Optimization in Continuous Time

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Three Approaches

- We are interested in optimization in continuous time, both in deterministic and stochastic environments.
- Elegant and powerful math (differential equations, stochastic processes...).
- Three approaches:
  1. Calculus of Variations.
  2. Optimal Control.
  3. Dynamic Programming.
- We will focus on the last two:
  1. Optimal control can do everything economists need from calculus of variations.
  2. Dynamic programming is better for the stochastic case.

Maximization Problem I

- Basic setup:

$$
\begin{gathered}
V(0, x(0)) = \max_{x(t), y(t)} \int_0^\infty f(t, x(t), y(t))\, dt \\
\text{s.t. } \dot{x} = g(t, x(t), y(t)) \\
x(0) = x_0, \ \lim_{t \to \infty} b(t) x(t) \geq x_1 \\
x(t) \in \mathcal{X}, y(t) \in \mathcal{Y}
\end{gathered}
$$

- $x(t)$ is a state variable.

- $y(t)$ is a control variable.

Maximization Problem II

- Admissible pair: $(x(t), y(t))$ s.t. the previous conditions are satisfied.

- Optimal pair: $(\widehat{x}(t), \widehat{y}(t))$ that reach $V(0, x(0)) < \infty$.
Then:

$$V(0, x(0)) = \int_0^\infty f(t, \widehat{x}(t), \widehat{y}(t))\, dt$$

- Two difficulties:

1. We need to find a whole function $y(t)$ of optimal choices.
2. The constraint is in the form of a differential equation.

Optimal Control

- Pontryagin and co-authors.

Principle of Optimality

If $\left(\widehat{x}(t), \widehat{y}(t)\right)$ is an optimal pair, then:

$$V\left(t_0, x\left(t_0\right)\right) = \int_{t_0}^{t_1} f\left(t, \widehat{x}(t), \widehat{y}(t)\right) dt + V\left(t_1, \widehat{x}\left(t_1\right)\right)$$

for all $t_1 \geq t_0$.

- We will assume that there is an optimal path.

- Proving existence is, however, not a trivial task.

Hamiltonian

- Define:

$$\mathcal{H}\left(t,x\left(t\right),y\left(t\right),\lambda\left(t\right)\right)=f\left(t,x\left(t\right),y\left(t\right)\right)+\lambda\left(t\right)g\left(t,x\left(t\right),y\left(t\right)\right)$$

where $\lambda\left(t\right)$ is the co-state multiplier.

- Necessary conditions:

$$\mathcal{H}_y\left(t,\widehat{x}\left(t\right),\widehat{y}\left(t\right),\lambda\left(t\right)\right)=0$$
$$\dot{\lambda}\left(t\right)=-\mathcal{H}_x\left(t,\widehat{x}\left(t\right),\widehat{y}\left(t\right),\lambda\left(t\right)\right)$$
$$\dot{x}=\mathcal{H}_\lambda\left(t,\widehat{x}\left(t\right),\widehat{y}\left(t\right),\lambda\left(t\right)\right)$$

plus $x\left(0\right)=x_0$ and $\lim_{t\rightarrow\infty}b\left(t\right)x\left(t\right)\geq x_1$.

Exponential Discounting Case I

- More specific form:

$$V\left(x\left(0\right)\right)=\max_{x(t),y(t)}\int_0^\infty e^{-\rho t}f\left(x\left(t\right),y\left(t\right)\right)dt$$

$$\text{s.t. } \dot{x}=g\left(x\left(t\right),y\left(t\right)\right)$$

$$x\left(0\right)=x_0,\ \lim_{t\to\infty}b\left(t\right)x\left(t\right)\geq x_1$$

$$x\left(t\right)\in \text{Int}\mathcal{X},\ y\left(t\right)\in\text{Int}\mathcal{Y}$$

- $g\left(x\left(t\right),y\left(t\right)\right)$ being autonomous is not needed but it helps to simplify notation.

Exponential Discounting Case II

- Hamiltonian:

$$\begin{aligned}\mathcal{H}\left(t, x\left(t\right), y\left(t\right), \lambda\left(t\right)\right) &= e^{-\rho t} f\left(x\left(t\right), y\left(t\right)\right) + \lambda\left(t\right) g\left(x\left(t\right), y\left(t\right)\right) \\ &= e^{-\rho t}\left[f\left(x\left(t\right), y\left(t\right)\right) + \mu\left(t\right) g\left(x\left(t\right), y\left(t\right)\right)\right]\end{aligned}$$

where

$$\mu\left(t\right) = e^{\rho t}\lambda\left(t\right)$$

- Current-Value Hamiltonian:

$$\widehat{\mathcal{H}}\left(x\left(t\right), y\left(t\right), \mu\left(t\right)\right) = f\left(x\left(t\right), y\left(t\right)\right) + \mu\left(t\right) g\left(x\left(t\right), y\left(t\right)\right)$$

Maximum Principle for Discounted Infinite-Horizon Problems

Theorem

Under some technical conditions, the optimal pair $(\widehat{x}(t), \widehat{y}(t))$ *satisfies the necessary conditions:*

1. $\widehat{\mathcal{H}}_y(x(t), y(t), \mu(t)) = 0$ *for* $\forall t \in \mathbb{R}_+$.
2. $\widehat{\mathcal{H}}_x(x(t), y(t), \mu(t)) = \rho\mu(t) - \dot{\mu}(t)$ *for* $\forall t \in \mathbb{R}_+$.
3. $\widehat{\mathcal{H}}_\mu(x(t), y(t), \mu(t)) = \dot{x}(t)$ *for* $\forall t \in \mathbb{R}_+$, $x(0) = x_0$ *and* $\lim_{t\to\infty} x(t) \geq x_1$.
4. $\lim_{t\to\infty} e^{-\rho t}\widehat{\mathcal{H}}(x(t), y(t), \mu(t)) = \lim_{t\to\infty} e^{-\rho t}\mu(t)\hat{x}(t) = 0$.

Sufficiency Conditions

- Previous theorem only delivers necessary conditions.

- However, we also need sufficient conditions.

Theorem

Mangasarian Sufficient Conditions for Discounted Infinite-Horizon Problems. The necessary conditions will be sufficient if f *and* g *are continuously differentiable and weakly monotone and* $\mathcal{H}\left(t, x\left(t\right), y\left(t\right), \lambda\left(t\right)\right)$ *is jointly concave in* $x\left(t\right)$ *and* $y\left(t\right)$ *for* $\forall t \in \mathbb{R}_{+}$.

- We will skip sufficiency arguments. They will be relevant later in models of endogenous growth.

Example I

- Consumption-savings problem:

$$V(a) = \max_{a,c} \int_0^\infty e^{-\rho t} u(c)\, dt$$
$$\dot{a} = ra + w - c$$

- Hamiltonian:

$$\widehat{\mathcal{H}}(a, c, \mu) = u(c) + \mu(ra + w - c)$$

- Necessary conditions:

$$\begin{aligned} \widehat{\mathcal{H}}_c(a, c, \mu) &= 0 \Rightarrow u'(c) - \mu = 0 \Rightarrow u'(c) = \mu \\ \widehat{\mathcal{H}}_a(a, c, \mu) &= \rho\mu - \dot{\mu} \Rightarrow r\mu = \rho\mu - \dot{\mu} \Rightarrow \frac{\dot{\mu}}{\mu} = -(r - \rho) \end{aligned}$$

Example II

- Then:

$$u''(c)\,\dot{c} = \dot{\mu} \Rightarrow$$
$$\frac{u''(c)}{u'(c)}\dot{c} = \frac{\dot{\mu}}{\mu} = -(r-\rho)$$

- Assume, for instance:

$$u(c) = \log c$$

 and we get:

$$\frac{\dot{c}}{c} = r - \rho$$

- Comparison with bang-bang solutions (linear returns and bounded controls).

Dynamic Programming

- Dynamic programming is a more flexible approach (for example, later, to introduce uncertainty).
- Instead of searching for an optimal path, we will search for decision rules.
- Cost: we will need to solve for PDEs instead of ODEs.
- But at the end, we will get the same solution.

Hamilton-Jacobi-Bellman (HJB) Equation

- When $V(t, x(t))$ is differentiable, $(\widehat{x}(t), \widehat{y}(t))$ satisfies:

$$f(t, \widehat{x}(t), \widehat{y}(t)) + \dot{V}(t, \widehat{x}(t)) + V_x(t, \widehat{x}(t)) g(t, \widehat{x}(t), \widehat{y}(t)) = 0$$

- Similar the Euler equation from a value function in discrete time.

- Other way to write the formula, closer to the Bellman equation:

$$-\dot{V}(t, \widehat{x}(t)) = \max_{x(t), y(t)} f(t, x(t), y(t)) + g(t, x(t), y(t)) V_x(t, x(t))$$

- Tight connection between $V_x(t, x(t))$ and $\mu(t)$.

Solution of the HJB Equation I

- The HJB equation allows for easy derivations.
- For exponential discount problems:

$$V\left(t,\widehat{x}\left(t\right)\right)=\int_{t}^{\infty}e^{-\rho s}f\left(\widehat{x}\left(s\right),\widehat{y}\left(s\right)\right)ds$$

- Note that:

$$V\left(t,\widehat{x}\left(t\right)\right)=e^{-\rho t}\int_{t}^{\infty}e^{-\rho\left(s-t\right)}f\left(\widehat{x}\left(s\right),\widehat{y}\left(s\right)\right)ds$$

and the integral on the right hand side does not depend on t.

Solution of the HJB Equation II

- Then:

$$\begin{aligned}\dot{V}\left(t,\widehat{x}\left(t\right)\right) &= -\rho e^{-\rho t}\int_{t}^{\infty}e^{-\rho\left(s-t\right)}f\left(\widehat{x}\left(s\right),\widehat{y}\left(s\right)\right)ds \\ &= -\rho\int_{t}^{\infty}e^{-\rho s}f\left(\widehat{x}\left(s\right),\widehat{y}\left(s\right)\right)ds \\ &= -\rho V\left(t,\widehat{x}\left(t\right)\right)\end{aligned}$$

- Simplyfing notation:

$$\rho V\left(x\right)=\max_{x,y}f\left(x,y\right)+g\left(x,y\right)V'\left(x\right)$$

Solution of the HJB Equation III

- Characterized by a necessary condition:

$$f_y(x,y) + g_y(x,y)V'(x) = 0$$

and an envelope condition:

$$(\rho - g_x(x,y))V'(x) - f_x(x,y) = g(x,y)V''(x)$$

- Then:

$$V'(x) = -\frac{f_y(x,y)}{g_y(x,y)} = h(x,y)$$

and

$$V''(x) = h_x(x,y) + h_y(x,y)\frac{dy}{dx}$$

Solution of the HJB Equation IV

- We can plug these two equations in the envelope condition, to get:

$$\begin{aligned} & (\rho - g_x(x, y))\, h(x, y) - f_x(x, y) \\ = & \left(h_x(x, y) + h_y(x, y) \frac{dy}{dx} \right) g(x, y) \end{aligned}$$

  an ODE on $\frac{dy}{dx}$.

- Analytical solutions?

- Standard numerical solution methods.

Solution of the HJB Equation V

- With our previous example:

$$\rho V(a) = \max_{a,c} u(c) + (ra + w - c) V'(a)$$

- Then:

$$h(a, c) = u'(c)$$

and:

$$-(r - \rho) u'(c) = u''(c) \frac{dc}{da} \dot{a} = u''(c) \dot{c}$$

- Therefore, as before:

$$\frac{u''(c)}{u'(c)} \dot{c} = -(r - \rho)$$

Comparison with Discrete Time

- HJB versus Bellman equation:

$$\begin{aligned} \rho V(a) &= \max_{a,c} u(c) + (ra + w - c) V'(a) \\ V(a) &= \max_{a,c} u(c) + \beta V((1+r)a + w - c) \end{aligned}$$

- Optimality conditions:

$$\begin{aligned} \frac{u''(c)}{u'(c)} \dot{c} &= -(r - \rho) \\ \frac{u'(c')}{u'(c)} &= \beta(1+r) \end{aligned}$$

Stochastic Case

- We move now into the stochastic case.
- Handling it with calculus of variations or optimal control is hard.
- At the same time, there are many problems in macro with uncertainty which are easy to formulate in continuous time.

Stochastic Problem

- Consider the problem:

$$V(x(0)) = \max_{x(t),y(t)} \mathbb{E} \int_0^\infty e^{-\rho t} f(x(t), y(t))\, dt$$
$$\text{s.t. } dx(t) = g(x(t), y(t))\, dt + \sigma(x(t))\, dW(t)$$

give some initial conditions.

- The evolution of the state is a controlled diffusion.

- If f is continuous and bounded, the integral is well defined.

Value Function and a Bellman-Type Property

- Given a small interval of time Δt, we get:

$$V(x(0)) \approx \max_{x,y} f(x(0), y(0))\Delta t + \frac{1}{1+\rho\Delta t}\mathbb{E}[V(x(0+\Delta t))]$$

- Multiply by $(1+\rho\Delta t)$ and substrate $V(x(0))$:

$$\rho V(x(0))\Delta t \approx \max_{x,y} f(x(0), y(0))\Delta t(1+\rho\Delta t) + \mathbb{E}[\Delta V]$$

- Divide by Δt

$$\rho V(x(0)) \approx \max_{x,y} f(x(0), y(0))(1+\rho\Delta t) + \frac{1}{\Delta t}\mathbb{E}[\Delta V]$$

- Letting $\Delta t \rightarrow 0$ and taking the limit:

$$\rho V(x(0)) = \max_{x,y} f(x(0), y(0)) + \frac{1}{dt}\mathbb{E}[dV]$$

Hamilton-Jacobi-Bellman (HJB) Equation I

- Given a small interval of time Δt, we get:

$$\rho V(x) = \max_{x,y} f(x,y) + \frac{1}{dt}\mathbb{E}[dV]$$

- Applying previous results:

$$\frac{1}{dt}\mathbb{E}[dV] = \left[gV' + \frac{1}{2}\sigma^2 V''\right]$$

we have

$$\rho V(x) = \max_{x,y} f(x,y) + g(x,y)V'(x) + \frac{1}{2}\sigma^2(x)V''(x) \;\; \forall x$$

- Important observation: thanks to Itō's lemma, the HJB is not stochastic.

Hamilton-Jacobi-Bellman (HJB) Equation II

- Comparison with deterministic case:

$$\rho V(x) = \max_{x,y} f(x,y) + g(x,y) V'(x)$$

$$\rho V(x) = \max_{x,y} f(x,y) + g(x,y) V'(x) + \frac{1}{2}\sigma^2(x) V''(x)$$

Extra term $\frac{1}{2}\sigma^2(x) V''(x)$ corrects for curvature.

- Concentrated HJB:

$$\rho V(x) = f(x, y(x)) + g(x, y(x)) V'(x) + \frac{1}{2}\sigma^2(x) V''(x)$$

is the Feynman–Kac formula that links parabolic partial differential equations (PDEs) and stochastic processes.

Hamilton-Jacobi-Bellman (HJB) Equation III

- Characterized by a necessary condition:

$$f_y(x,y) + g_y(x,y)\,V'(x) = 0$$

and an envelope condition:

$$(\rho - g_x(x,y))\,V'(x) - f_x(x,y) =$$
$$g(x,y)\,V''(x) + \frac{1}{2}\sigma^2(x)\,V'''(x) + \sigma(x)\,\sigma'(x)\,V''(x)$$

- Solutions:
  1. Theoretical: classical, viscosity, backward SDE, martingale duality.
  2. Numerical.

Real Business Cycle I

- Standard business cycle framework without labor choice:

$$\max_{\{c(t),k(t)\}_{t=0}^{\infty}} \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} u\left(c\left(t\right)\right) dt$$

$$\text{s.t. } \dot{k} = e^z k^\alpha - \delta k - c$$

$$dz = -\lambda z dt + \sigma z dW$$

- HJB:

$$\rho V\left(k, z\right) =$$

$$u\left(c\right) + \left(e^z k^\alpha - \delta k - c\right) V_1\left(k, z\right) - \lambda z V_2\left(k, z\right) + \frac{1}{2}\left(\sigma z\right)^2 V_{22}\left(k, z\right)$$

Real Business Cycle II

- Necessary condition

$$u'\left(c\left(t\right)\right)-V_1\left(k,z\right)=0$$

and envelope condition:

$$\begin{aligned}&\left(\rho-\left(\alpha e^{z}k^{\alpha-1}-\delta\right)\right)V_1\left(k,z\right)\\ =&\left(e^{z}k^{\alpha}-\delta k-c\right)V_{11}\left(k,z\right)-\lambda zV_{21}\left(k,z\right)\\ &+\frac{1}{2}\left(\sigma z\right)^2V_{221}\left(k,z\right)+\sigma^2zV_{22}\left(k,z\right)\end{aligned}$$