Asset Pricing

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February 12, 2016

Modern Asset Pricing

- How do we value an arbitrary stream of future cash-flows?
- Equilibrium approach to the computation of asset prices. Rubinstein (1976) and Lucas (1978) tree model.
- Absence of arbitrage: Harrison and Kreps (1979).
- Importance for macroeconomists:
  1. Quantities *and* prices.
  2. Financial markets equate savings and investment.
  3. Intimate link between welfare cost of fluctuations and asset pricing.
  4. Effect of monetary policy.
- We will work with a sequential markets structure with a complete set of Arrow securities.

Household Utility

- Representative agent.
- Preferences:

$$U(c) = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t(s^t))$$

- Budget constraints:

$$c_t(s^t) + \sum_{s_{t+1}|s^t} Q_t(s^t, s_{t+1}) a_{t+1}(s^t, s_{t+1}) \leq e_t(s^t) + a_t(s^t)$$

$$-a_{t+1}(s^{t+1}) \leq A_{t+1}(s^{t+1})$$

Problem of the Household

- We write the Lagrangian:

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \left\{ \begin{array}{c} \beta^t \pi(s^t) u(c_t(s^t)) \\ +\lambda_t\left(s^t\right) \left(\begin{array}{c} e_t(s^t) + a_t(s^t) - c_t(s^t) \\ -\sum_{s_{t+1}} Q_t(s^t, s_{t+1}) a_{t+1}(s^t, s_{t+1}) \end{array} \right) \\ +\upsilon_t\left(s^t\right) \left(A_{t+1}(s^{t+1}) + a_{t+1}(s^{t+1})\right) \end{array} \right\}$$

- We take first order conditions with respect to $c\left(s^t\right)$ and $a_{t+1}(s^t, s_{t+1})$ for all s^t.

- Because of an Inada condition on u, $\upsilon_t\left(s^t\right) = 0$.

Solving the Problem

- FOCs for all s^t:

$$\begin{aligned} \beta^t \pi\left(s^t\right) u'\left(c_t\left(s^t\right)\right) - \lambda_t\left(s^t\right) &= 0 \\ -\lambda_t\left(s^t\right) Q_t(s^t, s_{t+1}) + \lambda_{t+1}\left(s_{t+1}, s^t\right) &= 0 \end{aligned}$$

- Then:

$$Q_t(s^t, s_{t+1}) = \beta \pi\left(s_{t+1} \middle| s^t\right) \frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)}$$

- Fundamental equation of asset pricing.

- Intuition.

Interpretation

- The FOC is an equilibrium condition, not an explicit solution (we have endogenous variables in both sides of the equation).
- We need to evaluate consumption in equilibrium to obtain equilibrium prices.
- In our endowment set-up, this is simple.
- In production economies, it requires a bit more work.
- However, we already derived a moment condition that can be empirically implemented.

The j-Step Problem I

- How do we price claims further into the future?
- Create a new security $a_{t+j}(s^t, s_{t+j})$.
- For $j > 1$:

$$Q_t(s^t, s_{t+j}) = \beta^j \pi \left(s_{t+j} | \, s^t \right) \frac{u' \left(c_{t+j} \left(s^{t+j} \right) \right)}{u' \left(c_t \left(s^t \right) \right)}$$

- We express this price in terms of the prices of basic Arrow securities.

The j-Step Problem II

- Manipulating expression:

$$Q_t(s^t, s_{t+j}) =$$

$$= \beta^j \sum_{s_{t+1}|s^t} \pi\left(s_{t+1}|\, s^t\right) \pi\left(s_{t+j}|\, s^{t+1}\right) \frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)} \frac{u'\left(c_{t+j}\left(s^{t+j}\right)\right)}{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}$$

$$= \sum_{s_{t+1}|s^t} Q_t(s^t, s_{t+1}) Q_{t+1}(s^{t+1}, s_{t+j})$$

- Iterating:

$$Q_t(s^t, s_{t+j}) = \prod_{\tau=t}^{j-1} \sum_{s_{\tau+1}|s^\tau} Q_{t+\tau}(s^\tau, s_{\tau+1})$$

The Stochastic Discount Factor

- Stochastic discount factor (SDF):

$$m_t\left(s^t, s_{t+1}\right) = \beta \frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)}$$

- Note that:

$$\begin{aligned} \mathbb{E}_t m_t\left(s^t, s_{t+1}\right) &= \sum_{s_{t+1}|s^t} \pi\left(s_{t+1}|\, s^t\right) m_t\left(s^t, s_{t+1}\right) \\ &= \beta \sum_{s_{t+1}|s^t} \pi\left(s_{t+1}|\, s^t\right) \frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)} \end{aligned}$$

- Interpretation of the SDF: discounting corrected by asset-specific risk.

The Many Names of the Stochastic Discount Factor

The Stochastic discount factor is also known as:

1. Pricing kernel.
2. Marginal rate of substitution.
3. Change of measure.
4. State-dependent density.

Pricing Redundant Securities I

- With our framework we can price any security (the $j-$step pricing was one of those cases).

- Contract that pays $x_{t+1}\left(s^{t+1}\right)$ in event s^{t+1}:

$$
\begin{aligned}
p_t\left(s_{t+1}, s^t\right) &= \beta\pi\left(s_{t+1} | s^t\right)\frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)}x_{t+1}\left(s^{t+1}\right) \\
&= \pi\left(s_{t+1} | s^t\right) m_t\left(s^t, s_{t+1}\right) x_{t+1}\left(s^{t+1}\right) \\
&= Q_t(s^t, s_{t+1})x_{t+1}\left(s^{t+1}\right)
\end{aligned}
$$

Pricing Redundant Securities II

- Contract that pays $x_{t+1}\left(s^{t+1}\right)$ in each event s^{t+1} (sum of different contracts that pay in one event):

$$
\begin{aligned}
p_t\left(s^t\right) &= \beta \sum_{s_{t+1}|s^t} \pi\left(s_{t+1} \middle| s^t\right) \frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)} x_{t+1}\left(s^{t+1}\right) \\
&= \mathbb{E}_t m_t\left(s^t, s_{t+1}\right) x_{t+1}\left(s^{t+1}\right)
\end{aligned}
$$

- Note: we do not and we cannot take the expectation with respect to the price $Q_t(s^t, s_{t+1})$.

Example I: Uncontingent One-Period Bond at Discount

- Many bonds are auctioned or sold at discount:

$$\begin{aligned} b_t\left(s^t\right) &= \sum_{s_{t+1}|s^t} Q_t(s^t, s_{t+1}) = \beta \sum_{s_{t+1}|s^t} \pi\left(s_{t+1}|\, s^t\right) \frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)} \\ &= \mathbb{E}_t m_t\left(s^t, s_{t+1}\right) \end{aligned}$$

- Then, the risk-free rate:

$$R_t^f\left(s^t\right) = \frac{1}{b_t\left(s^t\right)} = \frac{1}{\mathbb{E}_t m_t\left(s^t, s_{t+1}\right)}$$

or $\mathbb{E}_t m_t\left(s^t, s_{t+1}\right) R^f\left(s^t\right) = 1$.

Example II: One-Period Bond

- Other bonds are sold at face value:

$$
\begin{aligned}
1 &= \beta \sum_{s_{t+1}|s^t} \pi\left(s_{t+1}|\, s^t\right) \frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)} R_t^b\left(s^t\right) \\
&= \mathbb{E}_t m_t\left(s^t, s_{t+1}\right) R_t^b\left(s^t\right)
\end{aligned}
$$

- As before, if the bond is risk-free:

$$
1 = \mathbb{E}_t m_t\left(s^t, s_{t+1}\right) R_t^f\left(s^t\right)
$$

Example III: Zero-Cost Portfolio

- Short-sell an uncontingent bond and take a long position in a bond:

$$
\begin{aligned}
0 &= \beta \sum_{s_{t+1}|s^t} \pi\left(s_{t+1}|\, s^t\right) \frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)} \left(R_t^b\left(s^t\right) - R_t^f\left(s^t\right)\right) \\
&= \mathbb{E}_t m_t\left(s^t, s_{t+1}\right) R_t^e\left(s^t\right)
\end{aligned}
$$

  where $R_t^e\left(s^t\right) = R_t^b\left(s^t\right) - R_t^f\left(s^t\right)$.

- $R_t^e\left(s^t\right)$ is known as the excess return. Key concept in empirical work.

- Why do we want to focus on excess returns? Different forces may drive the risk-free interest rate and the risk premia.

Example IV: Stock

- Buy at price $p_t\left(s^t\right)$, delivers a dividend $d_{t+1}\left(s^{t+1}\right)$, sell at $p_{t+1}\left(s^{t+1}\right)$:

$$p_t\left(s^t\right)=\beta\sum_{s_{t+1}|s^t}\pi\left(s_{t+1}|\,s^t\right)\frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)}\left(p_{t+1}\left(s^{t+1}\right)+d_{t+1}\left(s^{t+1}\right)\right)$$

- Often, we care about the price-dividend ratio (usually a stationary variable that we may want to forecast):

$$\frac{p_t\left(s^t\right)}{d_t\left(s^t\right)}=$$

$$\beta\sum_{s_{t+1}|s^t}\pi\left(s_{t+1}|\,s^t\right)\frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)}\left(\frac{p_{t+1}\left(s^{t+1}\right)}{d_{t+1}\left(s^{t+1}\right)}+1\right)\frac{d_{t+1}\left(s^{t+1}\right)}{d_t\left(s^t\right)}$$

Example V: Options

- Call option: right to buy an asset at price K_1. Price of asset $J\left(s^{t+1}\right)$

$$co_t\left(s^t\right) = \beta \sum_{s_{t+1}|s^t} \pi\left(s_{t+1}|\, s^t\right) \max\left(\left(J\left(s^{t+1}\right) - K_1\right)\frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)}, 0\right)$$

- Put option: right to sell an asset at price K_1. Price of asset $J\left(s^1\right)$

$$po_t\left(s^t\right) = \sum_{s_{t+1}|s^t} \pi\left(s_{t+1}|\, s^t\right) \max\left(\left(K_1 - J\left(s^{t+1}\right)\right)\frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)}, 0\right)$$

Example VI: Nominal Assets

- What happens if the price level, $P\left(s^t\right)$ changes over time?
- We can focus on real returns:

$$
\begin{aligned}
\frac{p_t\left(s^t\right)}{P_t\left(s^t\right)} &= \beta \sum_{s_{t+1}|s^t} \pi\left(s_{t+1}|\, s^t\right) \frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)} \frac{x_{t+1}\left(s^{t+1}\right)}{P_{t+1}\left(s^{t+1}\right)} \Rightarrow \\
p_t\left(s^t\right) &= \beta \sum_{s^1 \in S^1} \pi\left(s^1\right) \frac{u'\left(c\left(s^1\right)\right)}{u'\left(c\left(s_0\right)\right)} \frac{P_t\left(s^t\right)}{P_{t+1}\left(s^{t+1}\right)} x_{t+1}\left(s^{t+1}\right)
\end{aligned}
$$

Example VII: Term Structure of Interest Rates

- The risk-free rate j periods ahead is:

$$R_{tj}^{f}\left(s^{t}\right)=\left[\beta^{j}\mathbb{E}_{t}\frac{u'\left(c_{t+j}\left(s^{t+j}\right)\right)}{u'\left(c_{t}\left(s^{t}\right)\right)}\right]^{-1}$$

- And the yield to maturity is:

$$R_{tj}^{fy}\left(s^{t}\right)=\left(R_{tj}^{f}\left(s^{t}\right)\right)^{\frac{1}{j}}=\beta^{-1}\left[u'\left(c_{t}\left(s^{t}\right)\right)\left(\mathbb{E}_{t}u'\left(c_{t+j}\left(s^{t+j}\right)\right)\right)^{-1}\right]^{\frac{1}{j}}$$

- Structure of the yield curve:
  1. Average shape (theory versus data).
  2. Equilibrium dynamics.
- Equilibrium models versus affine term structure models.

Non Arbitrage

- A lot of financial contracts are equivalent.
- From previous results, we derive a powerful idea: absence of arbitrage.
- In fact, we could have built our theory from absence of arbitrage up towards equilibrium.
- Empirical evidence regarding non arbitrage.
- Possible limitations to non arbitrage conditions: liquidity constraints, short-sales restrictions, incomplete markets, ...
- Related idea: spanning of non-traded assets.

A Numerical Example

- Are there further economic insights that we can derive from our conditions?
- We start with a simple numerical example.
- $u(c) = \log c$.
- $\beta = 0.99$.
- $e\left(s^0\right) = 1,\ e\left(s_1 = high\right) = 1.1,\ e\left(s_1 = low\right) = 0.9$.
- $\pi\left(s_1 = high\right) = 0.5,\ \pi\left(s_2 = low\right) = 0.5$.

- Equilibrium prices:

$$
\begin{aligned}
q\left(s^0, s_1 = high\right) &= 0.99 * 0.5 * \frac{\frac{1}{1.1}}{\frac{1}{1}} = 0.45 \\
q\left(s^0, s_1 = low\right) &= 0.99 * 0.5 * \frac{\frac{1}{0.9}}{\frac{1}{1}} = 0.55 \\
q\left(s^0\right) &= 0.45 + 0.55 = 1
\end{aligned}
$$

- Note how the price is different from a naive adjustment by expectation and discounting:

$$
\begin{aligned}
q_{naive}\left(s^0, s_1 = high\right) &= 0.99 * 0.5 * 1 = 0.495 \\
q_{naive}\left(s^0, s_1 = low\right) &= 0.99 * 0.5 * 1 = 0.495 \\
q_{naive}\left(s^0\right) &= 0.495 + 0.495 = 0.99
\end{aligned}
$$

- Why is $q\left(s^0, s_1 = high\right) < q\left(s^0, s_1 = low\right)$?
  1. Discounting β.
  2. Ratio of marginal utilities: $\frac{u'\left(c\left(s^1\right)\right)}{u'\left(c\left(s_0\right)\right)}$.

Risk Correction

- We recall three facts:

1. $p_t\left(s^t\right) = \mathbb{E}_t m_t\left(s^t, s_{t+1}\right) x_{t+1}\left(s^{t+1}\right)$.
2. $cov_t(x,y) = \mathbb{E}_t(xy) - \mathbb{E}_t(x)\mathbb{E}_t(y)$.
3. $\mathbb{E}_t m_t\left(s^t, s_{t+1}\right) = 1/R_t^f\left(s^t\right)$.

- Then:

$$
\begin{aligned}
p_t\left(s^t\right) &= \mathbb{E}_t m_t\left(s^t, s_{t+1}\right) \mathbb{E}_t x_{t+1}\left(s^{t+1}\right) + cov_t\left(m_t\left(s^t, s_{t+1}\right), x_{t+1}\left(s^{t+1}\right)\right) \\
&= \frac{\mathbb{E}_t x_{t+1}\left(s^{t+1}\right)}{R_t^f\left(s^t\right)} + cov_t\left(m_t\left(s^t, s_{t+1}\right), x_{t+1}\left(s^{t+1}\right)\right) \\
&= \frac{\mathbb{E}_t x_{t+1}\left(s^{t+1}\right)}{R_t^f\left(s^t\right)} + cov_t\left(\beta \frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)}, x_{t+1}\left(s^{t+1}\right)\right) \\
&= \frac{\mathbb{E}_t x_{t+1}\left(s^{t+1}\right)}{R_t^f\left(s^t\right)} + \beta \frac{cov\left(u'\left(c_{t+1}\left(s^{t+1}\right)\right), x_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)}
\end{aligned}
$$

Covariance and Risk Correction I

Three cases:

1. If $cov_t\left(m_t\left(s^t, s_{t+1}\right), x_{t+1}\left(s^{t+1}\right)\right) = 0 \Rightarrow p_t\left(s^t\right) = \frac{\mathbb{E}_t x_{t+1}\left(s^{t+1}\right)}{R_t^f\left(s^t\right)}$, no adjustment for risk.

2. If $cov_t\left(m_t\left(s^t, s_{t+1}\right), x_{t+1}\left(s^{t+1}\right)\right) > 0 \Rightarrow p_t\left(s^t\right) > \frac{\mathbb{E}_t x_{t+1}\left(s^{t+1}\right)}{R_t^f\left(s^t\right)}$, premium for risk (insurance).

3. If $cov_t\left(m_t\left(s^t, s_{t+1}\right), x_{t+1}\left(s^{t+1}\right)\right) < 0 \Rightarrow p_t\left(s^t\right) < \frac{\mathbb{E}_t x_{t+1}\left(s^{t+1}\right)}{R_t^f\left(s^t\right)}$, discount for risk (speculation).

Covariance and Risk Correction II

- Risk adjustment is $cov_t\left(m_t\left(s^t, s_{t+1}\right), x_{t+1}\left(s^{t+1}\right)\right)$.
- Basic insight: risk premium is generated by covariances, no by variances.
- Why? Because of risk aversion. Investor cares about volatility of consumption, not about the volatility of asset.
- For an ε change in portfolio:

$$\sigma^2\left(c+\varepsilon x\right)=\sigma^2\left(c\right)+2\varepsilon cov\left(c,x\right)+\varepsilon^2\sigma^2\left(x\right)$$

Utility Function and the Risk Premium

- We also see how risk depends of marginal utilities:

  1. Risk-neutrality: if utility function is linear, you do not care about $\sigma^2(c)$.
  2. Risk-loving: if utility function is convex you want to increase $\sigma^2(c)$.
  3. Risk-averse: if utility function is concave you want to reduce $\sigma^2(c)$.

- It is plausible to assume that household are (basically) risk-averse.

A Small Detour

- Note that all we have said can be applied to the trivial case without uncertainty.

- In that situation, there is only one security, a bond, with price:

$$Q = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

- And the interest rate is:

$$R = \frac{1}{Q} = \frac{1}{\beta} \frac{u'(c_t)}{u'(c_{t+1})}$$

Pricing Securities in the Solow Model

- Assume CRRA utility, that we are in a BGP with growth rate g, and define $\beta = e^{-\delta}$.

- Then: $R = \frac{1}{\beta}\left(\frac{c}{(1+g)c}\right)^{-\gamma} = e^{\delta}(1+g)^{\gamma}$.

- Or in logs: $r \simeq \delta + \gamma g$, i.e., the real interest rate depends on the rate of growth of technology, the readiness of households to substitute intertemporally, and on the discount factor.

- Then, γ must be low to reconcile small international differences in the interest rate and big differences in g.

More on the Risk Free Rate I

- Assume that the growth rate of consumption is log-normally distributed.

- Note that with a CRRA utility function:

$$R_t^f\left(s^t\right)=\frac{1}{\mathbb{E}_t m_t\left(s^t,s_{t+1}\right)}=\frac{1}{\beta\mathbb{E}_t\left(\frac{c\left(s^{t+1}\right)}{c\left(s^t\right)}\right)^{-\gamma}}=\frac{1}{\beta\mathbb{E}_t\left(e^{-\gamma\Delta\log c\left(s^{t+1}\right)}\right)}$$

- Since $\mathbb{E}_t\left(e^z\right)=e^{\mathbb{E}_t(z)+\frac{1}{2}\sigma^2(z)}$ if z is normal:

$$R_t^f\left(s^t\right)=\left[\beta e^{-\gamma\mathbb{E}_t\Delta\log c\left(s^{t+1}\right)+\frac{1}{2}\gamma^2\sigma^2\left(\Delta\log c\left(s^{t+1}\right)\right)}\right]^{-1}$$

More on the Risk Free Rate II

- Taking logs:

$$r_t^f\left(s^t\right) = \delta + \gamma \mathbb{E}_t \Delta \log c\left(s^{t+1}\right) - \frac{1}{2}\gamma^2\sigma^2\left(\Delta \log c\left(s^{t+1}\right)\right)$$

- We can read this equation from right to left and from left to right!

- Rough computation (U.S. annual data, 1947-2005):

1. $\mathbb{E}_t \Delta \log c\left(s^{t+1}\right) = 0.0209.$
2. $\sigma\left(\Delta \log c\left(s^{t+1}\right)\right) = 0.011.$
3. Number for γ? benchmark log utility $\gamma = 1.$

Precautionary Savings

- Term $\frac{\gamma^2}{2}\sigma^2\left(\Delta \log c\left(s^{t+1}\right)\right)$ represents precautionary savings.
- Then, precautionary savings:

$$\frac{1^2}{2}(0.011)^2 = 0.00006 = 0.006\%$$

  decreases the interest rate by a very small amount.
- Why a decrease? General equilibrium effect: change in the ergodic distribution of capital.
- We will revisit this result when we talk about incomplete markets.
- Also, $\frac{\gamma^2}{2}\sigma^2\left(\Delta \log c\left(s^{t+1}\right)\right)$ is close to $\frac{\gamma}{2}\sigma^2\left(\log c\left(s^{t+1}\right)\right)$ (welfare cost of the business cycle):

$$\sigma^2\left(\Delta \log c\left(s^{t+1}\right)\right) \approx 0.33 * \sigma^2\left(\log c_{dev}\left(s^{t+1}\right)\right)$$

- We will come back to this in a few slides.

Quadratic Utility

- Precautionary term appears because we use a CRRA utility function.

- Suppose instead that we have a quadratic utility function (Hall, 1978)

$$-\frac{1}{2}\left(a-c\right)^2$$

- Then:

$$R_t^f\left(s^t\right)=\frac{1}{\mathbb{E}_t m_t\left(s^t,s_{t+1}\right)}=\frac{1}{\beta\mathbb{E}_t\left(\frac{a-c\left(s^{t+1}\right)}{a-c\left(s^t\right)}\right)}$$

Random Walk of Consumption I

- For a sufficiently big in relation with $c\left(s^{t+1}\right)$:

$$\frac{a-c\left(s^{t+1}\right)}{a-c\left(s^{t}\right)} \simeq 1-\frac{1}{a}\Delta c\left(s^{t+1}\right)$$

- Then:

$$R_t^f\left(s^t\right)=\frac{1}{e^{-\delta}\left(1-\frac{1}{a}\mathbb{E}_t\Delta c\left(s^{t+1}\right)\right)}$$

- Taking logs: $r_t^f\left(s^t\right)=\delta+\frac{1}{a}\mathbb{E}_t\Delta c\left(s^{t+1}\right)$.

Random Walk of Consumption II

- We derived Hall's celebrated result:

$$\mathbb{E}_t \Delta c\left(s^{t+1}\right) = a\left(r_t^f\left(s^t\right) - \delta\right)$$

- Consumption is a random walk (possibly with a drift).

- For the general case, we have a random walk in marginal utilities:

$$u'\left(c_t\left(s^t\right)\right) = \beta R_t^f\left(s^t\right)\mathbb{E}_t u'\left(c_{t+1}\left(s^{t+1}\right)\right)$$

  Harrison and Kreps (1979) equivalent martingale measure.

- Empirical implementation:

  1. GMM with additional regressors.
  2. Granger causality.

Precautionary Behavior

- Difference between risk-aversion and precautionary behavior. Leland (1968), Kimball (1990).
- Risk-aversion depends on the second derivative (concave utility).
- Precautionary behavior depends on the third derivative (convex marginal utility).
- Relation with linearization and certainty equivalence.

Random Walks I

- Random walks (or more precisely, martingales) are pervasive in asset pricing.

- Can we predict the market?

- Remember that the price of a share was:

$$p_t\left(s^t\right)=\beta\sum_{s_{t+1}|s^t}\pi\left(s_{t+1}|\,s^t\right)\frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)}\left(p_{t+1}\left(s^{t+1}\right)+d_{t+1}\left(s^{t+1}\right)\right)$$

or:

$$p_t\left(s^t\right)=\beta\mathbb{E}_t\frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)}\left(p_{t+1}\left(s^{t+1}\right)+d_{t+1}\left(s^{t+1}\right)\right)$$

Random Walks II

- Now, suppose that we are thinking about a short period of time ($\beta \approx 1$) and that firms do not distribute dividends (historically not a bad approximation because of tax reasons):

$$p_t\left(s^t\right) = \mathbb{E}_t \frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)} \left(p_{t+1}\left(s^{t+1}\right)\right)$$

- If in addition $\frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)}$ does not change (either because utility is linear or because of low volatility of consumption):

$$p_t\left(s^t\right) = \mathbb{E}_t p_{t+1}\left(s^{t+1}\right) = p_t\left(s^t\right) + \varepsilon_{t+1}$$

- Prices follow a random walk: the best forecast of the price of a share tomorrow is today's price.
- Can we forecast future movements of the market? No!
- We can generalize the idea to other assets.
- Empirical evidence. Relation with market efficiency.

A Second Look at Risk Correction

- We can restate the previous result about martingale risk correction in terms of returns.
- The pricing condition for a contract i with price 1 and yield $R_t^i\left(s^{t+1}\right)$ is:

$$1=\mathbb{E}_t m_t\left(s^t,s_{t+1}\right)R_t^i\left(s^{t+1}\right)$$

- Then:

$$1=\mathbb{E}_t m_t\left(s^t,s_{t+1}\right)\mathbb{E}_t R_t^i\left(s^{t+1}\right)+cov_t\left(m_t\left(s^t,s_{t+1}\right),R_t^i\left(s^{t+1}\right)\right)$$

- Multiplying by $-R_t^f\left(s^t\right)=-\left(\mathbb{E}_t m_t\left(s^t,s_{t+1}\right)\right)^{-1}$:

$$\begin{aligned}\mathbb{E}_t R_t^i\left(s^{t+1}\right)-R_t^f\left(s^t\right) &= -R_t^f\left(s^t\right)cov_t\left(m_t\left(s^t,s_{t+1}\right),R_t^i\left(s^{t+1}\right)\right)\\ &= -R_t^f\left(s^t\right)\beta\frac{cov\left(u'\left(c_{t+1}\left(s^{t+1}\right)\right),x_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)}\\ &= -\frac{cov\left(u'\left(c_{t+1}\left(s^{t+1}\right)\right),x_{t+1}\left(s^{t+1}\right)\right)}{\mathbb{E}_t u'\left(c_{t+1}\left(s^{t+1}\right)\right)}\end{aligned}$$

Beta-Pricing Model

- Note:

$$\mathbb{E}_t R_t^i\left(s^{t+1}\right) - R_t^f\left(s^t\right) = -R_t^f\left(s^t\right) cov_t\left(m_t\left(s^t, s_{t+1}\right), R_t^i\left(s^{t+1}\right)\right) \Rightarrow$$

$$\mathbb{E}_t R_t^i\left(s^{t+1}\right) = R_t^f\left(s^t\right) +$$

$$+\left(\frac{cov_t\left(m_t\left(s^t, s_{t+1}\right), R_t^i\left(s^{t+1}\right)\right)}{\sigma_t\left(m_t\left(s^t, s_{t+1}\right)\right)}\right)\left(-\frac{\sigma_t\left(m_t\left(s^t, s_{t+1}\right)\right)}{\mathbb{E}_t\left(m_t\left(s^t, s_{t+1}\right)\right)}\right)$$

$$= R_t^f\left(s^t\right) + \beta_{i,m,t}\lambda_{m,t}$$

- Interpretation:

1. $\beta_{i,m,t}$ is the quantity of risk of each asset (risk-free asset is the "zero-beta" asset).
2. $\lambda_{m,t}$ is the market price of risk (same for all assets).

Mean-Variance Frontier I

- Yet another way to look at the FOC:

$$1 = \mathbb{E}_t m_t\left(s^t, s_{t+1}\right)\mathbb{E}_t R_t^i\left(s^{t+1}\right) + cov_t\left(m_t\left(s^t, s_{t+1}\right), R_t^i\left(s^{t+1}\right)\right)$$

- Then:

$$1 = \mathbb{E}_t m_t\left(s^t, s_{t+1}\right)\mathbb{E}_t R_t^i\left(s^{t+1}\right)$$
$$+\frac{cov_t\left(m_t\left(s^t, s_{t+1}\right), R_t^i\left(s^{t+1}\right)\right)}{\sigma_t\left(m_t\left(s^t, s_{t+1}\right)\right)\sigma_t\left(R_t^i\left(s^{t+1}\right)\right)}\sigma_t\left(m_t\left(s^t, s_{t+1}\right)\right)\sigma_t\left(R_t^i\left(s^{t+1}\right)\right)$$

Mean-Variance Frontier II

- The coefficient of correlation between two random variables is:

$$\rho_{m,R_i,t} = \frac{cov_t\left(m_t\left(s^t, s_{t+1}\right), R_t^i\left(s^{t+1}\right)\right)}{\sigma_t\left(m_t\left(s^t, s_{t+1}\right)\right)\sigma_t\left(R_t^i\left(s^{t+1}\right)\right)}$$

- Then, we have:

$$\begin{aligned} 1 &= \mathbb{E}_t m_t\left(s^t, s_{t+1}\right)\mathbb{E}_t R_t^i\left(s^{t+1}\right) \\ &\quad + \rho_{m,R_i,t}\sigma_t\left(m_t\left(s^t, s_{t+1}\right)\right)\sigma_t\left(R_t^i\left(s^{t+1}\right)\right) \end{aligned}$$

- Or:

$$\mathbb{E}_t R_t^i\left(s^{t+1}\right) = R_t^f\left(s^t\right) - \rho_{m,R_i,t}\frac{\sigma_t\left(m_t\left(s^t, s_{t+1}\right)\right)}{\mathbb{E}_t m_t\left(s^t, s_{t+1}\right)}\sigma_t\left(R_t^i\left(s^{t+1}\right)\right)$$

Mean-Variance Frontier III

- Since $\rho_{m,R_i,t} \in [-1, 1]$:

$$\left| \mathbb{E}_t R_t^i \left(s^{t+1}\right) - R_t^f \left(s^t\right) \right| \leq \frac{\sigma_t \left(m_t \left(s^t, s_{t+1}\right)\right)}{\mathbb{E}_t m_t \left(s^t, s_{t+1}\right)} \sigma_t \left(R_t^i \left(s^{t+1}\right)\right)$$

- This relation is known as the *Mean-Variance frontier*: "How much return can you get for a given level of variance?"
- Any investor would hold assets within the mean-variance region.
- No assets outside the region will be hold.

Market Price of Risk I

- As we mentioned before, $\frac{\sigma_t(m_t(s^t, s_{t+1}))}{\mathbb{E}_t m_t(s^t, s_{t+1})}$ is the market price of risk.

- Can we find a good approximation for the market price of risk?

- Empirical versus model motivated pricing kernels.

- Assume a CRRA utility function. Then:

$$m_t\left(s^t, s_{t+1}\right) = \beta \left(\frac{c_{t+1}\left(s^{t+1}\right)}{c_t\left(s^t\right)} \right)^{-\gamma}$$

A Few Mathematical Results

- Note that if z is normal

$$\mathbb{E}\left(e^{z}\right) = e^{\mathbb{E}(z)+\frac{1}{2}\sigma^{2}(z)}$$

$$\sigma^{2}\left(e^{z}\right) = \left(e^{\sigma^{2}(z)} - 1\right) e^{2\mathbb{E}(z)+\sigma^{2}(z)}$$

  hence

$$\frac{\sigma\left(e^{z}\right)}{\mathbb{E}\left(e^{z}\right)} = \left(\frac{\sigma^{2}\left(e^{z}\right)}{\mathbb{E}\left(e^{z}\right)^{2}}\right)^{0.5} = \left(e^{\sigma^{2}(z)} - 1\right)^{0.5}$$

- Also $e^{x} - 1 \simeq x$.

Market Price of Risk II

- If we set $z = \frac{1}{\beta} \log m_t\left(s^t, s_{t+1}\right) = -\gamma \log\left(\frac{c_{t+1}\left(s^{t+1}\right)}{c_t\left(s^t\right)}\right)$, we have:

$$\frac{\sigma_t\left(m_t\left(s^t, s_{t+1}\right)\right)}{\mathbb{E}_t m_t\left(s^t, s_{t+1}\right)} = \left(e^{\gamma^2\sigma^2\left(\Delta \ln c\left(s^{t+1}\right)\right)} - 1\right)^{0.5}$$

$$\simeq \gamma\sigma\left(\Delta \ln c\left(s^{t+1}\right)\right)$$

- Price of risk depends on EIS and variance of consumption growth.

- This term already appeared in our formula for the risk-free rate:

$$r_t^f\left(s^t\right) = \delta + \gamma \mathbb{E}_t \Delta \log c\left(s^{t+1}\right) - \frac{1}{2}\gamma^2\sigma^2\left(\Delta \log c\left(s^{t+1}\right)\right)$$

- Also, a nearly identical term, $\frac{1}{2}\gamma\sigma^2\left(\ln c_{dev}\left(s^{t+1}\right)\right)$, was our estimate of the welfare cost of the business cycle.

Link with Welfare Cost of Business Cycle I

- This link is not casual: welfare costs of uncertainty and risk price are two sides of the same coin.

- We can coax the cost of the business cycle from market data.

- In lecture 1, we saw that we could compute the cost of the business cycle by solving:

$$\mathbb{E}_{t-1} u \left[(1 + \Omega_{t-1}) \, c\left(s^t\right) \right] = u\left(\mathbb{E}_{t-1} c\left(s^t\right) \right)$$

- Parametrize Ω_{t-1} as a function of $\alpha \in (0, 1)$. Then:

$$\mathbb{E}_{t-1} u \left[(1 + \Omega_{t-1}(\alpha)) \, c\left(s^t\right) \right] = \mathbb{E}_{t-1} u \left(\alpha \mathbb{E}_{t-1} c\left(s^t\right) + (1 - \alpha) \, c\left(s^t\right) \right)$$

Link with Welfare Cost of Business Cycle II

- Take derivatives with respect to α and evaluate at $\alpha = 0$

$$\Omega'_{t-1}(0) = \frac{\mathbb{E}_{t-1} u'\left(c\left(s^t\right)\right)\left(\mathbb{E}_{t-1} c\left(s^t\right) - c\left(s^t\right)\right)}{\mathbb{E}_{t-1} c\left(s^t\right) u'\left(c\left(s^t\right)\right)}$$

- Dividing by $\beta / u'\left(c\left(s^{t-1}\right)\right)$, we get $m\left(s^t\right)$

$$\Omega'_{t-1}(0) = \frac{\mathbb{E}_{t-1} m_t\left(s^{t-1}, s_t\right)\left(\mathbb{E}_{t-1} c\left(s^t\right) - c\left(s^t\right)\right)}{\mathbb{E}_{t-1} m_t\left(s^{t-1}, s_t\right) c\left(s^t\right)}$$

- Rearranging and using the fact that $\Omega_{t-1}(0) = 0$,

$$1 + \Omega'_{t-1}(0) = \frac{\mathbb{E}_{t-1} m_t\left(s^{t-1}, s_t\right) \mathbb{E}_{t-1} c\left(s^t\right)}{\mathbb{E}_{t-1} m_t\left(s^{t-1}, s_t\right) c\left(s^t\right)}$$

The Sharpe Ratio I

- Another way to represent the Mean-Variance frontier is:

$$\left| \frac{\mathbb{E}_t R_t^i \left(s^{t+1}\right) - R_t^f \left(s^t\right)}{\sigma_t \left(R_t^i \left(s^{t+1}\right)\right)} \right| \leq \frac{\sigma_t \left(m_t \left(s^t, s_{t+1}\right)\right)}{\mathbb{E}_t m_t \left(s^t, s_{t+1}\right)}$$

- This relation is known as the *Sharpe Ratio*.

- It answers the question: "How much more mean return can I get by shouldering a bit more volatility in my portfolio?"

- Note again the market price of risk bounding the excess return over volatility.

The Sharpe Ratio II

- For a portfolio at the Mean-Variance frontier:

$$\left| \frac{\mathbb{E}_t R_t^m \left(s^{t+1}\right) - R_t^f \left(s^t\right)}{\sigma_t \left(R_t^m \left(s^{t+1}\right)\right)} \right| = \frac{\sigma_t \left(m_t \left(s^t, s_{t+1}\right)\right)}{\mathbb{E}_t m_t \left(s^t, s_{t+1}\right)}$$

- Given a CRRA utility function, we derive before that, for excess returns at the frontier:

$$\left| \frac{\mathbb{E}_t R_t^{me} \left(s^{t+1}\right)}{\sigma_t \left(R_t^{me} \left(s^{t+1}\right)\right)} \right| \simeq \gamma \sigma \left(\Delta \ln c \left(s^{t+1}\right)\right)$$

- Alternatively (assuming $\mathbb{E}_t R_t^m \left(s^{t+1}\right) > R_t^f \left(s^t\right)$):

$$\mathbb{E}_t R_t^{me} \left(s^{t+1}\right) \simeq R_t^f \left(s^t\right) + \gamma \sigma \left(\Delta \ln c \left(s^{t+1}\right)\right) \sigma_t \left(R_t^m \left(s^{t+1}\right)\right)$$

The Equity Premium Puzzle I

- Let us go to the data and think about the stock market (i.e. $R_t^i\left(s^{t+1}\right)$ is the yield of an index) versus the risk free asset (the U.S. treasury bill).
- Average return from equities in XXth century: 6.7%. From bills 0.9%. (data from Dimson, Marsh, and Staunton, 2002).
- Standard deviation of equities: 20.2%.
- Standard deviation of $\Delta \ln c\left(s^{t+1}\right)$: 1.1%.

The Equity Premium Puzzle II

- Then:

$$\left|\frac{6.7\% - 0.9\%}{20.2\%}\right| = 0.29 \leq 0.011\gamma$$

  that implies a γ of at least 26!

- But we argued before that γ is at most 10.

- This observation is known as the Equity Premium Puzzle (Mehra and Prescott, 1985).

The Equity Premium Puzzle III

- We can also look at the equity premium directly.
- Remember the beta formula:

$$\mathbb{E}_t R_t^{me}\left(s^{t+1}\right) \simeq R_t^f\left(s^t\right) + \gamma\sigma\left(\Delta \ln c\left(s^{t+1}\right)\right)\sigma_t\left(R_t^m\left(s^{t+1}\right)\right)$$

- Then

$$\gamma\sigma\left(\Delta \ln c\left(s^{t+1}\right)\right)\sigma_t\left(R_t^m\left(s^{t+1}\right)\right) = 0.011 * 0.202 * \gamma = 0.0022 * \gamma$$

- For $\gamma = 3$, the equity premium should be 0.0066.

The Equity Premium Puzzle IV

- Things are actually worse than they look:
  1. Correlation between individual and aggregate consumption is not one.
  2. However, U.S. treasury bills are also risky (inflation risk).
- We can redo the derivation of the Sharpe Ratio in terms of excess returns:

$$\left| \frac{\mathbb{E}_t R_t^e \left(s^{t+1}\right)}{\sigma_t \left(R_t^e \left(s^{t+1}\right)\right)} \right| \leq \frac{\sigma_t \left(m_t \left(s^t, s_{t+1}\right)\right)}{\mathbb{E}_t m_t \left(s^t, s_{t+1}\right)}$$

The Equity Premium Puzzle V

- Build a excess return portfolio (Campbell, 2003):

  1. Mean: 8.1%

  2. Standard deviation: 15.3%

- Then

$$\left| \frac{8.1\%}{15.3\%} \right| = 0.53 \leq 0.011\gamma$$

that implies a γ of at least 50!

Raising Risk Aversion

- A naive answer will be to address the equity premium puzzle by raising γ (Kandel and Stambaugh, 1991).
- We cannot really go ahead and set $\gamma = 50$:
  1. Implausible intercountry differences in real interest rates.
  2. We would generate a risk-free rate puzzle (Weil, 1989).
  3. Problems in genera equilibrium.

The Risk-Free Rate Puzzle I

- Remember:

$$r_t^f\left(s^t\right)=\delta+\gamma\mathbb{E}_t\Delta\log c\left(s^{t+1}\right)-\frac{1}{2}\gamma^2\sigma^2\left(\Delta\log c\left(s^{t+1}\right)\right)$$

- $\Delta\log c\left(s^{t+1}\right)=0.0209$, $\sigma^2\left(\Delta\log c\left(s^{t+1}\right)\right)=(0.011)^2$ and $\gamma=10$:

$$\gamma\mathbb{E}_t\Delta\log c\left(s^{t+1}\right)-\frac{1}{2}\gamma^2\sigma^2\left(\Delta\log c\left(s^{t+1}\right)\right)$$
$$=10*2.09-0.5*100*(0.011)^2=20.4\%$$

- Hence, even with $r_t^f\left(s^t\right)=4\%$, we will need a $\delta=-16.4\%$: a $\beta\ggg 1$!

The Risk-Free Rate Puzzle II

- In fact, the risk-free rate puzzle is a problem by itself. Remember that rate of return on bills is 0.9%.

- $\Delta \log c\left(s^{t+1}\right) = 0.0209$, $\sigma^2\left(\Delta \log c\left(s^{t+1}\right)\right) = (0.011)^2$ and $\gamma = 1$:

$$0.009 = \delta + 0.0209 - \frac{1}{2}(0.011)^2$$

- This implies

$$\delta = 0.009 - 0.0209 + \frac{1}{2}(0.011)^2 = -0.0118$$

again, a $\beta > 1$!

Answers to Equity Premium Puzzle

1. Returns from the market have been odd. If return from bills had been around 4% and returns from equity 5%, you would only need a γ of 6.25. Some evidence related with the impact of inflation (this also helps with the risk-free rate puzzle).

2. There were important distortions on the market. For example regulations and taxes.

3. Habit persistence.

4. Separating EIS from risk-aversion: Epstein-Zin preferences.

5. The model is deeply wrong: behavioral.

Habit Persistence

- Assume that the utility function takes the form:

$$\frac{(c_t - hc_{t-1})^{1-\gamma} - 1}{1-\gamma}$$

- Interpretation. If $h = 0$ we have our CRRA function back.

- External versus internal habit persistence.

Why Does Habit Help? I

- Suppose $c_{t+1}\left(s^{t+1}\right)=1.01$, $c_t\left(s^t\right)=c_{t-1}\left(s^{t-1}\right)=1$, and $\gamma=2$:

$$\frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)}=\frac{\left(1.01-h\right)^{-2}}{\left(1-h\right)^{-2}}$$

- If $h=0$

$$\frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)}=\frac{\left(1.01\right)^{-2}}{\left(1\right)^{-2}}=0.9803$$

- If $h=0.95$

$$\frac{u'\left(c_{t+1}\left(s^{t+1}\right)\right)}{u'\left(c_t\left(s^t\right)\right)}=\frac{\left(1.01-0.95\right)^{-2}}{\left(0.05\right)^{-2}}=0.6944$$

Why Does Habit Help? II

- In addition, there is an indirect effect, since we can raise γ without generating a risk-free rate puzzle.

- We will have:

$$
\begin{aligned}
R_t^f\left(s^t\right) &= \frac{1}{\mathbb{E}_t m_t\left(s^t, s_{t+1}\right)} = \frac{1}{\beta \mathbb{E}_t\left(\frac{c(s^{t+1})-hc(s^t)}{c(s^t)-hc(s^{t-1})}\right)^{-\gamma}} \\
&= \frac{1}{\beta \mathbb{E}_t\left(e^{-\gamma \Delta \log(c(s^{t+1})-hc(s^t))}\right)}
\end{aligned}
$$

Why Does Habit Help? II

- Now:

$$r_t^f\left(s^t\right) = \delta + \gamma \mathbb{E}_t \Delta \log\left(c\left(s^{t+1}\right) - hc\left(s^t\right)\right) - \frac{1}{2}\gamma^2\sigma^2\left(\Delta \log\left(c\left(s^{t+1}\right) - hc\left(s^t\right)\right)\right)$$

- Note that for h close to 1

$$\mathbb{E}_t \Delta \log\left(c\left(s^{t+1}\right) - hc\left(s^t\right)\right) \approx \mathbb{E}_t \Delta \log\left(c\left(s^{t+1}\right)\right)$$

- So we basically get a higher variance term, with a negative sign.
- Hence, we can increase the γ that will let us have a reasonable risk-free interest rate.

Lessons from the Equity Premium Puzzle

We want to build DSGE models where the market price of risk is:

1. High.
2. Time-varying.
3. Correlated with the state of the economy.

We need to somehow fit together a low risk-free interest rate and a high return on risky assets.

Main Ideas of Asset Pricing

1. Non-arbitrage.
2. Risk-free rate is $r \simeq \delta + \gamma g +$precautionary behavior.
3. Risk is not important by itself: the key is covariance.
4. Mean-Variance frontier.
5. Equity Premium Puzzle.
6. Random walk of asset prices.