1. One dimensional potentials (20 points)

(a) $\nabla^2 V = -\rho/\epsilon_o$. Here since $V = V(x)$ we have

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} = 0$$

so $\rho = 0$ everywhere.

(b) $E_x = -\partial V/\partial x = -(V_2 - V_1)/d$ for $0 < x < d$ and zero otherwise. Thus we have a uniform nonzero field only in the region between $x = 0$ and $x = d$.

(c) The areal charge densities are found from the jump discontinuities in $E_x$ at the boundaries. This gives $\sigma = \pm \epsilon_o (V_2 - V_1)/d$ with a positive areal charge density on the boundary at the higher potential and the surface charges exactly compensate each other.

2. Electrostatic potentials for a nonuniformly charged shell (20 points)

(a) The total charge is the integral of $\sigma(\theta)$ over the surface of the sphere. One can solve this by writing $\sigma(\theta)$ in a Legendre series

$$\sigma(\theta) = \sigma_o \cos^2 \theta = \sigma_o \left( \frac{2}{3} P_2(\cos \theta) + \frac{1}{3} \right)$$

By the orthogonality of the Legendre polynomials, the total charge comes only from the isotropic $\sigma_o/3$ term (the surface integral contains the factor $\int^{1}_{-1} P_\ell(u)P_0(u) \, du$ which is nonzero only for $\ell = 0$). Therefore $Q = 4\pi \sigma_o R^2/3$. Alternatively, an explicit integration of $\sigma(\theta)$ over the surface of the sphere gives this result.

(b) The potential satisfies the Laplace equation, so the exponents give the descending powers of $1/r^{\ell+1}$ that multiply the Legendre polynomials $P_\ell$ in the separable exterior solutions to the Laplace equation

$$V^>(r, \theta) = \sum_\ell \frac{B^>_\ell}{r^{\ell+1}} P_\ell(\cos \theta)$$

Since $P_0 = 1$ and $P_1(u) = u$, one has $\lambda = 1$, $\mu = 2$, and $\nu = 3$.

(c) This boundary value problem is solved by matching the interior $(r < R)$ to the exterior $(r > R)$ solutions of the Laplace equation at $r = R$ augmented by the constraint that the normal derivatives have a jump discontinuity $\sigma(\theta)/\epsilon_o$ at every angle $\theta$. The interior solutions that match the solutions in Eqn. 1 are the products of the $P_\ell$ and ascending powers of $r$

$$V^<(r, \theta) = \sum_\ell A^<_\ell r^{\ell} P_\ell(\cos \theta)$$
Continuity of the potentials (2) and (3) requires
\[ A_\ell = \frac{B_\ell}{R^{2\ell+1}} \]
and using the Legendre series from part (a) the jump discontinuity condition reads
\[ -\left[ \frac{\partial V^>}{\partial r} \right]_{r=R} + \left[ \frac{\partial V^<}{\partial r} \right]_{r=R} = \sum_\ell \frac{(2\ell + 1)B_\ell}{R^{2\ell+2}} P_\ell(\cos \theta) = \frac{\sigma_o}{\epsilon_o} \left[ \frac{2P_2(\cos \theta) + P_0(\cos \theta)}{3} \right] \]
From which one can identify the coefficients \( A = B_0^> \), \( B = B_1^> \) and \( C = B_2^> \):
\[ A = B_0^> = \frac{\sigma_o R^2}{3\epsilon_o} \]
\[ B = B_1^> = 0 \]
\[ C = B_2^> = \frac{2\sigma_o R^4}{15\epsilon_o} \]
with all the higher coefficients \( B_\ell^> = 0 \). Note that the coefficient \( A \) gives the leading \( 1/r \) term in the exterior potential with an amplitude determined by the total charge \( Q \) found in part (a).

3. **Screening by a Spherical Conductor** (20 points)

(a) This can be solved by the method of images. The exterior electric field of the ball of charge is the field of a point charge of strength \( Q_{\text{ball}} = 4\pi \rho R^3/3 \) centered a distance \( a = R_1 + R_2 \) from the center of the conducting sphere. We can treat the exterior potential from this sphere as a point charge potential. Since the conducting sphere is isolated and neutral (note that it is not grounded) the conductor develops an induced surface charge density which for \( r > R_2 \) is seen as the superposition of two image charges: \( q_i \) and \( q_i' \) with \( b \) and \( b' \) measured with respect to the center of the conducting sphere. The formulas for these image charges give
\[ q_i = -Q \frac{R_2}{R_1 + R_2}; \quad b = \frac{R_2^2}{R_1 + R_2} \]
\[ q_i' = Q \frac{R_2}{R_1 + R_2}; \quad b' = 0 \text{ (at origin)} \]
where the charge in the second line is determined by the charge neutrality condition on the \( R_2 \) sphere. The electric potential is constant everywhere on the \( R_2 \) sphere and it can be easily evaluated at the point of contact \( P \)
\[ V_P = \frac{1}{4\pi \epsilon_o} \left[ \frac{Q}{R_1} + \frac{q_i}{R_2} - b + \frac{q_i'}{R_2} \right] \]
\[ = \frac{\rho}{3\epsilon_o R_1 + R_2} \]
Notice that the last line is the same as the potential from just \( q_i' \) since the sum of the potentials from \( Q \) and \( q_i \) gives zero at every point on the surface of the conductor.
(b) The electric field inside the conductor is zero since the conductor has a constant potential. Thus by Gauss’ law the (radial) electric field at point $P$ (i.e. at a distance $R_2^+$ from the center of the conducting sphere) gives the areal charge density $\sigma_P = \epsilon_o E_r(R_2^+)$. This field can be found by superposing three point charge fields, one from the exterior ball and two from the induced image charges:

$$E_r(R_2^+) = \frac{1}{4\pi \epsilon_o} \left[ \frac{Q}{R_1^2} + \frac{q_i}{R_2 - b} + \frac{q'_i}{R_2} \right]$$

$$= \frac{\rho R_1^2}{3\epsilon_o} \left[ -\frac{1}{R_1^2} - \frac{R_2}{R_1 + R_2} \left( \frac{R_2}{R_2 - \frac{R_1 R_2}{R_1 + R_2}} \right)^2 + \frac{R_3}{R_2} \right]$$

$$\sigma_P = -\frac{\rho}{3} \left[ R_1 + \frac{R_1(R_1 + R_2)}{R_2} - \frac{R_1^3}{R_2(R_1 + R_2)} \right]$$

$$= -\frac{\rho R_1^2}{3 R_1 + R_2}$$

where the last line follows after a bit of algebra. Notice that the normal component of the field has contributions from all three charges. Note also that $\sigma_P$ vanishes if $R_1 = 0$ since in this limit the volume of the $R_1$ sphere is zero and there is actually no exterior charge. Conversely, when $R_1 \neq 0$ but $R_2 \to 0$, $\sigma_P$ is the surface charge density that is required to quench the surface electric field of a solid ball of charge of charge density $\rho$ and radius $R_1$. 