Real Business Cycles

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Business Cycle

- U.S. economy fluctuates over time.

- How can we build models to think about it?

- Do we need different models than before to do so? Traditionally the answer was yes. Nowadays the answer is no.

- We will focus on equilibrium models of the cycle.

Business Cycles and Economic Growth

- How different are long-run growth and the business cycle?

Changes in Output per Worker	*Secular Growth*	*Business Cycle*
Due to changes in capital	1/3	0
Due to changes in labor	0	2/3
Due to changes in productivity	2/3	1/3

- We want to use the same models with a slightly different focus.

Stochastic Neoclassical Growth Model

- Cass (1965) and Koopmans (1965).
- Brock and Mirman (1972).
- Kydland and Prescott (1982).
- Hansen (1985).
- King, Plosser, and Rebelo (1988a,b).

References

- King, Plosser, and Rebelo (1988a,b).
- Chapter by Cooley and Prescott in Cooley's *Frontier of Business Cycle Research* (in fact, you want to read the whole book).
- Chapter by King and Rebelo (*Resurrection Real Business Cycle Models*) in *Handbook of Macroeconomics*.
- Chapter 12 in Ljungqvist and Sargent.

Preferences

- Preferences:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (1+n)^t u\left(c_t\left(s^t\right), l_t\left(s^t\right)\right)$$

$$\text{for } c_t\left(s^t\right) \geq 0,\ l_t\left(s^t\right) \in (0,1)$$

  where n is population growth.

- Standard technical assumptions (continuity, differentiability, Inada conditions, etc...).

- However, those still leave many degrees of freedom.

- Restrictions imposed by economic theory and empirical observation.

Restrictions on Preferences

Three observations:

1. Risk premium relatively constant⇒CRRA utility function.

2. Consumption grows at a roughly constant rate.

3. Stationary hours after the SWW⇒Marginal rate of substitution between labor and consumption must be linear in consumption.

$$\frac{u_c}{u_l} = w_t\left(s^t\right) \Rightarrow$$
$$c_t\left(s^t\right) f\left(l_t\left(s^t\right)\right) = w_t\left(s^t\right) \Rightarrow$$
$$\mu^t c_0 f\left(l_t\left(s^t\right)\right) = \mu^t w_0$$

Explanation: income and substitution effect cancel out.

Parametric Family

- Only parametric that satisfy conditions (King, Plosser, and Rebelo, 1988a,b):

$$\begin{aligned} \frac{(cv(l))^{1-\gamma}-1}{1-\gamma} \text{ if } \gamma &> 0,\ \gamma \neq 1 \\ \log c + \log v(l) \text{ if } \gamma &= 1 \end{aligned}$$

- Restrictions on $v(l)$:

1. $v \in C^2$
2. Depending on γ:
   (a) If $\gamma = 1$, $\log v(l)$ must be increasing and concave.
   (b) If $\gamma < 1$, $v^{1-\gamma}$ must be increasing and concave.
   (c) If $\gamma > 1$, $v^{1-\gamma}$ must be decreasing and convex.
3. $-\gamma v(l) v''(l) > (1-2\gamma)[v'(l)]^2$ to ensure overall concavity of u.

Three Useful Examples

1. CRRA-Cobb Douglass:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (1+n)^t \frac{\left(c_t\left(s^t\right)^{\theta}\left(1-l_t\left(s^t\right)\right)^{1-\theta}\right)^{1-\gamma}-1}{1-\gamma}$$

2. Log-log (limit as $\gamma \to 1$):

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (1+n)^t \left\{\log c_t\left(s^t\right)+\psi \log\left(1-l_t\left(s^t\right)\right)\right\}$$

3. Log-CRRA

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (1+n)^t \left\{\log c_t\left(s^t\right)-\psi \frac{l_t\left(s^t\right)^{1+\gamma}}{1+\gamma}\right\}$$

Household Problem

- Let me pick log-log for simplicity:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (1+n)^t \left\{ \log c_t \left(s^t\right) + \psi \log \left(1 - l_t \left(s^t\right)\right) \right\}$$

- Budget constraint:

$$c_t \left(s^t\right) + x_t \left(s^t\right) = w_t \left(s^t\right) l_t \left(s^t\right) + r_t \left(s^t\right) k_t \left(s^{t-1}\right), \ \forall \ t > 0$$

- Complete markets and Arrow securities.

- We can price any security.

Problem of the Firm I

- Neoclassical production function in per capita terms:

$$y_t\left(s^t\right) = e^{z_t} k_t\left(s^{t-1}\right)^{\alpha}\left((1+\mu)^t l_t\left(s^t\right)\right)^{1-\alpha}$$

- Note: labor-augmenting technological change (Phelps, 1966).

- We are setting up a model where the firm rents the capital from the household.

- However, we could also have a model where firms own the capital and the households own shares of the firms.

- Both environments are equivalent with complete markets.

Problem of the Firm II

- By profit maximization:

$$\begin{aligned} \alpha e^{z_t} k_t \left(s^{t-1}\right)^{\alpha-1} \left((1+\mu)^t l_t \left(s^t\right)\right)^{1-\alpha} &= r_t \left(s^t\right) \\ (1-\alpha) e^{z_t} k_t \left(s^{t-1}\right)^{\alpha} \left((1+\mu)^t l_t \left(s^t\right)\right)^{-\alpha} &= w_t \left(s^t\right) \end{aligned}$$

- Investment x_t induces a law of motion for capital:

$$(1+n) k_{t+1} \left(s^t\right) = (1-\delta) k_t \left(s^{t-1}\right) + x_t \left(s^t\right)$$

Evolution of the technology

- $s_t = z_t$

- z_t changes over time.

- It follows the AR(1) process:

$$z_t = \rho z_{t-1} + \sigma \varepsilon_t$$
$$\varepsilon_t \sim \mathcal{N}(0, 1)$$

- Interpretation of μ and ρ.

Arrow-Debreu Equilibrium

A Arrow-Debreu equilibrium are prices $\{\widehat{p}_t(s^t), \widehat{w}_t(s^t), \widehat{r}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$ and allocations $\{\hat{c}_t(s^t), \widehat{l}_t(s^t), \widehat{k}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$ such that:

1. Given $\{\hat{p}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$, $\{\hat{c}_t(s^t), \widehat{l}_t(s^t), \widehat{k}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$ solves

$$\max_{\{\hat{c}_t(s^t),\widehat{l}_t(s^t),\widehat{k}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (1+n)^t \left\{\log c_t\left(s^t\right) + \psi \log\left(1 - l_t\left(s^t\right)\right)\right\}$$

$$\text{s.t. } \sum_{t=0}^{\infty} \sum_{s^t\in S^t} \hat{p}_t(s^t)\left(c_t(s^t) + (1+n)\, k_{t+1}\left(s^t\right)\right)$$

$$\leq \sum_{t=0}^{\infty} \sum_{s^t\in S^t} \widehat{p}_t(s^t)\left(\widehat{w}_t(s^t) l_t(s^t) + \left(\widehat{r}_t(s^t) + 1 - \delta\right) k_{t+1}\left(s^t\right)\right)$$

$$c_t(s^t) \geq 0 \text{ for all } t$$

2. Firms pick $\{\widehat{l}_t(s^t), \widehat{k}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$ to minimize costs:

$$
\begin{aligned}
\alpha e^{z_t}\widehat{k}_t\left(s^{t-1}\right)^{\alpha-1}\left((1+\mu)^t\,\widehat{l}_t\left(s^t\right)\right)^{1-\alpha} &= \widehat{r}_t\left(s^t\right) \\
(1-\alpha)\,e^{z_t}\widehat{k}_t\left(s^{t-1}\right)^{\alpha}\left((1+\mu)^t\,\widehat{l}_t\left(s^t\right)\right)^{-\alpha} &= \widehat{w}_t\left(s^t\right)
\end{aligned}
$$

3. Markets clear:

$$
\widehat{c}_t(s^t) + (1+n)\,\widehat{k}_{t+1}\left(s^t\right) =
$$

$$
e^{z_t}\widehat{k}_t\left(s^{t-1}\right)^{\alpha}\left((1+\mu)^t\,\widehat{l}_t\left(s^t\right)\right)^{1-\alpha} + (1-\delta)\,\widehat{k}_t\left(s^{t-1}\right)
$$

for all t, all $s^t \in S^t$

Sequential Markets Equilibrium I.

- We introduce Arrow securities. ·

- Household problem: $\left\{c_t(s^t), l_t(s^t), k_t(s^t), \left\{a_{t+1}(s^t, s_{t+1})\right\}_{s_{t+1}\in S}\right\}_{t=0,s^t\in S^t}^{\infty}$ solve

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (1+n)^t \left\{\log c_t\left(s^t\right) + \psi \log\left(1 - l_t\left(s^t\right)\right)\right\}$$

$$\text{s.t. } c_t^i(s^t) + (1+n)\, k_{t+1}\left(s^t\right) + \sum_{s_{t+1}|s^t} \widehat{Q}_t(s^t, s_{t+1}) a_{t+1}(s^t, s_{t+1})$$

$$\leq \widehat{w}_t(s^t) l_t(s^t) + \left(\widehat{r}_t(s^t) + 1 - \delta\right) k_{t+1}\left(s^t\right) + a_t(s^t)$$

$$c_t(s^t) \geq 0 \text{ for all } t, s^t \in S^t$$

$$a_{t+1}(s^t, s_{t+1}) \geq -A_{t+1}(s^{t+1}) \text{ for all } t, s^t \in S^t$$

- Role of $A_{t+1}(s^{t+1})$.

Sequential Markets Equilibrium II

A SM equilibrium is prices for Arrow securities $\{\widehat{Q}_t(s^t, s_{t+1})\}_{t=0,s^t\in S^t,s_{t+1}\in S}^{\infty}$, allocations $\left\{\hat{c}_t^i(s^t), \widehat{l}_t(s^t), \widehat{k}_t(s^t), \left\{\hat{a}_{t+1}(s^t, s_{t+1})\right\}_{s_{t+1}\in S}\right\}_{t=0,s^t\in S^t}^{\infty}$ and input prices $\{\widehat{w}_t(s^t), \widehat{r}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$, such that:

1. Given $\{\widehat{Q}_t(s^t, s_{t+1})\}_{t=0,s^t\in S^t,s_{t+1}\in S}^{\infty}$ and $\{\widehat{w}_t(s^t), \widehat{r}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$

   $\left\{\hat{c}_t(s^t), \widehat{l}_t(s^t), \widehat{k}_t(s^t), \left\{\hat{a}_{t+1}(s^t, s_{t+1})\right\}_{s_{t+1}\in S}\right\}_{t=0,s^t\in S^t}^{\infty}$ solve the problem of the household.

2. Firms pick $\{\widehat{l}_t(s^t), \widehat{k}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$ to minimize costs:

$$\begin{aligned}
\alpha e^{z_t}\widehat{k}_t\left(s^{t-1}\right)^{\alpha-1}\left((1+\mu)^t\,\widehat{l}_t\left(s^t\right)\right)^{1-\alpha} &= \widehat{r}_t\left(s^t\right) \\
(1-\alpha)\,e^{z_t}\widehat{k}_t\left(s^{t-1}\right)^{\alpha}\left((1+\mu)^t\,\widehat{l}_t\left(s^t\right)\right)^{-\alpha} &= \widehat{w}_t\left(s^t\right)
\end{aligned}$$

3. Markets clear for all t, all $s^t \in S^t$

$$\widehat{c}_t(s^t)+(1+n)\,\widehat{k}_{t+1}\left(s^t\right) = e^{z_t}\widehat{k}_t\left(s^{t-1}\right)^{\alpha}\left((1+\mu)^t\,\widehat{l}_t\left(s^t\right)\right)^{1-\alpha}+(1-\delta)\,\widehat{k}_t\left(s^{t-1}\right)$$

Recursive Competitive Equilibrium

- Often, it is convenient to use a .third alternative competitive equilibrium concept: Recursive Competitive Equilibrium (RCE).

- Developed by Mehra and Prescott (1980).

- RCE emphasizes the idea of defining an equilibrium as a set of functions that depend on the state of the model.

- Two interpretation for states:

  1. Pay-off relevant states: capital, productivity, ...

  2. Other states: promised utility, reputation, ...

- Recursive notation: x and x'.

Value Function for the Household

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- Individual state: k.

- Aggregate states: K and z.

- Recursive problem:

$$v(k, K, z) = \max_{c,x,l} \left\{ \log c + \psi \log(1-l) + \beta(1+n)\mathbb{E}v\left(k', K', z'\right)|z \right\}$$

$$s.t.\ c + x = r(K, z)\,k + w(K, z)\,l$$

$$(1+n)\,k' = (1-\delta)\,k + x$$

$$(1+n)\,K' = (1-\delta)\,K + X(K, z)$$

$$z' = \rho z + \sigma\varepsilon'$$

Definition of Recursive Competitive Equilibrium

A RCE for our economy is a value function $v(k, K, z)$, households policy functions, $c(k, K, z), x(k, K, z)$, and $l(k, K, z)$, aggregate policy functions $C(K, z), X(K, z)$, and $L(K, z)$, and price functions $r(K, z)$ and $w(K, z)$ such that those functions satisfy:

1. Recursive problem of the household.

2. Firms maximize:

$$\begin{aligned} \alpha e^{z} K^{\alpha-1}\left((1+\mu) L(K, z)\right)^{1-\alpha} &= r(K, z) \\ (1-\alpha) e^{z} K^{\alpha}\left((1+\mu) L(K, z)\right)^{-\alpha} &= w(K, z) \end{aligned}$$

3. Consistency of individual and aggregate policy functions, $c(k, K, z) = C(K, z)$, $x(k, K, z) = X(K, z)$, $l(k, K, z) = L(K, z), \forall (K, z)$.

4. Aggregate resource constraint:

$$C(K, z) + X(K, z) = e^{z} K^{\alpha}\left((1+\mu) L(K, z)\right)^{1-\alpha}, \ \forall (K, z)$$

Equilibrium Conditions

$$\frac{1}{c_t\left(s^t\right)}=\beta\mathbb{E}_t\frac{1}{c_{t+1}\left(s^{t+1}\right)}\left(r_{t+1}\left(s^{t+1}\right)+1-\delta\right)$$

$$\psi\frac{c_t\left(s^t\right)}{1-l_t\left(s^t\right)}=w_t\left(s^t\right)$$

$$r_t\left(s^t\right)=\alpha e^{z_t}k_t\left(s^{t-1}\right)^{\alpha-1}\left((1+\mu)^t l_t\left(s^t\right)\right)^{1-\alpha}$$

$$w_t\left(s^t\right)=(1-\alpha)\,e^{z_t}k_t\left(s^{t-1}\right)^{\alpha}(1+\mu)^{(1-\alpha)t}\,l_t\left(s^t\right)^{-\alpha}$$

$$c_t\left(s^t\right)+(1+n)\,k_{t+1}\left(s^t\right)=$$

$$e^{z_t}k_t\left(s^{t-1}\right)^{\alpha}\left((1+\mu)^t l_t\left(s^t\right)\right)^{1-\alpha}+(1-\delta)\,k_t\left(s^{t-1}\right)$$

$$z_t=\rho z_{t-1}+\sigma\varepsilon_t$$

Scaling the Economy I

- Economy has long-run growth rate equal to $(n + \mu)$.

- Per capita terms, the economy grows at a rate μ.

- Hence, the model is non-stationary and we need to rescale it.

- General condition: transform every non-stationary variable into a stationary one by dividing it by $(1 + \mu)^t$

$$\widetilde{x}_t\left(s^t\right) = \frac{x_t\left(s^t\right)}{(1 + \mu)^t}$$

Scaling the Economy II

We can rewrite the preferences (and adding a suitable constant):

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (1+n)^t \frac{\left(c_t\left(s^t\right) v\left(l_t\left(s^t\right)\right)\right)^{1-\gamma}-1}{1-\gamma}$$

$$=\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (1+n)^t \frac{\left((1+\mu)^t \widetilde{c}_t\left(s^t\right) v\left(l_t\left(s^t\right)\right)\right)^{1-\gamma}-1}{1-\gamma} \Rightarrow$$

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (1+n)^t (1+\mu)^{t(1-\gamma)} \frac{\left(\widetilde{c}_t\left(s^t\right) v\left(l_t\left(s^t\right)\right)\right)^{1-\gamma}-1}{1-\gamma}$$

Scaling the Economy III

We can rewrite the preferences (and adding a suitable constant):and

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (1+n)^t \left\{ \log c_t\left(s^t\right) + \log v\left(l_t\left(s^t\right)\right) \right\}$$

$$= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (1+n)^t \left\{ \log (1+\mu)^t \widetilde{c}_t\left(s^t\right) + \log v\left(l_t\left(s^t\right)\right) \right\} \Rightarrow$$

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (1+n)^t \left\{ \log \widetilde{c}_t\left(s^t\right) + \log v\left(l_t\left(s^t\right)\right) \right\}$$

Scaling the Economy IV

- The resource constraint, diving both sides by $(1+\mu)^t$

$$\widetilde{c}_t\left(s^t\right)+(1+n)(1+\mu)\widetilde{k}_{t+1}\left(s^t\right)=$$
$$e^{z_t}\widetilde{k}_t\left(s^{t-1}\right)^{\alpha}l_t\left(s^t\right)^{1-\alpha}+(1-\delta)\widetilde{k}_t\left(s^{t-1}\right)$$

- Input prices:

$$r_t\left(s^t\right)=\alpha e^{z_t}\widetilde{k}_t\left(s^{t-1}\right)^{\alpha-1}l_t\left(s^t\right)^{1-\alpha}$$
$$\widetilde{w}_t\left(s^t\right)=(1-\alpha)e^{z_t}\widetilde{k}_t\left(s^{t-1}\right)^{\alpha}l_t\left(s^t\right)^{-\alpha}$$

A New Competitive Equilibrium

- We can define a competitive equilibrium in the rescaled economy.
- Equilibrium conditions (log case):

$$\frac{(1+\mu)}{\widetilde{c}_t(s^t)} = \beta \mathbb{E}_t \frac{1}{\widetilde{c}_{t+1}\left(s^{t+1}\right)} \left(r_{t+1}\left(s^{t+1}\right) + 1 - \delta\right)$$

$$\psi \frac{\widetilde{c}_t\left(s^t\right)}{1 - l_t(s^t)} = \widetilde{w}_t\left(s^t\right)$$

$$r_t\left(s^t\right) = \alpha e^{z_t} \widetilde{k}_t\left(s^{t-1}\right)^{\alpha-1} l_t\left(s^t\right)^{1-\alpha}$$

$$\widetilde{w}_t\left(s^t\right) = (1-\alpha) e^{z_t} \widetilde{k}_t\left(s^{t-1}\right)^{\alpha} l_t\left(s^t\right)^{-\alpha}$$

$$\widetilde{c}_t\left(s^t\right) + (1+n)(1+\mu)\widetilde{k}_{t+1}\left(s^t\right) = e^{z_t}\widetilde{k}_t\left(s^{t-1}\right)^{\alpha} l_t\left(s^t\right)^{1-\alpha} + (1-\delta)\widetilde{k}_t\left(s^{t-\right.$$

$$z_t = \rho z_{t-1} + \sigma \varepsilon_t$$

Existence and Welfare Theorems

- There is a unique equilibrium in this economy once we impose the right transversality condition.
- Both welfare theorems hold.
- We can move back and forth between the market equilibrium and the social planner's problem.

Behavior of the Model

- We want to characterize behavior of the model.

- Three type of dynamics:

  1. Balanced growth path.
  2. Transitional dynamics (Cass, 1965, and Koopmans, 1965).
  3. Ergodic behavior.

Stochastic Behavior

- We have an initial shock: productivity changes.

- We have a transmission mechanism: intertemporal substitution and capital accumulation.

- We can look at a simulation from this economy.

- Why only a simulation?

- To simulate the model we need:

  1. To select parameter values.

  2. To compute the solution of the model.

Selecting Parameter Values

- How do we determine the parameter values?

- Two main approaches:

  1. Calibration.

  2. Statistical methods: Methods of Moments, ML, Bayesian.

- Advantages and disadvantages.

Calibration as an Empirical Methodology

- Emphasized by Lucas (1980) and Kydland and Prescott (1982).

- Two sources of information:

  1. Well accepted microeconomic estimates.
  2. Matching long-run properties of the economy.

- Problems of 1. and 2.

- References:

  1. Browning, Hansen and Heckman (1999) chapter in *Handbook of Macroeconomics*.
  2. Debate in Journal of Economic Perspectives, Winter 1996: Kydland and Prescott, Hansen and Heckman, Sims.

Calibration of the Standard Model

- Parameters: β, ψ, α, δ, μ, n, ρ, σ.

- n: population growth in the data.

- μ: per capita long run growth.

- α: capital income. Proprietor's income?

Balanced Growth Path

- Equilibrium conditions in the BGP:

$$\frac{1+\mu}{\widetilde{c}} = \beta\frac{1}{\widetilde{c}}\left(r+1-\delta\right)$$

$$\psi\frac{\widetilde{c}}{1-l} = \widetilde{w}$$

$$r = \alpha\widetilde{k}^{\alpha-1}l^{1-\alpha}$$

$$\widetilde{w} = \left(1-\alpha\right)\widetilde{k}^{\alpha}l^{-\alpha}$$

$$\widetilde{c}+\left(1+n\right)\left(1+\mu\right)\widetilde{k} = \widetilde{k}^{\alpha}l^{1-\alpha}+\left(1-\delta\right)\widetilde{k}$$

- A system of 5 equations on 5 unknowns.

Three Conditions of the Balanced Growth Path

- First:

$$r = \alpha \frac{\widetilde{y}}{\widetilde{k}} = \frac{1+\mu}{\beta} - 1 + \delta$$

- Also:

$$(1+n)(1+\mu)\widetilde{k} = (1-\delta)\widetilde{k} + \widetilde{x} \Rightarrow$$
$$\delta = \frac{\widetilde{x}}{\widetilde{k}} + 1 - (1+n)(1+\mu)$$

- Finally,

$$\psi \frac{\widetilde{c}}{1-l} = (1-\alpha)\frac{\widetilde{y}}{l} \Rightarrow \frac{\widetilde{c}}{\widetilde{y}} = \frac{1-\alpha}{\psi}\frac{1-l}{l}$$

Using the Three Conditions to Calibrate the Model

- First, we use data on hours of work to find

$$\psi = (1-\alpha)\frac{\widetilde{y}}{\widetilde{c}}\frac{1-l}{l}$$

- Second, give data and

$$\delta = \frac{\widetilde{x}}{\widetilde{k}} + 1 - (1+n)(1+\mu)$$

  we determine δ.

- Finally, we get β:

$$\beta = (1+\mu)\left(\alpha\frac{\widetilde{y}}{\widetilde{k}} + 1 - \delta\right)^{-1}$$

Frisch Elasticity I

- Define the Frisch Elasticity as:

$$\left.\frac{d\log l}{d\log w}\right|_{c \text{ constant}}$$

- For our parametric family:

1. $\frac{\left(c^{\theta}(1-l)^{1-\theta}\right)^{1-\gamma}-1}{1-\gamma}$: $\frac{1-l}{l}$.

2. $\log c+\psi\log(1-l)$: $\frac{1-l}{l}$.

3. $\log c-\psi\frac{l^{1+\gamma}}{1+\gamma}$: $1/\gamma$.

Frisch Elasticity II

- Empirical evidence is that $l \approx 1/3$ (Ghez and Becker, 1975).

- Then, our Frisch Elasticity is 2.

- Empirical evidence:

  1. Traditional view: MaCurdy (1981), Altonji (1986), Browning, Deaton and Irish (1985) between 0 and 0.5.

  2. Revisionist view: between 0.5 and 1.6 (Browning, Hansen, and Heckman, 1999). Some estimates (Imai and Keane, 2004) even higher (3.8).

Equivalence between Utility Functions

- With $\log c_t + \psi \log(1 - l_t)$, the static FOC is:

$$\psi \frac{c_t}{1 - l_t} = w_t$$

while with $\log c_t - \psi \frac{l_t^{1+\gamma}}{1+\gamma}$, the static FOC is

$$\psi c_t l_t^{\gamma} = w_t$$

- Loglinearize both expressions:

$$\psi \frac{c}{1 - l} \widehat{c}_t + \psi \frac{cl}{(1 - l)^2} \widehat{l}_t = w \widehat{w}_t \Rightarrow$$
$$\widehat{c}_t + \frac{l}{1 - l} \widehat{l}_t = \widehat{w}_t$$

$$\begin{aligned} \psi c l^{\gamma} \left(\widehat{c}_t + \gamma \widehat{l}_t \right) &= w \widehat{w}_t \Rightarrow \\ \widehat{c}_t + \gamma \widehat{l}_t &= \widehat{w}_t \end{aligned}$$

- If we calibrate the model to $l \approx 1/3$:

$$\widehat{c}_t + \frac{1}{2} \widehat{l}_t = \widehat{w}_t$$

  and hence, both utility functions are equivalent if we make $\gamma = \frac{l}{1-l}$. In the case $l \approx 1/3$, $\gamma = 1/2$.

Solow Residual

- Last step is to calibrate

$$z_t = \rho z_{t-1} + \sigma \varepsilon_t$$

- Obtain the Solow residual after a time trend has been removed.

- Estimate ρ and σ by OLS.

- Problems of estimate.

Solution Methods

- Value function iteration.

- Projection.

- Perturbation:

  1. Generalization of linearization.

  2. `Dynare`.

General Structure of Linearized System

- There are many linear solvers. Fundamental equivalence.

- "A Toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily" by Harald Uhlig.

- Given m states x_t, n controls y_t, and k exogenous stochastic processes z_{t+1}, we have:

$$
\begin{aligned}
Ax_t + Bx_{t-1} + Cy_t + Dz_t &= 0 \\
E_t \left(Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t \right) &= 0 \\
E_t z_{t+1} &= Nz_t
\end{aligned}
$$

  where C is of size $l \times n$, $l \geq n$ and of rank n, that F is of size $(m + n - l) \times n$, and that N has only stable eigenvalues.

Policy Functions I

We guess policy functions of the form:

$$x_t = Px_{t-1} + Qz_t$$
$$y_t = Rx_{t-1} + Sz_t$$

where P, Q, R, and S are matrices such that the computed equilibrium is stable.

Policy Functions I

For simplicity, suppose $l = n$. See Uhlig for general case (I have never be in the situation where $l = n$ did not hold).

Then:

1. P satisfies the matrix quadratic equation:

   $$\left(F - JC^{-1}A\right)P^2 - \left(JC^{-1}B - G + KC^{-1}A\right)P - KC^{-1}B + H = 0$$

   The equilibrium is stable iff $\max\left(abs\left(eig\left(P\right)\right)\right) < 1$.

2. R is given by:

   $$R = -C^{-1}\left(AP + B\right)$$

3. Q satisfies:

$$N' \otimes \left(F - JC^{-1}A\right) + I_k \otimes \left(JR + FP + G - KC^{-1}A\right) vec\,(Q) \\ = vec\left(\left(JC^{-1}D - L\right) N + KC^{-1}D - M\right)$$

4. S satisfies:

$$S = -C^{-1}\,(AQ + D)$$

How to Solve Quadratic Equations

To solve

$$\Psi P^2 - \Gamma P - \Theta = 0$$

for the $m \times m$ matrix P:

1. Define the $2m \times 2m$ matrices:

$$\Xi = \begin{bmatrix} \Gamma & \Theta \\ I_m & 0_m \end{bmatrix}, \text{ and } \Delta = \begin{bmatrix} \Psi & 0_m \\ 0_m & I_m \end{bmatrix}$$

2. Let s be the generalized eigenvector and λ be the corresponding generalized eigenvalue of Ξ with respect to Δ. Then we can write $s' = [\lambda x', x']$ for some $x \in \Re^m$.

3. If there are m generalized eigenvalues $\lambda_1, \lambda_2, ..., \lambda_m$ together with generalized eigenvectors $s_1, ..., s_m$ of Ξ with respect to Δ, written as $s' = \left[\lambda x_i', x_i'\right]$ for some $x_i \in \Re^m$ and if $(x_1, ..., x_m)$ is linearly independent, then:

$$P = \Omega \Lambda \Omega^{-1}$$

is a solution to the matrix quadratic equation where $\Omega = [x_1, ..., x_m]$ and $\Lambda = [\lambda_1, ..., \lambda_m]$. The solution of P is stable if $\max |\lambda_i| < 1$. Conversely, any diagonalizable solution P can be written in this way.

Comparison with US economy

- Simulated Economy output fluctuations are around 70% as big as observed fluctuations.

- Consumption is less volatile than output.

- Investment is much more volatile.

- Behavior of hours.

Assessment of the Basic Real Business Model

- It accounts for a substantial amount of the observed fluctuations.
- It accounts for the covariances among a number of variables.
- It has some problems accounting for the behavior of the hours worked.
- More important question: where do productivity shocks come from?

Negative Productivity Shocks

- The model implies that half of the quarters we have negative technology shocks.

- Is this plausible? What is a negative productivity shocks?

- Role of trend: negative shocks also include growth of technology below the trend.

- s.d. of shocks is 0.007. Mean quarter productivity growth is 0.0047 (to give us a 1.9% growth per year).

- As a consequence, we would only observe negative technological shocks when $\varepsilon_t < -0.0047$.

- This happens in the model around 25% of times. Comparison with the data.

Some Policy Implications

- The basic model is Pareto-efficient.
- Fluctuations are the optimal response to a changing environment.
- Fluctuations are not a sufficient condition for inefficiencies or for government intervention.
- In fact in this model the government can only worsen the allocation.
- Recessions have a “cleansing” effect.

Asset Market Implications I

- We will have the fundamental asset pricing equation:

$$Q_t(s^t, s_{t+1}) = \beta\pi\left(s_{t+1} \middle| s^t\right) \frac{u'\left(c_{t+1}\left(s^{t+1}\right), l_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right), l_t\left(s^t\right)\right)}$$

- If utility is separable and log in consumption:

$$Q_t(s^t, s_{t+1}) = \beta\pi\left(s_{t+1} \middle| s^t\right) \frac{c_t\left(s^t\right)}{c_{t+1}\left(s^{t+1}\right)}$$

- Now, $c_t\left(s^t\right)$ is the equilibrium consumption.

- Since $c_t\left(s^t\right)$ is smooth in the model, $Q_t(s^t, s_{t+1})$ will also be smooth. Hence, we will have the standard equity premium puzzle.

Asset Market Implications II

- Return to invest in an uncontingent bond sold at face value 1:

$$\mathbb{E}_t \beta \frac{c_t\left(s^t\right)}{c_{t+1}\left(s^{t+1}\right)} R_t^b\left(s^t\right)$$

- Return to invest in capital:

$$\mathbb{E}_t \beta \frac{c_t\left(s^t\right)}{c_{t+1}\left(s^{t+1}\right)}\left(r_{t+1}\left(s^{t+1}\right)+1-\delta\right)$$

- By non-arbitrage:

$$\mathbb{E}_t \beta \frac{c\left(s^t\right)}{c_{t+1}\left(s^{t+1}\right)} R_t^b\left(s^t\right)=\mathbb{E}_t \beta \frac{c_t\left(s^t\right)}{c_{t+1}\left(s^{t+1}\right)}\left(r_{t+1}\left(s^{t+1}\right)+1-\delta\right)$$

- Presence of capital ties down returns.

Further Extensions

- We can extend our model in several directions.

- Two objectives:

  1. Fix empirical problems.

  2. Address additional questions.

- Examples:

  1. Indivisible labor supply.

  2. Capacity utilization.

  3. Investment Specific technological change.

  4. Monopolistic Competition.

Lotteries

- Our first extension is to introduce lotteries: Rogerson (1988) and Hansen (1985).
- General procedure to deal with non-convexities.
- For example, an agent can either work 0 hours or l^* hours. Why?
- Extensive versus intensive margin.
- Then, expected utility:

$$pu(c_1, l^*) + (1 - p)\, u(c_2, 0)$$

- Resource constrain in the economy (law of large numbers):

$$pc_1 + (1 - p)\, c_2 = c$$

Aggregation

- First order condition: $u_c(c_1, l^*) = u_c(c_2, 0)$.

- For our log-log utility function $\log c + \psi \log(1-l)$, we have

$$c = c_1 = c_2$$

- Also, In the aggregate, we have that $l = pl^*$.

- Then, expected utility is

$$\log c + p\psi \log(1-l^*) + (1-p)\log 1 \Rightarrow \log c + Al$$

where $A = \psi \frac{\log(1-l^*)}{l^*}$.

- Note that this utility function belongs to the class $\log c - \psi \frac{l^{1+\gamma}}{1+\gamma}$ with $\gamma = 0$, i.e., with infinite Frisch elasticity.

Capacity Utilization

- In benchmark model, the short run elasticity of capital is zero while in the long run is infinite.

- Empirical evidence of use of machinery, number of shifts, or electricity consumption.

- Modified production function:

$$y_t\left(s^t\right)=e^{z_t}\left(u_t\left(s^t\right)k_t\left(s^{t-1}\right)\right)^{\alpha}\left((1+\mu)^t l_t\left(s^t\right)\right)^{1-\alpha}$$

  where u_t is the utilization rate.

- Depreciation:

$$(1+n)\,k_{t+1}\left(s^t\right)=\left(1-\delta\left(u_t\left(s^t\right)\right)\right)k_t\left(s^{t-1}\right)+x_t\left(s^t\right)$$

Combining Both Extensions

- We can generate 70 percent of aggregate fluctuations with a s.d. of 0.003.

- How do we look at the Solow residual in this model?

- This implies negative technological growth in around 5 percent of quarters, roughly observation in the data.

Investment-Specific Technological Change

- Greenwood, Herkowitz, and Krusell (1997 and 2000): importance of technological change specific to new investment goods for understanding postwar U.S. growth and aggregate fluctuations.

- Observation: fall in the relative price of capital.

- Implications for NIPA.

- A simple way to model it:

$$(1+n)\, k_{t+1}\left(s^{t}\right) = (1-\delta)\, k_{t}\left(s^{t-1}\right) + v_{t} x_{t}\left(s^{t}\right)$$

  where v_t is the inverse of the relative price of capital.

- Two different technological shocks with different implications.

Monopolistic Competition

- Final good producer with competitive behavior.
- Continuum of intermediate good producers with market power.
- Alternative formulations: continuum of goods in the utility function.
- Otherwise, the model is the same as the standard RBC model.

The Final Good Producer

- Production function:

$$y_t(s_t) = \left(\int_0^1 (y_{it}(s_t))^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

  where ε controls the elasticity of substitution.

- Final good producer is perfectly competitive and maximize profits, taking as given all intermediate goods prices $p_{ti}(s_t)$ and the final good price $p_t(s_t)$.

Maximization Problem

- Thus, its maximization problem is:

$$\max_{y_{it}(s_t)} p_t(s_t)\, y_t(s_t) - \int_0^1 p_{it}(s_t)\, y_{it}(s_t)\, di$$

- First order conditions are for $\forall i$:

$$p_t \frac{\varepsilon}{\varepsilon - 1}\left(\int_0^1 (y_{it}(s_t))^{\frac{\varepsilon-1}{\varepsilon}}\, di\right)^{\frac{\varepsilon}{\varepsilon-1}-1} \frac{\varepsilon - 1}{\varepsilon}(y_{it}(s_t))^{\frac{\varepsilon-1}{\varepsilon}-1} - p_{it}(s_t) = 0$$

Working with the First Order Conditions

- Dividing the first order conditions for two intermediate goods i and j, we get:

$$\frac{p_{it}(s_t)}{p_{jt}(s_t)} = \left(\frac{y_{it}(s_t)}{y_{jt}(s_t)}\right)^{-\frac{1}{\varepsilon}}$$

or:

$$p_{jt}(s_t) = \left(\frac{y_{it}(s_t)}{y_{jt}(s_t)}\right)^{\frac{1}{\varepsilon}} p_{it}(s_t)$$

- Hence:

$$p_{jt}(s_t)\, y_{jt}(s_t) = p_{it}(s_t)\, y_{it}(s_t)^{\frac{1}{\varepsilon}} \left(y_{jt}(s_t)\right)^{\frac{\varepsilon-1}{\varepsilon}}$$

- Integrating out:

$$\int_0^1 p_{jt}(s_t)\, y_{jt}(s_t)\, dj = p_{it}(s_t)\, y_{it}(s_t)^{\frac{1}{\varepsilon}} \int_0^1 y_{jt}^{\frac{\varepsilon-1}{\varepsilon}} dj = p_{it}(s_t)\, y_{it}(s_t)^{\frac{1}{\varepsilon}} \left(y_{jt}(s_t)\right)^{\frac{\varepsilon-1}{\varepsilon}}$$

Input Demand Function

- By zero profits ($p_t(s_t) y_t(s_t) = \int_0^1 p_{jt}(s_t) y_{jt}(s_t) dj$), we get:

$$
\begin{aligned}
p_t(s_t) y_t(s_t) &= p_{it}(s_t) y_{it}(s_t)^{\frac{1}{\varepsilon}} \left(y_{jt}(s_t)\right)^{\frac{\varepsilon-1}{\varepsilon}} \\
&\Rightarrow p_t(s_t) = p_{it}(s_t) y_{it}(s_t)^{\frac{1}{\varepsilon}} y_t(s_t)^{-\frac{1}{\varepsilon}}
\end{aligned}
$$

- Consequently, the input demand functions associated with this problem are:

$$
y_{it}(s_t) = \left(\frac{p_{it}(s_t)}{p_t(s_t)}\right)^{-\varepsilon} y_t(s_t) \qquad \forall i
$$

- Interpretation.

Price Level

- By the zero profit condition $p_t(s_t)\, y_t(s_t) = \int_0^1 p_{it}(s_t)\, y_{it}(s_t)\, di$ and plug-in the input demand functions:

$$\begin{aligned} p_t(s_t)\, y_t(s_t) &= \int_0^1 p_{it}(s_t) \left(\frac{p_{it}(s_t)}{p_t(s_t)}\right)^{-\varepsilon} y_t(s_t)\, di \\ &\Rightarrow p_t(s_t)^{1-\varepsilon} = \int_0^1 p_{it}(s_t)^{1-\varepsilon}\, di \end{aligned}$$

- Thus:

$$p_t(s_t) = \left(\int_0^1 p_{it}(s_t)^{1-\varepsilon}\, di\right)^{\frac{1}{1-\varepsilon}}$$

Intermediate Good Producers

- Continuum of intermediate goods producers.

- No entry/exit.

- Each intermediate good producer i has a production function

$$y_{it}(s_t) = A_t k_{it}(s_t)^{\alpha} l_{it}(s_t)^{1-\alpha}$$

- A_t follows the AR(1) process:

$$\begin{aligned} \log A_t &= \rho \log A_{t-1} + z_t \\ z_t &\sim \mathcal{N}(0, \sigma_z) \end{aligned}$$

Maximization Problem I

- Intermediate goods producers solve a two-stages problem.

- First, given w_t and r_t, they rent l_{it} and k_{it} in perfectly competitive factor markets in order to minimize real cost:

$$\min_{l_{it}(s_t),k_{it}(s_t)} \{w_t(s_t)\, l_{it}(s_t) + r_t(s_t)\, k_{it}(s_t)\}$$

  subject to their supply curve:

$$y_{it} = A_t k_{it}(s_t)^{\alpha}\, l_{it}(s_t)^{1-\alpha}$$

First Order Conditions

- The first order conditions for this problem are:

$$
\begin{aligned}
w_t(s_t) &= \varrho(1-\alpha) A_t k_{it}(s_t)^{\alpha} l_{it}(s_t)^{-\alpha} \\
r_t(s_t) &= \varrho\alpha A_t k_{it}(s_t)^{\alpha-1} l_{it}(s_t)^{1-\alpha}
\end{aligned}
$$

  where ϱ is the Lagrangian multiplier or:

$$
k_{it}(s_t) = \frac{\alpha}{1-\alpha}\frac{w_t(s_t)}{r_t(s_t)} l_{it}(s_t)
$$

- Note that ratio capital-labor only is the same for all firms i.

Real Cost

- The real cost of optimally using l_{it} is:

$$\left(w_t\left(s_t\right)l_{it}\left(s_t\right)+\frac{\alpha}{1-\alpha}w_t\left(s_t\right)l_{it}\left(s_t\right)\right)$$

- Simplifying:

$$\left(\frac{1}{1-\alpha}\right)w_t\left(s_t\right)l_{it}\left(s_t\right)$$

Marginal Cost I

- The firm has constant returns to scale.

- Then, we can find the real marginal cost $mc_t(s_t)$ by setting the level of labor and capital equal to the requirements of producing one unit of good $A_t k_{it}(s_t)^{\alpha} l_{it}(s_t)^{1-\alpha} = 1$

- Thus:

$$
\begin{aligned}
A_t k_{it}(s_t)^{\alpha} l_{it}(s_t)^{1-\alpha} &= A_t \left(\frac{\alpha}{1-\alpha} \frac{w_t(s_t)}{r_t(s_t)} l_{it}(s_t) \right)^{\alpha} l_{it}(s_t) \\
&= A_t \left(\frac{\alpha}{1-\alpha} \frac{w_t(s_t)}{r_t(s_t)} \right)^{\alpha} l_{it}(s_t) = 1
\end{aligned}
$$

Marginal Cost II

- Then:

$$
\begin{aligned}
mc_t(s_t) &= \left(\frac{1}{1-\alpha}\right) w_t(s_t) \frac{1}{A_t} \left(\frac{\alpha}{1-\alpha} \frac{w_t(s_t)}{r_t(s_t)}\right)^{-\alpha} \\
&= \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^{\alpha} \frac{1}{A_t} w_t(s_t)^{1-\alpha} r_t(s_t)^{\alpha}
\end{aligned}
$$

- Note that the marginal cost does not depend on i.

- Also, from the optimality conditions of input demand, input prices must satisfy:

$$
k_t(s_t) = \frac{\alpha}{1-\alpha} \frac{w_t(s_t)}{r_t(s_t)} l_t(s_t)
$$

Maximization Problem II

- The second part of the problem is to choose price that maximizes discounted real profits, i.e.,

$$\max_{p_{it}(s_t)} \left\{ \left(\frac{p_{it}(s_t)}{p_t(s_t)} - mc_t(s_t) \right) y_{it}(s_t) \right\}$$

subject to

$$y_{it}(s_t) = \left(\frac{p_{it}(s_t)}{p_t(s_t)} \right)^{-\varepsilon} y_t(s_t),$$

- First order condition:

$$\left(\frac{p_{it}(s_t)}{p_t(s_t)} \right)^{-\varepsilon} \frac{y_t(s_t)}{p_t(s_t)} - \varepsilon \left(\frac{p_{it}(s_t)}{p_t(s_t)} - mc_t(s_t) \right) \left(\frac{p_{it}(s_t)}{p_t(s_t)} \right)^{-\varepsilon-1} \frac{y_t(s_t)}{p_t(s_t)} = 0$$

Mark-Up Condition

- From the fist order condition:

$$1 - \varepsilon \left(\frac{p_{it}(s_t)}{p_t(s_t)} - mc_t(s_t) \right) \left(\frac{p_{it}(s_t)}{p_t(s_t)} \right)^{-1} = 0 \Rightarrow$$
$$p_{it}(s_t) = \varepsilon \left(p_{it}(s_t) - mc_t(s_t) p_t(s_t) \right) \Rightarrow$$
$$p_{it}(s_t) = \frac{\varepsilon}{\varepsilon - 1} mc_t(s_t) p_t(s_t)$$

- Mark-up condition.

- Reasonable values for ε.

Aggregation I

- To derive an expression for aggregate output, remember that:

$$\frac{k_{it}(s_t)}{l_{it}(s_t)} = \frac{\alpha}{1-\alpha}\frac{w_t(s_t)}{r_t(s_t)}$$

- Since this ratio is equivalent for all intermediate firms, it must also be the case that:

$$\frac{k_{it}(s_t)}{l_{it}(s_t)} = \frac{k_t(s_t)}{l_t(s_t)} = \frac{\alpha}{1-\alpha}\frac{w_t(s_t)}{r_t(s_t)}$$

- If we substitute this condition in the production function of the intermediate good firm $A_t k_{it}(s_t)^{\alpha} l_{it}(s_t)^{1-\alpha}$ we derive:

$$y_{it} = A_t \left(\frac{k_{it}(s_t)}{l_{it}(s_t)}\right)^{\alpha} l_{it}(s_t) = A_t \left(\frac{k_t(s_t)}{l_t(s_t)}\right)^{\alpha} l_{it}(s_t)$$

Aggregation II

- The demand function for the firm is:

$$y_{it}(s_t) = \left(\frac{p_{it}(s_t)}{p_t(s_t)}\right)^{-\varepsilon} y_t(s_t) \qquad \forall i,$$

- Thus, we find the equality:

$$\left(\frac{p_{it}(s_t)}{p_t(s_t)}\right)^{-\varepsilon} y_t(s_t) = A_t \left(\frac{k_t(s_t)}{l_t(s_t)}\right)^{\alpha} l_{it}(s_t)$$

- If we integrate in both sides of this equation:

$$y_t(s_t) \int_0^1 \left(\frac{p_{it}(s_t)}{p_t(s_t)}\right)^{-\varepsilon} di = A_t \left(\frac{k_t(s_t)}{l_t(s_t)}\right)^{\alpha} \int_0^1 l_{it}(s_t)\, di = A_t k_t(s_t)^{\alpha} l_t(s_t)^{1-\alpha}$$

Aggregation III

- Then:

$$y_t(s_t) = \frac{A_t \cdot}{v_t(s_t)} k_t(s_t)^{\alpha} l_t(s_t)^{1-\alpha}$$

where

$$v_t(s_t) = \int_0^1 \left(\frac{p_{it}(s_t)}{p_t(s_t)}\right)^{-\varepsilon} di = \frac{j_t(s_t)^{-\varepsilon}}{p_t(s_t)^{-\varepsilon}}$$

- But note that:

$$p_t(s_t) = \left(\int_0^1 p_{it}(s_t)^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}} = p_{it}(s_t)$$

since all intermediate good producers charge the same price.

- Then: $v_t(s_t) = \int_0^1 \left(\frac{p_{it}(s_t)}{p_t(s_t)}\right)^{-\varepsilon} di = 1$ and:

$$y_t = A_t k_t(s_t)^{\alpha} l_t(s_t)^{1-\alpha}$$

Behavior of the Model

- Presence of monopolistic competition is, by itself, pretty irrelevant.

- Why? Constant mark-up.

- Similar to a tax.

- Solutions:

  1. Shocks to mark-up (maybe endogenous changes).

  2. Price rigidities.

Further Extensions

- We can extend our model in many other directions.

- Examples we are not going to cover:

  1. Fiscal Policy shocks (McGrattan, 1994).

  2. Agents with Finite Lives (Ríos-Rull, 1996).

  3. Home Production (Benhabib, Rogerson, and Wright, 1991).