# Measure Theory

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## Why Bother with Measure Theory?

- Kolmogorov (1933).

- Foundation of modern probability.

- Deals easily with:

  1. Continuous versus discrete probabilities. Also mixed probabilities.
  2. Univariate versus multivariate.
  3. Independence.
  4. Convergence.

## Introduction to Measure Theory

- Measure theory is an important field for economists.

- We cannot do in a lecture what it will take us (at least) a whole semester.

- Three sources:

  1. Read chapters 7 and 8 in SLP.

  2. Excellent reference: *A User's Guide to Measure Theoretic Probability*, by David Pollard.

  3. Take math classes!!!!!!!!!!!!!

## $\sigma-$**Algebra**

- Let $S$ be a set and let $\mathcal{S}$ be a family of subsets of $S$. $\mathcal{S}$ is a $\sigma-$algebra if

  1. $\emptyset, S \in \mathcal{S}$.

  2. $A \in \mathcal{S} \Rightarrow A^c = S \backslash A \in \mathcal{S}$.

  3. $A_n \in \mathcal{S}$, $n = 1, 2, ..., \Rightarrow \cup_{n=1}^{\infty} A_n \in \mathcal{S}$.

- $(S, \mathcal{S})$: measurable space.

- $A \in \mathcal{S}$: measurable set.

## Borel Algebra

- Define a collection $\mathcal{A}$ of subsets of $S$.

- $\sigma-$algebra generated by $\mathcal{A}$: the intersection of all $\sigma-$algebra containing $\mathcal{A}$ is a $\sigma-$algebra.

- $\sigma-$algebra generated by $\mathcal{A}$ is the smallest $\sigma-$algebra containing $\mathcal{A}$.

- Example: let $\mathcal{B}$ be the collection of all open balls (or rectangles) of $\mathbb{R}^l$ (or a restriction of).

- Borel algebra: the $\sigma-$algebra generated by $\mathcal{B}$.

- Borel set: any set in $\mathcal{B}$.

## Measures

- Let $(S, \mathcal{S})$ be a measurable space.

- Measure: an extended real-valued function $\mu : \mathcal{S} \rightarrow \mathbb{R}_{\infty}$ such that:

  1. $\mu(\emptyset) = 0$.

  2. $\mu(A) \geq 0$, $\forall A \in \mathcal{S}$.

  3. If $\{A_n\}_{n=1}^{\infty}$ is a countable, disjoint sequence of subsets in $\mathcal{S}$, then $\mu\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$.

- If $\mu(S) < \infty$, then $\mu$ is finite.

- $(S, \mathcal{S}, \mu)$: measurable space.

## Probability Measures

- Probability measure: $\mu$ such that $\mu(S) = 1$.

- Probability space: $(S, \mathcal{S}, \mu)$ where $\mu$ is a probability measure.

- Event: each $A \in \mathcal{S}$.

- Probability of an event: $\mu(A)$.

- $B(S, \mathcal{S})$: space all probability measures on $(S, \mathcal{S})$.

## Almost Everywhere

- Given $(S, \mathcal{S}, \mu)$, a proposition holds almost $\mu-$everywhere ($\mu-$a.e.), if $\exists$ a set $A \in \mathcal{S}$ with $\mu(A) = 0$, such that the proposition holds on $A^c$.

- If $\mu$ is a probability measure, we often use the phrase almost surely (a.s.) instead of almost everywhere.

## Completion

- Let $(S, \mathcal{S}, \mu)$ be a measure space.

- Define the family of subsets of any set with measure zero:

$$\mathcal{C} = \{C \subset S : C \subseteq A \text{ for some } A \in \mathcal{S} \text{ with } \mu(A) = 0\}$$

- Completion of $\mathcal{S}$ is the family $\mathcal{S}'$:

$$\mathcal{S}' = \left\{B' \subseteq S : B' = (B \cup C_1) \backslash C_2,\ B \in \mathcal{S},\ C_1, C_2 \in \mathcal{C}\right\}$$

- $\mathcal{S}'(\mu)$: completion of $\mathcal{S}$ with respect to measure $\mu$.

## Universal $\sigma-$Algebra

- $\mathcal{U} = \cap_{\mu \in B(S,\mathcal{S})} \mathcal{S}'(\mu)$.

- Note:

  1. $\mathcal{U}$ is a $\sigma-$algebra.

  2. $\mathcal{B} \subset \mathcal{U}$.

- Universally measurable space is a measurable space with its universal $\sigma-$algebra.

- Universal $\sigma-$algebras avoid a problem of Borel $\sigma-$algebras: projection of Borel sets are not necessarily measurable with respect to $\mathcal{B}$.

## Measurable Function

- Measurable function into $\mathbb{R}$: given a measurable space $(S, \mathcal{S})$, a real-valued function $f : S \rightarrow \mathbb{R}$ is measurable with respect to $\mathcal{S}$ (or $\mathcal{S}-$measurable) if

$$\{s \in \mathcal{S} : f(s) \leq a\} \in \mathcal{S}, \ \forall \ a \in \mathbb{R}$$

- Measurable function into a measurable space: given two measurable spaces $(S, \mathcal{S})$ and $(T, \mathcal{T})$, the function $f : S \rightarrow T$ is measurable if:

$$\{s \in \mathcal{S} : f(s) \in A\} \in \mathcal{S}, \ \forall \ A \in \mathcal{T}$$

- If we set $(T, \mathcal{T}) = (\mathbb{R}, \mathcal{B})$, the second definition nests the first.

- Random variable: a measurable function in a probability space.

## Measurable Selection

- Measurable selection: given two measurable spaces $(S, \mathcal{S})$ and $(T, \mathcal{T})$ and a correspondence Γ of $S$ into $T$, the function $h : S \to T$ is a measurable selection from Γ is $h$ is measurable and:

$$h(s) \in \Gamma(s), \forall\ s \in \mathcal{S}$$

- Measurable Selection Theorem: Let $S \subseteq \mathbb{R}^l$ and $T \subseteq \mathbb{R}^m$ and $\mathcal{S}$ and $\mathcal{T}$ be their universal $\sigma-$algebras. Let Γ:$S \to T$ be a (nonempty) compact-valued and u.h.c. correspondence. Then, $\exists$ a measurable selection from Γ.

## Measurable Simple Functions

- $M(S, \mathcal{S})$: space of measurable, extended real-valued functions on $S$.

- $M^+(S, \mathcal{S})$: subset of nonnegative functions.

- Measurable simple function:

$$\phi(s) = \sum_{i=1}^{n} a_i \chi_{A_i}(s)$$

- Importance: for any measurable function $f$, $\exists \{\phi_n\}$ such that $\phi_n(s) \to f$ pointwise.

## Integrals

- Integral of $\phi$ with respect to $\mu$:

$$\int_S \phi(s)\,\mu(ds) = \sum_{i=1}^{n} a_i \mu(A_i)$$

- Integral of $f \in M^+(S, \mathcal{S})$ with respect to $\mu$:

$$\int_S f(s)\,\mu(ds) = \sup_{\phi(s) \in M^+(S,\mathcal{S})} \int_S \phi(s)\,\mu(ds)$$

  such that $0 \leq \phi \leq f$.

- Integral of $f \in M^+(S, \mathcal{S})$ over $A$ with respect to $\mu$:

$$\int_A f(s)\,\mu(ds) = \int_S f(s)\,\chi_A(s)\,\mu(ds)$$

## Positive and Negative Parts

- We define the previous results with positive functions.

- How do we extend to the general case?

- $f^+$: positive part of a function

$$f^+(s) = \begin{cases} f(s) \text{ if } f(s) \geq 0 \\ 0 \text{ if } f(s) < 0 \end{cases}$$

- $f^-$: negative part of a function

$$f^-(s) = \begin{cases} -f(s) \text{ if } f(s) \leq 0 \\ 0 \text{ if } f(s) > 0 \end{cases}$$

## Integrability

- Let $(S, \mathcal{S}, \mu)$ be a measure space and let $f$ be measurable, real-valued function on $S$. If $f^+$ and $f^-$ both have finite integrals with respect to $\mu$, then $f$ is integrable and the integral is given by

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

- If $A \in \mathcal{S}$, the integral of $f$ over $A$ with respect to $\mu$:

$$\int_A f d\mu = \int_A f^+ d\mu - \int_A f^- d\mu$$

## Transition Functions

- Transition function: given a measurable space $(Z, \mathcal{Z})$, a function $Q : Z \times \mathcal{Z} \longrightarrow [0,1]$ such that:

  1. For $\forall\ z \in Z$, $Q(z, \cdot)$ is a probability measure on $(Z, \mathcal{Z})$.

  2. For $\forall\ A \in \mathcal{Z}$, $Q(\cdot, A)$ is $\mathcal{Z}-$measurable.

- $\left(Z^t, \mathcal{Z}^t\right) = (Z \times ... \times Z, \mathcal{Z} \times ... \times \mathcal{Z})$ ($t$ times).

- Then, for any rectangle $B = A_1 \times ... \times A_t \in \mathcal{Z}^t$, define:

$$\mu^t(z_0, B) = \int_{A_1} ... \int_{A_{t-1}} \int_{A_t} Q(z_{t-1}, dz_t)\, Q(z_{t-2}, dz_{t-1}) ... Q(z_0, dz_1)$$

## Two Operators

- For any $\mathcal{Z}-$measurable function $f$, define:

$$(Tf)(z) = \int f\left(z'\right) Q\left(z, dz'\right), \forall s \in \mathcal{S}$$

Interpretation: expected value of $f$ next period.

- For any probability measure $\lambda$ on $(Z, \mathcal{Z})$, define:

$$(T^{*}\lambda)(A) = \int Q(z, A)\, \lambda(dz), \forall A \in \mathcal{Z}$$

Interpretation: probability that the state will be in $A$ next period.

## Basic Properties

- $T$ maps the space of bounded $\mathcal{Z}$-measurable functions, $B(Z, \mathcal{Z})$, into itself.

- $T^*$ maps the space of probability measures on $(Z, \mathcal{Z})$, $\Lambda(Z, \mathcal{Z})$, into itself.

- $T$ and $T^*$ are adjoint operators:

$$\int (Tf)(z)\,\lambda(dz) = \int f\left(z'\right)(T^*\lambda)\left(dz'\right), \ \forall \lambda \in \Lambda(Z, \mathcal{Z})$$

for any function $f \in B(Z, \mathcal{Z})$.

## Two Properties

- A transition function $Q$ on $(Z, \mathcal{Z})$ has the Feller property if the associated operator $T$ maps the space of bounded continuous function on $Z$ into itself.

- A transition function $Q$ on $(Z, \mathcal{Z})$ is monotone if for every nondecreasing function $f$, $Tf$ is also non-decreasing.

## Consequences of our Two Properties

- If $Z \subset \mathbb{R}^l$ is compact and $Q$ has the Feller property, then $\exists$ a probability measure $\lambda^*$ that is invariant under $Q$:

$$\lambda^* = (T^* \lambda^*)(A) = \int Q(z, A) \lambda^*(dz)$$

- Weak convergence: a sequence $\{\lambda_n\}$ converges weakly to $\lambda$ ($\lambda_n \Rightarrow \lambda$) if

$$\lim_{n \to \infty} \int f d\lambda_n = \int f d\lambda, \forall f \in C(S)$$

- If $Q$ is monotone, has the Feller property, and there is enough "mixing" in the distribution, there is a unique invariant probability measure $\lambda^*$, and $T^{*n} \lambda_0 \Rightarrow \lambda^*$ for $\forall \lambda_0 \in \Lambda(Z, \mathcal{Z})$.