Introduction to Uncertainty

Jesús Fernández-Villaverde

University of Pennsylvania

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Uncertainty in Macroeconomics

- Modern macro studies stochastic processes of observed variables.

- Two elements:

  1. Dynamics.

  2. Uncertainty.

- We will introduce some basic concepts by building a pure exchange economy with stochastic endowments.

- In this lecture, we will present the expected discounted utility and use it to assess the welfare cost of the business cycle.

Time

- Discrete time $t \in \{0, 1, 2, ...\}$.
- Why discrete time?
  1. Economic data is discrete.
  2. Easier math.
- Comparison with continuous time:
  1. Discretize observables.
  2. More involved math (stochastic calculus), but often we have extremely powerful results.
- Calendar versus planning time.

Events

- One event s_t happens in each period.

- $s_t \in S = \{1, 2, ..., N\}$.

- Note:

  1. S is a finite set. We will later talk about measure theory.

  2. S does not depend on time.

- Event history $s^t = (s_0, s_1, ..., s_t) \in S \times ... \times S = S^{t+1}$.

Probabilities

- Probability of s^t is $\pi(s^t)$.
- Conditional probability of s_{t+1} is $\pi\left(s_{t+1} \middle| s^t\right)$.
- At this moment, we are not imposing any transition probability among states across time.
- Our notation allows the *particular* cases:

$$\begin{aligned} \pi\left(s_{t+1} \middle| s^t\right) &= \pi\left(s_{t+1}\right) \\ \pi\left(s_{t+1} \middle| s^t\right) &= \pi\left(s_{t+1} \middle| s_t\right) \end{aligned}$$

Commodity Space

- One good in the economy.
- However, good indexed by event history over infinite time. Hence our commodity space is slightly more complicated (see chapter 15 in SLP).
- Commodity space: $(C, \|\cdot\|)$.
- We pick l_∞, i.e., the space of sequences $c = (c_0, c_1, ...)$, $c_n \in \mathbb{R}$ that are bounded in the norm:

$$\|c\|_\infty = \sup_i |c_i|$$

Household Preferences

- Preferences admit a representation:

$$U(c) = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t(s^t))$$

- This is known as the von Neumann-Morgenstern expected utility function.

- Remember:

1. Key assumptions: continuity and independence axioms.
2. Linear in probabilities.
3. Cardinal utility: unique only up to an affine transformation.

Facts about Utility Function I: Time Separability

- Total utility c equals the expected discounted sum of period (or instantaneous) utility $u(c_t(s^t))$.

- The period utility at time t only depends on consumption in period t and not on consumption in other periods.

- This formulation rules out, among other things, habit persistence.

- However, it is easy to relax: recursive utility functions.

Facts about Utility Function II: Time Discounting

- $\beta < 1$ indicates that agents are impatient.
- β is called the (subjective) time discount factor.
- The subjective time discount rate ρ is defined by $\beta = \frac{1}{1+\rho}$.
- Assumption: constant over time$\rightarrow$exponential discounting.
- Alternatives: hyperbolic discounting, endogenous discounting, ...

Facts about Utility Function III: Risk Aversion

- Arrow-Pratt Absolute Risk Aversion:

$$ARA = -\frac{u''(c)}{u'(c)}$$

Why do we divide by $u'(c)$?

- Arrow-Pratt Relative Risk Aversion:

$$RRA = -\frac{u''(c)}{u'(c)}c$$

Interpretation.

Common Utility Functions

- Constant Absolute Risk Aversion (CARA):

$$-e^{-ac}$$

- Constant Relative Risk Aversion (CRRA):

$$\begin{aligned} \frac{c^{1-\gamma}-1}{1-\gamma} \text{ for } \gamma &\neq 1 \\ \log c \text{ for } \gamma &= 1 \end{aligned}$$

(you need to take limits and apply L'Hôpital's rule).

- Why CRRA Utility Functions?

1. Market price of risk has been roughly constant over the last two centuries.
2. This observation suggests that risk aversion should be relatively constant over wealth levels.

CRRA Utility Functions

- γ plays a dual role controlling risk-aversion and intertemporal substitution.

- Coefficient of Relative Risk-aversion:

$$-\frac{u''(c)}{u'(c)}c = \gamma$$

- Elasticity of Intertemporal Substitution:

$$-\frac{u(c_2)/u(c_1)}{c_2/c_1}\frac{d\left(c_2/c_1\right)}{d\left(u(c_2)/u(c_1)\right)} = \frac{1}{\gamma}$$

- Advantages and disadvantages.

Cost of Business Cycles

- Simple CRRA utility function already answers many questions.
- Lucas (1987), *Models of Business Cycles*: What is the welfare cost of business cycles?
- Importance of question:
  1. Limits of stabilization policy.
  2. Macroeconomic priorities.

A Process for Consumption

- Assume that consumption evolves over time as:

$$c_t = \mu^t (1+\lambda) e^{-\frac{1}{2}\sigma_z^2} z_t c$$

where $\log z_t \sim \mathcal{N}\left(0, \sigma_z^2\right)$.

- The moment generating function of a lognormal distribution implies:

$$\mathbb{E}\left(z_t^m\right) = e^{\frac{m^2 \sigma_z^2}{2}}$$

- Then:

$$\mathbb{E}\left(e^{-\frac{1}{2}\sigma_z^2} z_t\right) = 1$$

$$\mathbb{E}\left(z_t^{1-\gamma}\right) = e^{\frac{1}{2}(1-\gamma)^2 \sigma_z^2}$$

A Compensating Differential

- We want to find the value of λ such that:

$$\mathbb{E}\frac{c_t^{1-\gamma}-1}{1-\gamma}=\frac{(\mu^t c)^{1-\gamma}-1}{1-\gamma}$$

- If this condition is true period by period and event by event, it should also be true when we sum up.

- Moreover, the converse is also true: λ is the smallest number that makes total utilities over time to be equal. Why? Because of the CRRA and the i.i.d. structure of z_t.

- Interpretation: λ is the welfare cost of uncertainty, i.e., by how much we need to raise consumption in every period and state.

Finding Compensating Differential

- Dropping irrelevant constants, λ solves:

$$\mathbb{E}\left((1+\lambda)\left(e^{-\frac{1}{2}\sigma_z^2}z_t\right)\right)^{1-\gamma}=1\Rightarrow$$

$$(1+\lambda)\,e^{-\frac{1}{2}\sigma_z^2}\left(\mathbb{E}z_t^{1-\gamma}\right)^{\frac{1}{1-\gamma}}=1\Rightarrow$$

$$(1+\lambda)\,e^{-\frac{1}{2}\sigma_z^2+\frac{1}{2}(1-\gamma)\sigma_z^2}=1\Rightarrow$$

$$(1+\lambda)\,e^{-\frac{1}{2}\gamma\sigma_z^2}=1$$

- Taking logs: $\lambda\approx\frac{1}{2}\gamma\sigma_z^2$.

- Let us put some numbers here. Using quarterly U.S. data 1947-2006, $\sigma_z^2=(0.033)^2$. What is γ?

Size of Risk Aversion

- Most evidence suggests that γ is low, between 1 and 3. At most 10.
- Types of evidence:
  1. Questionnaires.
  2. Experiments.
  3. Econometric estimates from observed behavior.
- Two powerful arguments from growth theory international comparisons. We will revisit these points when we talk about asset pricing.
- Rabin (2000): "Risk Aversion and Expected-Utility Theory: A Calibration Theorem.", *Econometrica*.

An Estimate of the Cost of the Business Cycle

- Let us take $\gamma = 1$ as a benchmark number. Then, we have:

$$\lambda \approx \frac{1}{2}\gamma\sigma_z^2 = \frac{1}{2} * 1 * (0.033)^2 = 0.0005$$

- Even if we take $\gamma = 10$ as an upper bound:

$$\lambda \approx \frac{1}{2}\gamma\sigma_z^2 = \frac{1}{2} * 10 * (0.033)^2 = 0.005$$

- These are extremely small numbers.

- Later we will see how this finding is intimately linked with the Equity premium puzzle.

- How could we turn around this result?

Alternatives Routes

- We assumed:

  1. Representative agent.
  2. Exogenous lognormal consumption.
  3. Expected utility.

- How important are each of these three assumptions?

Representative Agent

- Representative agent: fluctuations are at the margin.
- Lucas is very explicit about the possible costs of inequality.
- We will see in the next lecture that, with complete markets, we will have perfect risk sharing.
- But the interesting question is the effects of business cycles with incomplete markets and heterogeneity.
- Krusell and Smith (2002), loss of 0.001 of average consumption, 65% of households *lose* when business cycles are removed.

Exogenous Lognormal Consumption

- A combined hypothesis: exogenous consumption+lognormal consumption.
- Exogenous consumption⇒Cho and Cooley (2001), business cycles *may* increase welfare: mean versus spread effect. Same answer if we have New Keynesian models Galí, Gertler, and López-Salido (2007).
- Lognormal consumption⇒great depressions? Chaterjee and Corbae (2005): welfare cost of 0.0187. They calibrate a great depression every 87 years.
- A nonparametric approach by Álvarez and Jermann (2004) suggests costs between 0.0008 and 0.0049.

Problems of Expected Utility

- We have representation:

$$U(c) = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t(s^t))$$

- Three strong assumptions:

  1. Intertemporal elasticity of substitution and risk aversion are determined by just one parameter.
  2. Temporal separability.
  3. Expected utility.

- All are problematic and they may affect our calculations.

Recursive Utility

- Espstein-Zin preferences (1989):

$$U_t = \left[(1-\beta)\, c_t^{\rho} + \beta \left(\mathbb{E}_t U_{t+1}^{\alpha} \right)^{\frac{\rho}{\alpha}} \right]^{\frac{1}{\rho}}$$

  separates elasticity of substitution:

$$\gamma = \frac{1}{1-\rho}$$

  from risk-aversion α.

- Applied to evaluate cost of business cycles by Tallarini (2000).

- Risk in the long run:

  1. Bansal and Yaron (2004): difficult to distinguish a long run component from a random walk.
     Implications for the equity premium.

  2. Croce (2006): cost of business cycle.

Temporal Anomalies

- Present-bias. Frederich, Lowenstein, and O'Donoghue (2002), "Time Discounting and Time Preference: A Critical Review". *Journal of Economic Literature*.

- Explanations:

  1. Hyperbolic discounting (Phelps and Pollack, 1968, Laibson, 1996):

$$\sum_{t=0}^{\infty} \delta \beta^t u(c_t)$$

  2. Temptation: Gul and Pesendorfer (2003).

Uncertainty Anomalies

1. Framing effects (Kahneman and Tversky).

2. Allais paradox. Three prizes in a lottery: $\{0, 1, 10\}$

   Problem 1: $L_1 = (0, 1, 0)$ versus $L_2 = (0.01, 0.89, 0.1)$.

   Problem 2: $L_3 = (0.89, 0.11, 0)$ versus $L_4 = (0.9, 0, 0.1)$.

3. Ellsberg paradox.

Ambiguity Aversion

- Knight (1921) risk versus uncertainty.
- Gilboa and Schmidler (1989):

$$\min_{Q \in \mathcal{P}} \mathbb{E}_Q u(c)$$

- Two possible extensions:
  1. Choice over time.
  2. General class of ambiguity aversion.

Choice over Time

- Epstein and Schneider (2003):

$$\min_{Q \in \mathcal{P}} \mathbb{E}_Q \sum_{t=0}^{\infty} \beta^t u(c_t)$$

- Difficult technical assumption⇒rectangularity.

Ambiguity and the Variational Representation of Preferences

- Maccheroni, Marinacci, and Rustichini (2006):

$$\min_{Q\in\mathcal{P}}\left\{\mathbb{E}_Q u(c)+\phi\left(Q\right)\right\}$$

- The function u represents risk attitudes while the index c captures ambiguities attitudes.

- They extend it to the intertemporal case.

- One particular example:

$$\min_{Q\in\mathcal{P}}\left\{\mathbb{E}_Q u(c)+\theta R\left(Q\|P\right)\right\}$$

- Hansen and Sargent's (2006) research program on robust control.