H8.P1 Line Charge In a Cylindrical Cavity

A uniformly charged line with charge per unit length \( \lambda \) is in the cavity of a grounded \((V = 0)\) cylindrical conducting pipe with inner radius \( a \). The line charge is oriented parallel to the long axis of the pipe, but it displaced from the center (along the \( x \) direction) by perpendicular offset \( d \) (note \( d < a \)). The line charge induces a charge distribution on the interior wall of the pipe, which we calculate in this problem.

(a) Find the electric potential from the induced charge density everywhere in the interior of the pipe. You can express your result as a power series in the dimensionless parameters \( s = a \) and \( d = s \).

(b) Use the result of (a) to calculate the surface charge density \( \sigma(\phi) \) on the inside of the pipe. Verify that your result satisfies the perfect screening condition. The series expression for \( \sigma \) can be summed exactly (it’s a fairly straightforward sum), so you should be able to obtain an exact analytic formula for \( \sigma(\phi) \). Make a plot of the induced surface charge density as a a function of \( \phi \) for the cases \( d/a = 0 \) (this should be a uniform charge density) and for \( d/a = 0.9 \) (line charge close to the wall).

Useful hint: when \( x < 1 \)

\[
- \ln \sqrt{1 - 2x \cos \phi + x^2} = \sum_{m=1}^{\infty} \frac{1}{m} x^m \cos(m\phi)
\]

H8.P2 Energy for Surface Charge Distributions

A sphere of radius \( R \) has a surface charge density

\[
\sigma(\theta) = \sum_{\ell} \sigma_{\ell} P_{\ell}(\cos \theta)
\]

(a) Find the interior and exterior electric potentials for this charge distribution.

(b) Find the total electrostatic energy of this charge distribution. Verify that your answer reduces to our known results for the limiting cases with \( \sigma_{\ell} \neq 0 \) for only \( \ell = 0 \) and \( \ell = 1 \).

1 If you express your result as a Legendre series, show that this series terminates at a finite number of terms and find the highest order Legendre polynomial in this expansion.

2 After you find the charge density \( \sigma(\theta) \) calculate the electrostatic energy of this system.

3 Also find the (constant) potential inside the sphere. There are many routes to solving this problem. A very good one is to break the exterior potential into a part from the external sources \( V_{\text{ext}} \) and a part from the induced surface charge on the sphere \( V_R \).
4 Find the charge density $\sigma(\theta)$ on the surface of the conducting sphere. The surface charge density changes sign on the surface only for sufficiently large external field $E_0$. Find $E_0$.

5 Verify that the $\ell = 0$ term in the expansion gives the Coulomb potential of a point source with total charge $Q = \sigma \pi R^2$. 