
Comparing the necessary conditions

• Following very similar steps than in the problem of the consumer we find:

\[
\begin{align*}
\dot{c} &= \frac{1}{\sigma} (\alpha k^{\alpha-1} - \delta - \rho) \\
\dot{k} &= k^\alpha - c - (n + \delta) k \\
\lim_{t \to \infty} e^{-\rho t} \mu(t) \hat{k}(t) &= 0
\end{align*}
\]

• From the household problem:

\[
\begin{align*}
\dot{c} &= \frac{1}{\sigma} (\alpha k^{\alpha-1} - \delta - \rho) \\
\dot{k} &= k^\alpha - c - (n + \delta) k \\
\lim_{t \to \infty} e^{-\rho t} \mu(t) \hat{k}(t) &= 0
\end{align*}
\]

• Both problems have the same necessary conditions!