Topics in OLG models

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Population growth

- Consider the simple model without money ($m = 0$).
- Population grows at constant rate n, so that for each old person in a given period there are $(1 + n)$ young people around.
- Equilibrium conditions are the same, except resource feasibility:

$$c_t^{t-1} + (1 + n)c_t^t = e_t^{t-1} + (1 + n)e_t^t$$

  or, in terms of excess demands:

$$z(p_{t-1}, p_t) + (1 + n)y(p_t, p_{t+1}) = 0$$

- Hence, in our offer curve diagram, the slope of the resource line is not -1 anymore, but $-(1 + n)$.
- Without any government intervention, the unique equilibrium is the autarkic equilibrium.

Social Security

Social Security

- Is a pay-as-you-go social security system welfare-improving?
- We assume stationary endowments $e_t^t = w_1$ and $e_{t+1}^t = w_2$ for all t.
- The social security system: the young pay social security taxes of $\tau \in [0, w_1)$ and receive social security benefits b when old.
- We assume that the social security system balances its budget in each period, so that benefits are given by

$$b = \tau(1+n)$$

- The new unique competitive equilibrium is again autarkic with endowments $(w_1 - \tau, w_2 + \tau(1+n))$ and interest rates:

$$1 + r_{t+1} = 1 + r = \frac{U'(w_1 - \tau)}{\beta U'(w_2 + \tau(1+n))}$$

Welfare analysis, I

- For any $\tau > 0$, the initial old generation receives a windfall transfer of $\tau(1+n) > 0$ and hence it is unambiguously better off.
- For all other generations, the equilibrium lifetime utility as a function of the social security system is

$$V(\tau) = U(w_1 - \tau) + \beta U(w_2 + \tau(1+n))$$

- The introduction of a small social security system is welfare improving if and only if $V'(\tau)$, evaluated at $\tau = 0$, is positive.
- Since $V'(0) = -U'(w_1) + \beta U'(w_2)(1+n)$, $V'(0) > 0$ if and only if:

$$n > \frac{U'(w_1)}{\beta U'(w_2)} - 1 = \bar{r}$$

where $\bar{r}$ is the autarkic interest rate.

Welfare analysis, II

- The introduction of a (marginal) pay-as-you-go social security system is welfare improving if and only if the population growth rate exceeds the equilibrium (autarkic) interest rate.
- Social security has the same function as money in our economy: it is a social institution that transfers resources between generations (backward in time) that do not trade among each other in equilibrium.
- Pareto improvement because the private marginal rate of substitution $1 + \bar{r}$ (at the autarkic allocation) falls short of the social intertemporal rate of transformation $1 + n$.

Optimal size of Social Security

- The optimal size of social security τ^* is such that the resulting autarkic equilibrium interest rate is at least equal to the population growth rate, or

$$1 + n \leq \frac{U'(w_1 - \tau^*)}{\beta U'(w_2 + \tau^*(1+n))}$$

- Note, however, that any $\tau > \tau^*$ satisfying $\tau \leq w_1$ generates a Pareto optimal allocation, too: the representative generation would be better off with a smaller system, but the initial old generation would be worse off.

- Current system: reform and political economy.

Ricardian equivalence

Ricardian equivalence

- How should the government finance a given stream of government expenditures, say, for a war?
- Two ways:
  1. Tax current generations (as a tax or as seigniorage).
  2. Issue government debt.
- Which are the consequences of each option?
- Ricardian equivalence: it makes no difference.
- We can call it the Modigliani-Miller theorem of public finance.

Infinite lifetime horizon and borrowing constraints, I

- Ricardian equivalence is most easily demonstrated within the Arrow-Debreu market structure of infinite horizon models.

- Consider the a infinite horizon pure exchange model and introduce a government that has to finance a given exogenous stream of government expenditures (in real terms) denoted by $\{g_t\}_{t=1}^{\infty}$.

- Government expenditures do not yield any utility to the agents or they enter in a separable manner in the utility function (this assumption is not at all restrictive).

- Let p_t denote the Arrow-Debreu price at date 0 of one unit of the consumption good delivered at period t.

Infinite lifetime horizon and borrowing constraints, II

- The government has initial outstanding real debt of B_1 that is held by the public.
- Let b_1^i denote the initial endowment of government bonds of agent i.
- Then

$$\sum_{i \in I} b_1^i = B_1$$

- To finance its expenditures, the government levies lump-sum taxes: let τ_t^i denote the taxes that agent i pays in period t, denoted in terms of the period t consumption good.

Arrow-Debreu Equilibrium

Given a sequence of government spending $\{g_t\}_{t=1}^{\infty}$ and initial government debt B_1 and $(b_1^i)_{i\in I}$ an Arrow-Debreu equilibrium are allocations $\{(\hat{c}_t^i)_{i\in I}\}_{t=1}^{\infty}$, prices $\{\hat{p}_t\}_{t=1}^{\infty}$ and taxes $\{(\tau_t^i)_{i\in I}\}_{t=1}^{\infty}$ such that

1. Given prices $\{\hat{p}_t\}_{t=1}^{\infty}$ and taxes $\{(\tau_t^i)_{i\in I}\}_{t=1}^{\infty}$ for all $i \in I$, $\{\hat{c}_t^i\}_{t=1}^{\infty}$ solves:

$$\max_{\{c_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} U(c_t^i)$$

$$\text{s.t.} \quad \sum_{t=1}^{\infty} \hat{p}_t (c_t + \tau_t^i) \leq \sum_{t=1}^{\infty} \hat{p}_t e_t^i + \hat{p}_1 b_1^i$$

2. Given prices $\{\hat{p}_t\}_{t=1}^{\infty}$ the tax policy satisfies:

$$\sum_{t=1}^{\infty} \hat{p}_t g_t + \hat{p}_1 B_1 = \sum_{t=1}^{\infty} \sum_{i\in I} \hat{p}_t \tau_t^i$$

3. For all $t \geq 1$:

$$\sum_{i\in I} \hat{c}_t^i + g_t = \sum_{i\in I} e_t^i$$

Key intuition

- In an Arrow-Debreu definition of equilibrium, the government, as the agent, faces a single intertemporal budget constraint which states that the total value of tax receipts is sufficient to finance the value of all government purchases plus the initial government debt.
- From the definition it is clear that, with respect to government tax policies, the only thing that matters is the total value of taxes $\sum_{t=1}^{\infty} \hat{p}_t \tau_t^i$ that the individual has to pay, but not the timing of taxes.
- Thus, it is then straightforward to prove the Ricardian equivalence theorem for this economy.

Theorem

Take as given a sequence of government spending $\{g_t\}_{t=1}^{\infty}$ and initial government debt $B_1, (b_1^i)_{i \in I}$. Suppose that allocations $\{(\hat{c}_t^i)_{i \in I}\}_{t=1}^{\infty}$, prices $\{\hat{p}_t\}_{t=1}^{\infty}$ and taxes $\{(\tau_t^i)_{i \in I}\}_{t=1}^{\infty}$ form an Arrow-Debreu equilibrium. Let $\{(\hat{\tau}_t^i)_{i \in I}\}_{t=1}^{\infty}$ be an arbitrary alternative tax system satisfying

$$\sum_{t=1}^{\infty} \hat{p}_t \tau_t^i = \sum_{t=1}^{\infty} \hat{p}_t \hat{\tau}_t^i \text{ for all } i \in I$$

Then $\{(\hat{c}_t^i)_{i \in I}\}_{t=1}^{\infty}$, $\{\hat{p}_t\}_{t=1}^{\infty}$ and $\{(\hat{\tau}_t^i)_{i \in I}\}_{t=1}^{\infty}$ form an Arrow-Debreu equilibrium.

Proof

The budget constraint of individuals does not change, hence the optimal consumption choice at the old equilibrium prices does not change. Obviously resource feasibility is satisfied. The government budget constraint is satisfied by assumption.

Remarks

- The sequence of government expenditures is taken as fixed and exogenously given. Ricardian equivalence does NOT say that the timing of government expenditures are irrelevant, only how we finance them.
- The condition in the theorem rules out redistribution among individuals.
- The new tax system has the same cost to each individual at the old equilibrium prices (but not necessarily at alternative prices).
- We can redo the proof in a sequential markets equilibrium.
- With borrowing constraints or non lump-sum taxes, Ricardian equivalence fails.

Finite horizon and operative bequest motives

- There is only a very limited Ricardian equivalence theorem for OLG economies.
- Any change in the timing of taxes that redistributes among generations is in general not neutral in the Ricardian sense.
- Barro's (1974) "Are Government Bonds Net Wealth?": under certain conditions finitely lived agents will behave as if they had infinite lifetime.
- Key mechanism: current generations are connected to future generations by altruistically motivated transfers.
- These may be transfers from old to young via bequests or from young to old via social security programs.

Environment, I

- Standard pure exchange OLG model with two-period lived agents.
- No population growth.
- Agents have endowment $e_t^t = w$ when young and no endowment when old.
- a_t^t: savings of currently young people for the second period of their lives.
- a_{t+1}^t: savings of the currently old people for the next generation (the old people's bequests).
- We require bequests to be nonnegative, that is, $a_{t+1}^t \geq 0$.
- In our previous OLG models, $a_{t+1}^t = 0$ was the only optimal choice since individuals were completely selfish.
- We will see below how to induce positive bequests when discussing individuals' preferences.

Environment, II

- There is a government that, for simplicity, has 0 government expenditures but initial outstanding government debt B.
- This debt is denominated in terms of the period 1 consumption good.
- The initial old generation is endowed with the B units of bonds.
- Government bonds are zero coupon bonds with maturity of one period.
- Government keeps its outstanding government debt constant and we assume a constant one-period real interest rate r on these bonds.
- In order to finance the interest payments on government debt the government taxes the currently young people.

Budget constraints and market clearing

- The government budget constraint:

$$\frac{B}{1+r} + \tau = B$$

- The representative generation budget constraints:

$$\begin{aligned} c_t^t + \frac{a_t^t}{1+r} &= w - \tau \\ c_{t+1}^t + \frac{a_{t+1}^t}{1+r} &= a_t^t + a_t^{t-1} \end{aligned}$$

- We can consolidate the two budget constraints to obtain:

$$c_t^t + \frac{c_{t+1}^t}{1+r} + \frac{a_{t+1}^t}{(1+r)^2} = w + \frac{a_t^{t-1}}{1+r} - \tau$$

Budget constraints and market clearing

- The budget constraint of the initial old generation is given by:

$$c_1^0 + \frac{a_1^0}{1+r} = B$$

- The equilibrium conditions for the goods and the asset market are, respectively:

$$\begin{aligned} c_t^{t-1} + c_t^t &= w \text{ for all } t \geq 1 \\ a_t^{t-1} + a_t^t &= B \text{ for all } t \geq 1 \end{aligned}$$

Preferences, I

- Individuals are altruistic and care about the well-being of their descendant.
- The agent cares only about her immediate descendant, but (possibly) not at all about grandchildren.
- No strategic bequest to induce actions of the children that yield utility to the parents.
- This strategic bequest motive does not necessarily help to reestablish Ricardian equivalence, as Bernheim, Shleifer and Summers (1985) show.

Preferences, II

- Preferences of generation t are represented by:

$$u_t(c_t^t, c_{t+1}^t, a_{t+1}^t) = U(c_t^t) + \beta U(c_{t+1}^t) + \alpha V_{t+1}(e_{t+1})$$

  where $V_{t+1}(e_{t+1})$ is the maximal utility generation $t+1$ can attain with lifetime resources $e_{t+1} = w + \frac{a_{t+1}^t}{1+r} - \tau$, which are a function of bequests $a_{t+1}^t$from generation t.

- We make no assumption about the size of α as compared to β, but assume $\alpha \in (0,1)$.

- The initial old generation has preferences represented by

$$u_0(c_1^0, a_1^0) = \beta U(c_1^0) + \alpha V_1(e_1)$$

Optimization problem initial generation

$$V_0(B) = \max_{c_1^0, a_1^0 \geq 0} \left\{ \beta U(c_1^0) + \alpha V_1(e_1) \right\}$$

$$\text{s.t. } c_1^0 + \frac{a_1^0}{1+r} = B$$

$$e_1 = w + \frac{a_1^0}{1+r} - \tau$$

- The two constraints can be consolidated to

$$c_1^0 + e_1 = w + B - \tau$$

- This yields optimal decision rules $c_1^0(B)$ and $a_1^0(B)$ (or $e_1(B)$).
- From now on we assume $a_1^0(B) > 0$.

Experiment

- Consider the experiment: increase initial government debt marginally by ΔB and repay this additional debt by levying higher taxes on the first young generation.
- To repay the ΔB, taxes for the young have to increase by:

$$\Delta \tau = \Delta B$$

- The optimal choices for c_1^0 and e_1 do not change.
- The initial old generation receives additional transfers of bonds of magnitude ΔB from the government and increases its bequests a_1^0 by $(1+r)\Delta B$ so that lifetime resources available to their descendants (and hence their allocation) is left unchanged.
- Altruistically motivated bequest motives just undo the change in fiscal policy. Ricardian equivalence is restored.

General problem

$$V_0(B) = \max_{c_1^0, a_1^0 \geq 0} \left\{ \beta U(c_1^0) + \alpha V_1(a_1^0) \right\}$$

$$\text{s.t. } c_1^0 + \frac{a_1^0}{1+r} = B$$

$$= \max_{c_1^0, a_1^0 \geq 0} \left\{ \beta U(c_1^0) + \alpha \left\{ \begin{array}{c} \max\limits_{c_1^1, c_2^1, a_2^1 \geq 0, a_1^1} \left\{ U(c_1^1) + \beta U(c_2^1) + \alpha V_2(a_2^1) \right\} \\ \text{s.t. } c_1^1 + \frac{a_1^1}{1+r} = w - \tau \\ c_2^1 + \frac{a_2^1}{1+r} = a_1^1 + a_1^0 \end{array} \right\} \right\}$$

$$\text{s.t. } c_1^0 + \frac{a_1^0}{1+r} = B$$

Rewriting the maximization problem

$$\max_{c_1^0, a_1^0, c_1^1, c_2^1, a_2^1 \geq 0, a_1^1} \left\{ \beta U(c_1^0) + \alpha U(c_1^1) + \alpha\beta U(c_2^1) + \alpha^2 V_2(a_2^1) \right\}$$

$$\text{s.t. } c_1^0 + \frac{a_1^0}{1+r} = B$$

$$c_1^1 + \frac{a_1^1}{1+r} = w - \tau$$

$$c_2^1 + \frac{a_2^1}{1+r} = a_1^1 + a_1^0$$

Iterating

$$\max_{\{(c_t^{t-1}, c_t^t, a_t^{t-1})\}_{t=1}^{\infty} \geq 0} \left\{ \beta U(c_1^0) + \sum_{t=1}^{\infty} \alpha^t \left(U(c_t^t) + \beta U(c_{t+1}^t) \right) \right\}$$

$$s.t.\ c_1^0 + \frac{a_1^0}{1+r} = B$$

$$c_t^t + \frac{c_{t+1}^t}{1+r} + \frac{a_{t+1}^t}{(1+r)^2} = w - \tau + \frac{a_t^{t-1}}{1+r}$$

- Consumer problem of an infinitely lived agent.

Irrelevance theorems

Irrelevance theorems

- Fiscal policy determines the path of future debt of the government.
- The government will finance such path with a portfolio of currency and real assets.
- In many environments, the particular structure of the government portfolio is irrelevant.
- Natural application of Modigliani-Miller's insight to public finance. Ricardian equivalence would be just a particular case.
- First shown by Wallace (1981).
- Extended by Chamley and Polemarchakis (1984) and Peled (1985).

Environment

- Time is discrete, $t = 1, 2, 3, \dots$ and the economy (but not its households) lasts forever.
- In each period, a i.i.d. random variable x_t is realized with value $x_i \geq 0$ for $i = 1, ..., I$ and

$$p\left(x_t = x_i\right) = \pi_i$$

$$\sum_{i=1}^{I} \pi_i = 1$$

  Furthermore, x_t is observed at the start of the period, before any t, decision has been undertaken.

- In each period there is a single consumption good and a storage technology.
- A unit of good stored in time in period t delivers x_{t+1} in period $t+1$.
- Notation: we will use the subindex i in a random variable only when we want to be specific about its realization.

Endowments and consumption

- In each time period a new generation (of measure 1) is born, which we index by its date of birth.
- Households live for two periods and then die.
- $(e_t^t, e_{i,t+1}^t)$: generation t's after tax endowment of the consumption good in the first and second period of their live.
- $(c_t^t, c_{i,t+1}^t)$: consumption allocation of generation t.
- Utility function:

$$\sum_{i=1}^{I} \pi_i u\left(c_t^t, c_{i,t+1}^t\right)$$

where $u(\cdot, \cdot)$ satisfies standard assumptions.

Problem of the household, I

- Households can transfer goods across time and states of nature using three mechanisms:

    1. Trade $b_{i,t}$ Arrow securities at period t at price $q_{i,t}$ that pay 1 unit of good at period $t+1$ and state i. Also, denote $q_t = (q_{i,t}, ..., q_{I,t})$ and note that $b_{i,t}$ must be in zero net supply.
    2. Store k_t units of good at period t and get $x_{t+1} k_t$ at period $t+1$.
    3. Buy m_t units of currency at price p_t (the reciprocal of the price level).

- Thus, the budget constraints are:

$$c_t^t + k_t + \sum_{i=1}^{I} q_{i,t} b_{i,t} + p_t m_t = e_t^t$$

$$c_{i,t+1}^t = e_{i,t+1}^t + x_i k_t + p_{i,t+1} m_t + b_{i,t} \text{ for all } i$$

Problem of the household, II

- We can sum all constraints after multiplying the $t+1$ ones by $q_{i,t}$:

$$c_t^t + \sum_{i=1}^{I} q_{i,t} c_{i,t+1}^t =$$

$$e_t^t + \sum_{i=1}^{I} q_{i,t} e_{i,t+1}^t + \left(\sum_{i=1}^{I} q_{i,t} x_i - 1 \right) k_t + \left(\sum_{i=1}^{I} q_{i,t} p_{i,t+1} - p_t \right) m_t$$

- By lack of arbitrage

$$\begin{aligned} \sum_{i=1}^{I} q_{i,t} x_i - 1 &\leq 0 \text{ with equality if } k_t > 0 \\ \sum_{i=1}^{I} q_{i,t} p_{i,t+1} - p_t &\leq 0 \text{ with equality if } m_t > 0, \end{aligned}$$

which implies that the last two terms of the budget constraint disappear.

Problem of the household, III

- Thus, problem of household is:

$$\sum_{i=1}^{I} \pi_i u\left(c_t^t, c_{i,t+1}^t\right)$$

$$\text{s.t. } c_t^t + \sum_{i=1}^{I} q_{i,t} c_{i,t+1}^t = e_t^t + \sum_{i=1}^{I} q_{i,t} e_{i,t+1}^t$$

- Necessary conditions:

$$q_{i,t} \sum_{i=1}^{I} \pi_i u_1\left(c_t^t, c_{i,t+1}^t\right) = \pi_i u_2\left(c_t^t, c_{i,t+1}^t\right) \text{ for all } i$$

Government

- Budget constraint given some storage k_t^g, some government consumption $g_{i,t}$, and some money creation $m_t - m_{t-1}$:

$$k_t^g + g_{i,t} = tax_{i,t} + x_i k_{t-1}^g + p_i \left(m_t - m_{t-1}\right)$$

where the net lump-sum taxes satisfy

$$tax_{i,t} = y_t - e_t^t - e_t^{t-1}$$

for an aggregate endowment y_t.

- Initial conditions k_0^g, e_0^1, and m_0 are given.

Equilibrium conditions, I

- From the households:

$$q_{i,t} \sum_{i=1}^{I} \pi_i u_1 \left(c_t^t, c_{i,t+1}^t\right) = \pi_i u_2 \left(c_t^t, c_{i,t+1}^t\right) \text{ for all } i \tag{1}$$

$$c_t^t + \sum_{i=1}^{I} q_{i,t} c_{i,t+1}^t = e_t^t + \sum_{i=1}^{I} q_{i,t} e_{i,t+1}^t \tag{2}$$

$$\sum_{i=1}^{I} q_{i,t} x_i - 1 \leq 0 \text{ with equality if } k_t > 0 \tag{3}$$

$$\sum_{i=1}^{I} q_{i,t} p_{i,t+1} - p_t \leq 0 \text{ with equality if } m_t > 0 \tag{4}$$

Equilibrium conditions, II

- From the government:

$$k_t^g + g_{i,t} = tax_{i,t} + x_i k_{t-1}^g + p_i \left(m_t - m_{t-1} \right) \tag{5}$$

$$tax_{i,t} = y_t - e_t^t - e_t^{t-1} \tag{6}$$

- From market clearing:

$$c_{i,t+1}^t = e_{i,t+1}^t + x_i k_t + p_{i,t+1} m_t \text{ for all } i \tag{7}$$

- Initial conditions:

$$k_0^g, e_0^1, m_0$$

Equilibrium determination

- Notice that in equilibrium we have 12 sequences of random variables, $\left\{c_t^t, c_{t+1}^t, e_t^t, e_{t+1}^t, k_t, k_t^g, y_t, m_t, g_t, tax_t, q_t, p_t\right\}_{t=1}^{\infty}$, but only 7 sequences of equilibrium conditions.
- Therefore, there is a variety of partition between endogenous and exogenous variables that can be entertained.
- For example, we can take $\left\{e_t^t, e_{t+1}^t, k_t^g, y_t, g_t\right\}_{t=1}^{\infty}$ as given and use the equilibrium conditions to solve for $\left\{c_t^t, c_{t+1}^t, k_t, m_t, tax_t, q_t, p_t\right\}_{t=1}^{\infty}$.
- In this case, we use (5) and (6) and solve directly for:

$$tax_{i,t} = y_t - e_t^t - e_t^{t-1}$$

$$m_t = m_{t-1} + \frac{1}{p_i}\left(k_t^g + g_{i,t} - tax_{i,t} - x_i k_{t-1}^g\right)$$

so we can focus on the subset $\left\{c_t^t, c_{t+1}^t, k_t, p_t, q_t\right\}_{t=1}^{\infty}$.

The theorems

Preliminaries

- Define a vector that stacks the initial conditions and an exogenous process for aggregate endowment, $\Omega = \left[k_0^g, e_0^1, m_0, \{\overline{y}_t\}_{t=1}^{\infty}\right]$.
- Also, let us limit our investigation to equilibria where (3) and (4) hold with equality.
- We will write a theorem where, given a government policy $\left\{\overline{e}_t^t, \overline{e}_{t+1}^t, \overline{k}_t^g, \overline{m}_t\right\}_{t=1}^{\infty}$ that implies an equilibrium, we will create an alternative policy that satisfies certain conditions and that brings an equilibrium with the same allocation and asset prices, perhaps with a different price path.
- Our result can be easily generalized to less restrictive environments as long as we ensure that households will voluntarily hold the different assets in the economy in equilibrium.

Theorem

Take Ω and a policy $\left\{\overline{e}^t_t, \overline{e}^t_{t+1}, \overline{k}^g_t, \overline{m}_t\right\}_{t=1}^{\infty}$ as given and let the equilibrium they imply be $\left\{\overline{c}^t_t, \overline{c}^t_{t+1}, \overline{k}_t, \overline{g}_t, \overline{tax}_t, \overline{q}_t, \overline{p}_t\right\}_{t=1}^{\infty}$. Then, the alternative policy $\left\{\widehat{e}^t_t, \widehat{e}^t_{t+1}, \widehat{k}^g_t, \widehat{m}_t\right\}_{t=1}^{\infty}$ such that for all t and i:

$$\widehat{e}^t_t + \sum_{i=1}^{I} \overline{q}_{i,t}\widehat{e}^t_{i,t+1} = \overline{e}^t_t + \sum_{i=1}^{I} \overline{q}_{i,t}\overline{e}^t_{i,t+1}$$

$$\widehat{p}_t\widehat{m}_t - \overline{p}_t\overline{m}_t = \widehat{k}^g_t - \overline{k}^g_t + \widehat{e}^t_t - \overline{e}^t_t$$

$$\widehat{e}^t_{i,t+1} - \overline{e}^t_{i,t+1} =$$

$$\left(\widehat{k}^g_t - \overline{k}^g_t\right)\left(x_i - \frac{\widehat{p}_{i,t+1}}{\widehat{p}_t}\right) + \left(\frac{\overline{p}_{i,t+1}}{\overline{p}_t} - \frac{\widehat{p}_{i,t+1}}{\widehat{p}_t}\right)\overline{p}_t\overline{m}_t - \frac{\widehat{p}_{i,t+1}}{\widehat{p}_t}\left(\widehat{e}^t_t - \overline{e}^t_t\right)$$

$$0 \leq \widehat{k}^g_t \leq \overline{k}^g_t + \overline{k}_t$$

implies the same allocation and asset prices $\left\{\overline{c}^t_t, \overline{c}^t_{t+1}, \overline{k}_t, \overline{m}_t, \overline{tax}_t, \overline{q}_t\right\}_{t=1}^{\infty}$ and a possibly different price path $\{\widehat{p}_t\}_{t=1}^{\infty}$ where $\widehat{p}_1 = \overline{p}_1$.

Comments

- Notice the key role of complete markets in allowing the government to move up and down revenue from different periods.

- Take the first two conditions

$$\widehat{p}_t \widehat{m}_t - \overline{p}_t \overline{m}_t = \widehat{k}_t^g - \overline{k}_t^g - \sum_{i=1}^{I} \overline{q}_{i,t} \left(\widehat{e}_{i,t+1}^t - \overline{e}_{i,t+1}^t \right)$$

  This equation states a "real bills" property: the difference in the value of base money depends on the differences on backing $(\widehat{k}_t^g - \overline{k}_t^g)$ and on discounted future taxes.

- Introducing indexed-bonds does not change the main result (Peled, 1985).

Three versions of the theorem

Wallace's Modigliani-Miller theorem for open-market operations

If in addition to the conditions in the basic theorem, we also fix $\widehat{e}_t^t = \overline{e}_t^t$, we have that $\widehat{p}_t = \overline{p}_t$ for all t.

Chamley-Polemarchakis' neutrality theorem

If in addition to the conditions in the basic theorem, we also fix $\widehat{e}_t^t = \overline{e}_t^t$ and $\widehat{e}_{i,t+1}^t = \overline{e}_{i,t+1}^t$, we have that $\widehat{p}_t \neq \overline{p}_t$ for all t.

Ricardian equivalence theorem

If in addition to the conditions in the basic theorem, we also fix $\widehat{k}_t^g = \overline{k}_t^g$ and $\widehat{p}_t = \overline{p}_t$, we have that $\widehat{e}_t^t \neq \overline{e}_t^t$ and $\widehat{e}_{i,t+1}^t \neq \overline{e}_{i,t+1}^t$ for all t.

A simpler case of Wallace's Theorem, I

- Simple case $I = 1$ (so we can drop the subindex).
- Also let us look at equilibria where money and storage happen.
- This requires that $x < 1$.
- Then, equilibrium conditions of household are:

$$q_t = \frac{1}{x}$$

$$\frac{p_{t+1}}{p_t} = \frac{1}{x}$$

$$\frac{u_2\left(c_t^t, c_{t+1}^t\right)}{u_1\left(c_t^t, c_{t+1}^t\right)} = q_t$$

$$c_t^t + q_t c_{t+1}^t = e_t^t + q_t e_{t+1}^t$$

A simpler case of Wallace's Theorem, II

- Now, with Wallace's extra condition of $\widehat{e}_t^t = \overline{e}_t^t$, we can find that, the first condition on alternative fiscal policies is that $\widehat{e}_{t+1}^t = \overline{e}_{t+1}^t$ and thus

$$\widehat{p}_t \widehat{m}_t - \overline{p}_t \overline{m}_t = \widehat{k}_t^g - \overline{k}_t^g$$
$$\left(\frac{\overline{p}_{i,t+1}}{\overline{p}_t} - \frac{\widehat{p}_{i,t+1}}{\widehat{p}_t} \right) \overline{p}_t \overline{m}_t = 0$$
$$0 \leq \widehat{k}_t^g \leq \overline{k}_t^g + \overline{k}_t$$

  which clearly shows that prices must remain unchanged.

- One example: $\overline{k}_t^g = 0$ for all t. Pick a new $\widehat{k}_t^g > 0$ and issue currency to pay for it:

$$\overline{p}_t \left(\widehat{m}_t - \overline{m}_t \right) = \widehat{k}_t^g$$

A simpler case of Wallace's Theorem, III

- Intuition:

  1. Government issues more money to buy more of the good to store.
  2. The young households get the new currency (since the rates of return of storage and currency are both exogenously equal to x they just re-balance their portfolios) and store less of the good.
  3. In the next period, the government transfers to the households carrying the extra currency the return of storage.
  4. The extra income from the transfer is exactly equal to the lower income from the lower private storage.
  5. Hence, no component of the allocation or of prices changes.