Physics and Mathematics of F-theory
Harvard University

ArXiv 1805.03214, 1802.04802, 1712.02337

Coulomb Phases and Anomalies: Geometric Approach to 5d/6d Theories

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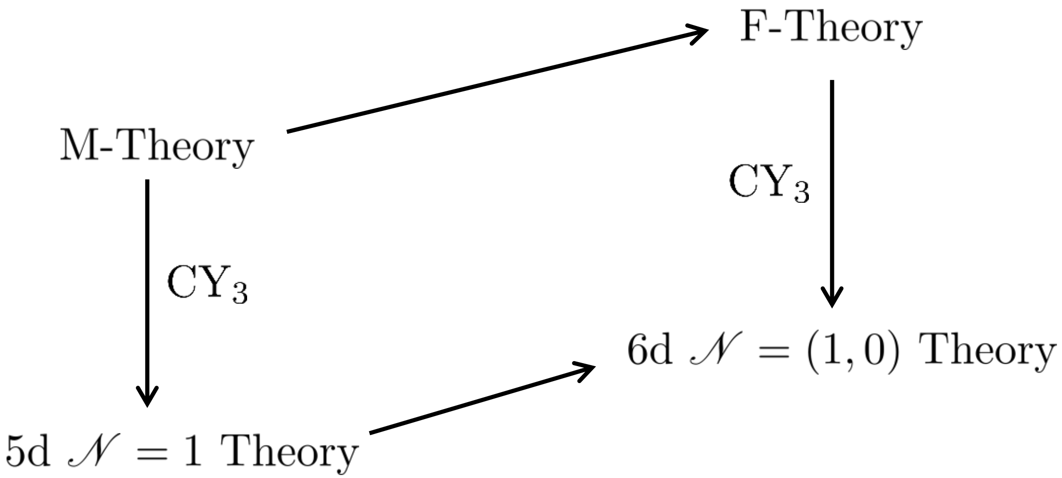
BASED ON WORKS WITH MBOYO ESOLE AND SHING-TUNG YAU

Motivation

- Elliptic Fibrations
  - Geometrically engineered gauge theories
  - Captures global aspect of the gauge theory



- 5d/6d Field Contents

5d $\mathcal{N} = 1$ Theory	6d $\mathcal{N} = (1,0)$ Theory
Gravity multiplets $(g_{\mu\nu}, \psi_{\mu I}, A_\mu)$	Gravity multiplets $(g_{\mu\nu}, B^+_{\mu\nu}, \psi^-_\mu)$
	Tensor multiplets $(B^-_{\mu\nu}, \phi, \chi^+)$
Vector multiplets $(A^A_\mu, \lambda^A_I, \phi^A)$	Vector multiplets (A_μ, λ^-)
Hypermultiplets (ζ^m, A^m_I)	Hypermultiplets (q, η^+)

Elliptic Fibrations and Gauge Theories

- (Semi-simple) Lie group G, Lie algebra $\mathfrak{g}$, Representation $\mathbf{R}$
- Dictionary between the elliptic fibration and the gauge theory

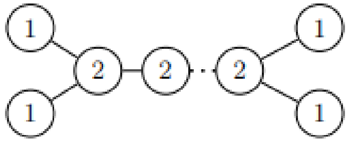
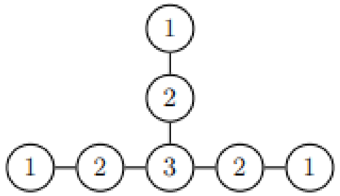
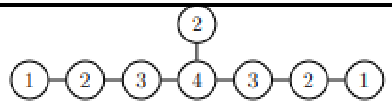
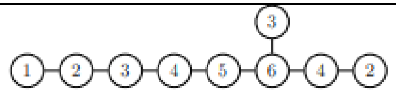
Elliptic Fibration	Gauge Theory
Codimension 1 singularities	Gauge algebra ($\mathfrak{g}$)
Codimension 2 singularities	Representation ($\mathbf{R}$)
Crepant resolutions	Coulomb phases
Flops	Phase transitions
Triple intersection polynomial	5d prepotential
Mordell-Weil group	The fundamental group of the gauge group ($\pi_1(G)$)

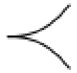
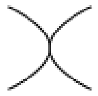
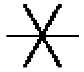
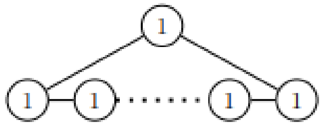
Elliptic Fibration

- Weierstrass model: $y^2 z = x^3 + fxz^3 + gz^3$
  - Projective Bundle: $\pi : X_0 = P_B[\mathscr{O}_B \oplus \mathscr{L}^{\otimes 2} \oplus \mathscr{L}^{\otimes 3}] \to B$

An elliptic fibration $\varphi : Y \to B$ cut out by the zero locus of a section of the line bundle $\mathscr{O}(3) \oplus \pi^* \mathscr{L}^{\otimes 6}$ in X_0, where $\mathscr{L}$ is a line bundle over a quasi-projective variety B.

  - Section: $z = x = 0$
  - Projective coordinates: $[z : x : y]$ of a $\mathbb{P}^2$
  - Discriminant: $\Delta = 4f^3 + 27g^2$
  - j-invariant: $j = 1728 \frac{4f^3}{\Delta}$

Type	$v(c_4)$	$v(c_6)$	$v(\Delta)$	j	Monodromy	Fiber	Dual Graph
I_n^*	2	≥ 3	$n+6$	∞	$\begin{pmatrix}-1 & -n\\ 0 & -1\end{pmatrix}$		$\widetilde{D}_{n+4}$
	≥ 2	3	$n+6$				
IV^*	≥ 3	4	8	0	$\begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix}$		$\widetilde{E}_6$
III^*	3	≥ 5	9	1728	$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$		$\widetilde{E}_7$
II^*	≥ 4	5	10	0	$\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}$		$\widetilde{E}_8$

Type	$v(c_4)$	$v(c_6)$	$v(\Delta)$	j	Monodromy	Fiber	Dual Graph
I_0	≥ 0	≥ 0	0	$\mathbb{C}$	I_2	Smooth	-
I_1	0	0	1	∞	$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$		$\widetilde{A}_0$
II	≥ 1	1	2	0	$\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}$		$\widetilde{A}_0$
III	1	≥ 2	3	1728	$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$		$\widetilde{A}_1$
IV	≥ 2	2	4	0	$\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}$		$\widetilde{A}_2$
I_n	0	0	$n > 1$	∞	$\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$		$\widetilde{A}_{n-1}$

Kodaira Classification and Tate’s Algorithm

Algorithm To Get Geometric Data

Step 1. Determine a singular Weierstrass model with Kodaira fibers associated to the desired Lie group G.

Step 2. Determine a crepant resolution of the singular Weierstrass model.

Step 3. Compute the pushforward formulas to push the total Chern class of the resolved elliptic fibration to its base. Then, the generating function of Euler characteristics is computed.

- For a d-dimensional base, the Euler characteristic is given by the coefficient of t^d in a power series expansion.
- Compute the Euler characteristics for Calabi-Yau threefolds.

Step 4. Compute the Hodge numbers using the fact that the base is a rational surface and Shioda-Tate-Wazir theorem.

Step 5. Determine the fiber structure of the resolved Weierstrass Model.

Step 6. Determine the representations by computing the geometric weights of the irreducible components of the singular fibers over codimension-two points.

Step 7. Compute the triple intersection polynomial.

Simple Groups

- Euler characteristics and Hodge numbers of Calabi-Yau threefolds are computed for various simple groups. [Esole, Jefferson, MJK]

Algebra	Group	Kodaira Fiber	$h^{1,1}(Y_3)$	$h^{2,1}(Y_3)$	$\chi(Y_3)$
–	$\{e\}$	I_1	$11-K^2$	$11+29K^2$	$-60K^2$
A_1	$SU(2)$	I_2^s, I_2^{ns}, III, IV^{ns}	$12-K^2$	$12+29K^2+15KS+3S^2$	$-60K^2-30KS-6S^2$
A_2 G_2	SU(3) G_2	I_3^s, IV^s I_0^{*ns}	$13-K^2$	$13+29K^2+24KS+6S^2$	$-60K^2-48KS-12S^2$
A_3 B_3	SU(4) Spin(7)	I_4^s I_0^{*ss}	$14-K^2$	$14+29K^2+32KS+10S^2$	$-60K^2-64KS-20S^2$
D_4 F_4	Spin(8) F_4	I_0^{*s} IV^{*ns}	$15-K^2$	$15+29K^2+36KS+12S^2$	$-60K^2-72KS-24S^2$
A_4	SU(5)	I_5^s	$15-K^2$	$15+29K^2+40KS+15S^2$	$-60K^2-80KS-30S^2$
D_5	Spin(10)	I_1^{*s}	$16-K^2$	$16+29K^2+42KS+16S^2$	$-60K^2-84KS-32S^2$
E_6	E_6	IV^{*s}	$17-K^2$	$17+29K^2+45KS+18S^2$	$-60K^2-90KS-36S^2$
E_7	E_7	III^*	$18-K^2$	$18+29K^2+49KS+21S^2$	$-60K^2-98KS-42S^2$
E_8	E_8	II^*	$19-K^2$	$19+29K^2+60KS+30S^2$	$-60K^2-120KS-60S^2$
A_1	SO(3)	I_2^{ns}	$12-K^2$	$12+17K^2$	$-36K^2$
B_2	SO(5)	I_4^{ns}	$14-K^2$	$14+9K^2$	$-20K^2$
A_3	SO(6)	I_4^s	$14-K^2$	$14+5K^2$	$-12K^2$

Simple Groups

- Euler characteristics and Hodge numbers are computed for various simple groups [Esole, Jefferson, MJK].
- Step 1-5 are studied for SU(n) for n≤5 [Esole, Yau] [Grimm, Hayashi] [Esole, Shao, Yau] [Esole, Shao].
- All Step1-7 are studied for G_2, Spin(7), Spin(8) [Esole, Jagadeesan, MJK] and F_4 [Esole, Jefferson, MJK].

Semi-simple Groups

- Semi-simple Lie algebras
  - The discriminant of the fibration contains at least two irreducible components Δ_1 and Δ_2 (for which the dual graph of the singular fiber over the generic point is reducible).
  - “*Collisions of singularities*”
- Studied for various models
  - SO(4) and Spin(4) [Esole, MJK]
  - SU(2)×G2 [Esole, MJK]
  - SU(2)×SU(4), (SU(2)×SU(4))/Z_2, SU(2)×Sp(4), (SU(2)×Sp(4))/Z_2, [Esole, MJK, Yau]
  - SU(2)×SU(3) [To appear; Esole, Jagadeesan, MJK]
- Organize the collisions of singularities by the rank of the associated Lie algebra

The simplest: $D_2 = A_1 + A_1$

- Collisions of two singular fibers with dual graphs $\tilde{A}_1$.
- Two possible gauge groups:
  - Spin(4) = SU(2)×SU(2) Trivial Mordell-Weil
  - SO(4) = (SU(2)×SU(2))/Z_2 Mordell-Weil group Z_2

[Esole, MJK]

SO(4)	$\mathbf{R} = (\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{2},\mathbf{2})$ $\chi(CY_3) = -4(9K^2 + 4KT + T^2)$ $h^{1,1}(CY_3) = 13 - K^2, \quad h^{2,1}(CY_3) = 13 + 17K^2 + 8KT + 2T^2$ $\mathscr{F}^{+}_{triple} = -2\left(24L^2 - 10LT + T^2\right)\psi_1^3 + 6T(T-4L)\psi_1\phi_1^2 + 4T(L-T)\phi_1^3$
Spin(4)	$\mathbf{R} = (\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{2},\mathbf{2}) \oplus (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$ $\chi(CY_3) = -2\left(30K^2 + 15KS + 15KT + 3S^2 + 4ST + 3T^2\right)$ $\mathscr{F}^{+}_{triple} = 2T(-2L + S - T)\phi_1^3 - 6ST\psi_1\phi_1^2 - 2S(2L+S)\psi_1^3$ $h^{1,1}(CY_3) = 13 - K^2, \quad h^{2,1}(CY_3) = 13 + 29K^2 + 15KS + 15KT + 3S^2 + 4ST + 3T^2$

Next simplest:

- Next simplest: SU(2)×G with G a simple Lie group of rank 2
- Three possible gauge groups:
  - SU(2)×SU(3) : Non-abelian sector of the standard model.
  - SU(2)×Sp(4) } QCD-like theories obtained by replacing SU(3) by
  - $SU(2)\times G_2$ } another simple and simply-connected group of rank 2.

SU(2)×Sp(4)
[Esole, MJK, Yau]

Models	Algebraic data	# Flops
$I_2^{ns}+I_4^{ns}$ $MW=\mathbb{Z}_2$	$F=y^2z-(x^3+a_2x^2z+st^2xz^2)$ $\Delta=s^2t^4(a_2^2-4st^2)$ $G=(SU(2)\times Sp(4))/\mathbb{Z}_2$ $\mathbf{R}=(\mathbf{3},\mathbf{1})\oplus(\mathbf{1},\mathbf{10})\oplus(\mathbf{2},\mathbf{4})\oplus(\mathbf{1},\mathbf{5})$ $\chi=-4(9K^2+8K\cdot T+3T^2)$	3
$I_2^{ns}+I_4^{ns}$ $MW=\{1\}$	$F=y^2z-(x^3+a_2x^2z+\widetilde{a}_4st^2xz^2+\widetilde{a}_6s^2t^4z^3)$ $\Delta=s^2t^4(4a_2^3\widetilde{a}_6-a_2^2\widetilde{a}_4^2-18a_2\widetilde{a}_4\widetilde{a}_6st^2+4a_4^3st^2+27\widetilde{a}_6^2s^2t^4)$ $G=SU(2)\times Sp(4)$ $\mathbf{R}=(\mathbf{3},\mathbf{1})\oplus(\mathbf{1},\mathbf{10})\oplus(\mathbf{2},\mathbf{4})\oplus(\mathbf{1},\mathbf{5})\oplus(\mathbf{2},\mathbf{1})\oplus(\mathbf{1},\mathbf{4})$ $\chi=-2(30K^2+15K\cdot S+30K\cdot T+3S^2+8S\cdot T+10T^2)$	3

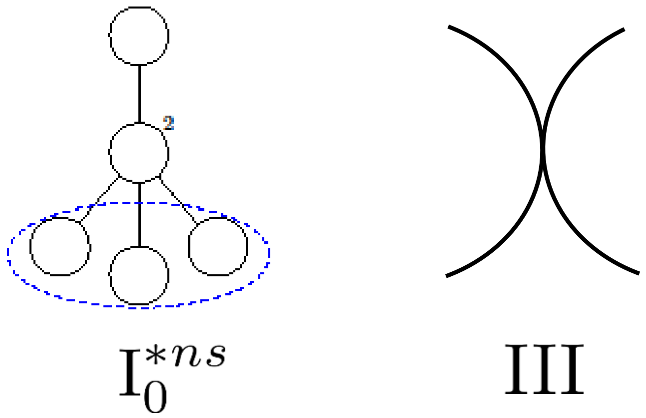
$SU(2)\times G_2$ is an important model

- M/F-theory: Naturally appears in the study of non-Higgsable clusters.
  - Over non-compact bases, collisions of singularities are used to classify 6d N=(1,0) Superconformal field theories using elliptic fibrations.
  - Ex: Such a non-Higgsable model is produced when the discriminant locus containing two rational curves with self-intersection -3 and -2 intersecting transversally or these rational curves which form a chain of curves intersecting transversally at a point with self intersections (-3,-2,-2).
  - The non-Higgsable cluster (-2,-3) is an important example of an $SU(2)\times G_2$-model.
- Birational Geometry: Naturally appears as a key model due to the simplicity of its fiber structure.

Step1 of the algorithm

Step 1. Determine a singular Weierstrass model with Kodaira fibers associated to the desired Lie group G.

- Five possibilities: $\mathrm{I}_2^s + \mathrm{I}_0^{*ns}$, $\mathrm{I}_2^{ns} + \mathrm{I}_0^{*ns}$, $\boxed{\mathrm{III} + \mathrm{I}_0^{*ns}}$, $\mathrm{I}_3^{ns} + \mathrm{I}_0^{*ns}$, $\mathrm{IV}^{ns} + \mathrm{I}_0^{*ns}$.
- Non-Higgsable model studied in the literature: $\mathrm{III} + \mathrm{I}_0^{*ns}$.
- Weierstrass equation: $y^2 z = x^3 + f s t^2 x z^2 + g s^2 t^3 z^3$.
- Discriminant locus: $\Delta = s^3 t^6 (4 f^3 + 27 g^2 s)$.



Step2 of the algorithm

Step 2. Determine a crepant resolution of the singular Weierstrass model.

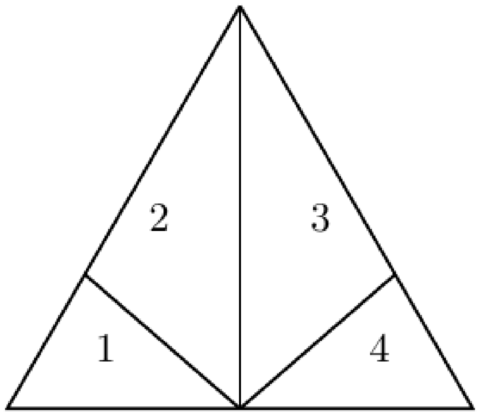
- There are four total independent crepant resolutions for SU(2)×G_2-model.

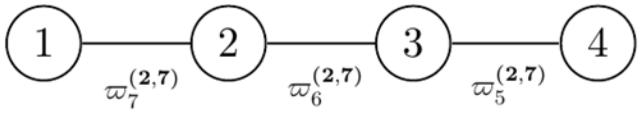
$$\textbf{Resolution I:}\quad X_0 \xleftarrow{(x,y,s|e_1)} X_1 \xleftarrow{(x,y,t|w_1)} X_2 \xleftarrow{(y,w_1|w_2)} X_3\,,$$

$$\textbf{Resolution II:}\quad X_0 \xleftarrow{(x,y,p_0|p_1)} X_1 \xleftarrow{(y,p_1,t|w_1)} X_2 \xleftarrow{(p_0,t|w_2)} X_3\,,$$

$$\textbf{Resolution III:}\quad X_0 \xleftarrow{(x,y,t|w_1)} X_1 \xleftarrow{(x,y,s|e_1)} X_2 \xleftarrow{(y,w_1|w_2)} X_3\,,$$

$$\textbf{Resolution IV:}\quad X_0 \xleftarrow{(x,y,t|w_1)} X_1 \xleftarrow{(y,w_1|w_2)} X_2 \xleftarrow{(x,y,s|e_1)} X_3\,.$$





$$\varpi_5^{(\mathbf{2,7})}=(1;-2,1),\ \varpi_6^{(\mathbf{2,7})}=(1;1,-1),\ \varpi_7^{(\mathbf{2,7})}=(1;-1,0).$$

Step3 of the algorithm

Step 3. Compute the pushforward formulas to push the total Chern class of the resolved elliptic fibration to its base. Then we get the generating function of the Euler characteristics and total Chern classes.

- Three blowup maps to get a crepant resolution:

$$f_1 : X_1 \to X_0, \quad f_2 : X_2 \to X_1, \quad \text{and} \quad f_3 : X_3 \to X_2.$$

- Using pushforward theorems: For $L = c_1(\mathscr{L})$ and $H = c_1\Big(\mathscr{O}_{X_0}(1)\Big)$,

$$\chi(Y) = \int_Y c(TY) \cap [Y] = \int_B \pi_* f_{1*} f_{2*} f_{3*} c(TY) \cap [Y]$$

$$= 6\,\frac{S^2 - 2L - 3SL + 2(S^2 - 3SL + S - 2L)T + (3S+2)T^2}{(1+S)(1+T)(-1-6L+2S+3T)}\, c(TB).$$

$$([Y] = (f_3^* f_2^* f_1^*(3H + 6\pi^* L) - 2 f_3^* f_2^* E_1 - 2 f_3^* W_1 - W_2) \cap [X_3])$$

- Now for the Calabi-Yau threefolds:
  - Calabi-Yau condition: $c_1 = L = -K.$
  - Expand: $L \to Lt,\ S \to St,\ T \to Tt,\ c(TB) \to c_t(TB) = c_1 + c_2 t + c_3 t^2 + \cdots.$
  - Euler characteristic of CY3: $\chi(Y_3) = -6(10K^2 + S^2 + 5SK + 2ST + 8KT + 2T^2).$

Step4 of the algorithm

Step 4. Compute the Hodge numbers using the fact that the base is a rational surface and Shioda-Tate-Wazir theorem.

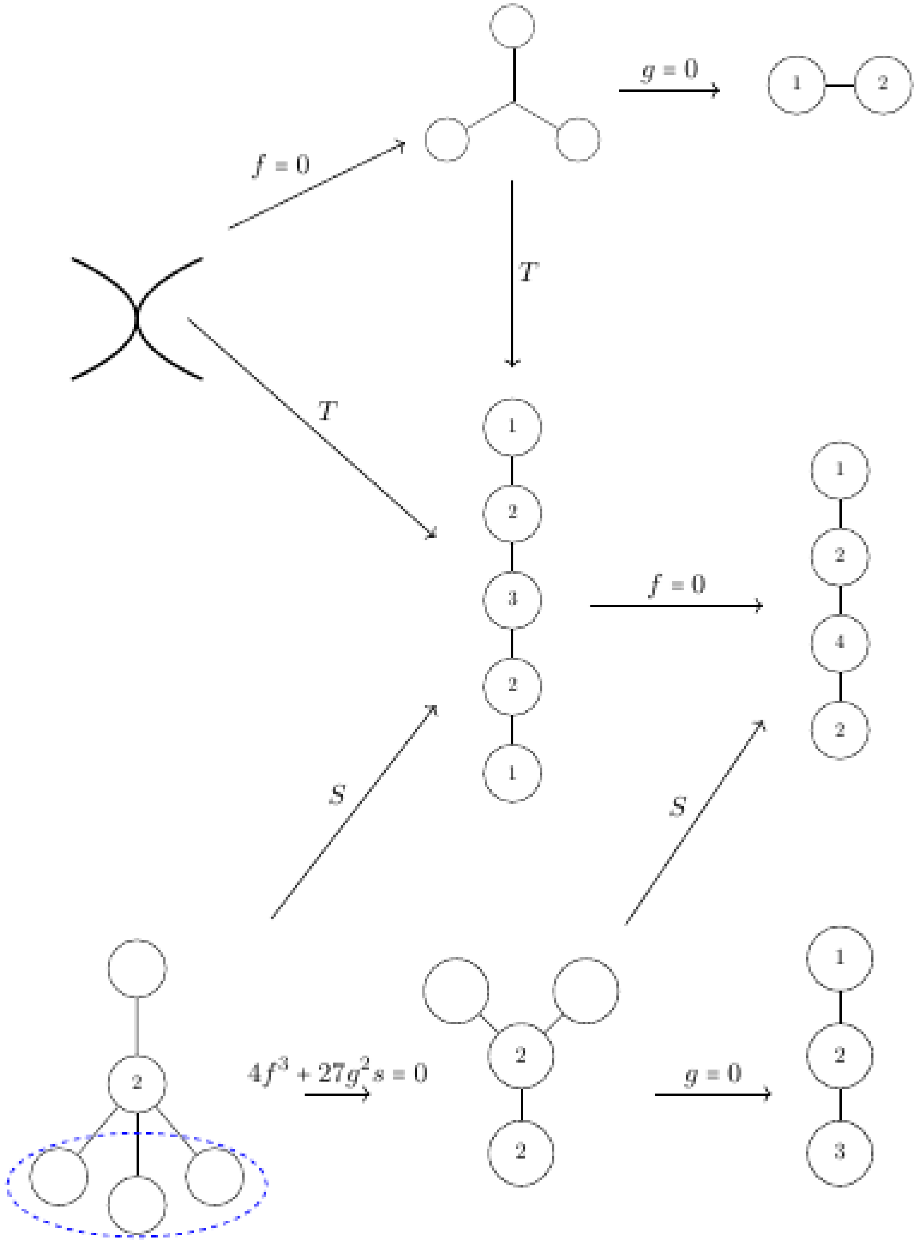
- Since B is a rational surface, $h^{0,1}(B) = h^{0,2}(B) = 0.$
- Shioda-Tate-Wazir theorem:
  - $h^{1,1}(Y) = h^{1,1}(B) + f + 1,\ h^{2,1}(Y) = h^{1,1}(Y) - \frac{1}{2}\chi(Y).$
  - f is the number of geometrically irreducible fibral divisors not touching the zero section.
- Hodge numbers for SU(2), G_2, and SU(2)×G_2-model

Algebra	Group	Kodaira Fiber	$h^{1,1}$	$h^{2,1}$
A_1	$SU(2)$	I_2^s, I_2^{ns}, III, IV^{ns}	$12 - K^2$	$12 + 29K^2 + 15KS + 3S^2$
$\mathfrak{g}_2$	G_2	I_0^{*ns}	$13 - K^2$	$13 + 29K^2 + 24KT + 6T^2$
$A_1 + \mathfrak{g}_2$	$SU(2)\times G_2$	$III + I_0^{*ns}$	$14 - K^2$	$14 + 29K^2 + 15KS + 3S^2 + 24KT + 6T^2 + 6ST$

Step5 of the algorithm

Step 5. Determine the fiber structure of the resolved Weierstrass Model.

- Example: Resolution I
- Codim-two, over both divisors S and T:
  - Fiber of type III* (dual graph $\widetilde{E}_7$ with contracted nodes)
- Codim-three, over both divisors S and T:
  - Fiber of type IV* (dual graph $\widetilde{E}_8$ with contracted nodes)



Step6 of the algorithm-1

Step 6. Determine the representations by computing the geometric weights of the irreducible components of the singular fibers over codimension-two points.

- Computed weights and representations of the curves composing the fiber.
- All four crepant resolutions yield the same representation:

$$\mathbf{R} = (\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{17}) \oplus (\mathbf{2},\mathbf{7}) \oplus (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{7}).$$

  - Each fundamentals are over a divisor S or T.
  - Bifundamentals are over both divisors S and T.
  - Adjoints as expected.

Representation	Weights		
$(\mathbf{2},\mathbf{1})$	$\varpi_1^{(\mathbf{2},\mathbf{1})} = (1;0,0)$	$\varpi_2^{(\mathbf{2},\mathbf{1})} = (-1;0,0)$	
$(\mathbf{1},\mathbf{7})$	$\varpi_1^{(\mathbf{1},\mathbf{7})} = (0;1,0)$	$\varpi_2^{(\mathbf{1},\mathbf{7})} = (0;-1,1)$	$\varpi_3^{(\mathbf{1},\mathbf{7})} = (0;2,-1)$
		$\varpi_4^{(\mathbf{1},\mathbf{7})} = (0;0,0)$	
	$\varpi_5^{(\mathbf{1},\mathbf{7})} = (0;-2,1)$	$\varpi_6^{(\mathbf{1},\mathbf{7})} = (0;1,-1)$	$\varpi_7^{(\mathbf{1},\mathbf{7})} = (0;-1,0)$
$(\mathbf{2},\mathbf{7})$	$\varpi_1^{(\mathbf{2},\mathbf{7})} = (1;1,0)$	$\varpi_2^{(\mathbf{2},\mathbf{7})} = (1;-1,1)$	$\varpi_3^{(\mathbf{2},\mathbf{7})} = (1;2,-1)$
		$\varpi_4^{(\mathbf{2},\mathbf{7})} = (1;0,0)$	
	$\varpi_5^{(\mathbf{2},\mathbf{7})} = (1;-2,1)$	$\varpi_6^{(\mathbf{2},\mathbf{7})} = (1;1,-1)$	$\varpi_7^{(\mathbf{2},\mathbf{7})} = (1;-1,0)$
	$\varpi_8^{(\mathbf{2},\mathbf{7})} = (-1;1,0)$	$\varpi_9^{(\mathbf{2},\mathbf{7})} = (-1;-1,1)$	$\varpi_{10}^{(\mathbf{2},\mathbf{7})} = (-1;2,-1)$
		$\varpi_{11}^{(\mathbf{2},\mathbf{7})} = (-1;0,0)$	
	$\varpi_{12}^{(\mathbf{2},\mathbf{7})} = (-1;-2,1)$	$\varpi_{13}^{(\mathbf{2},\mathbf{7})} = (-1;1,-1)$	$\varpi_{14}^{(\mathbf{2},\mathbf{7})} = (-1;-1,0)$

- The fundamental Weyl chamber is the half cone defined by the positivity of the linear form by the simple roots:

$$\psi_1 > 0, \quad 2\phi_1 - \phi_2 > 0, \quad -3\phi_1 + 2\phi_2 > 0.$$

(ψ_1: coroot of A_1, $\phi_{1,2}$: the basis of fundamental coroots of G_2)

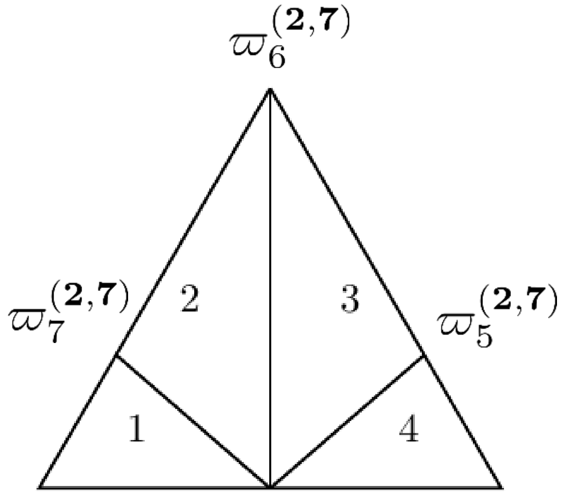
- Three hyperplanes intersecting the interior of the fundamental Weyl chamber:
  - $(\varpi_5^{(\mathbf{2},\mathbf{7})}, \varpi_6^{(\mathbf{2},\mathbf{7})}, \varpi_7^{(\mathbf{2},\mathbf{7})}) = (\psi_1 - 2\phi_1 - \phi_2, \psi_1 + \phi_1 - \phi_2, \psi_1 - \phi_2).$

Step6 of the algorithm-2

Step 6. Compute the geometric weights of the irreducible components of the singular fibers over codimension-two points.

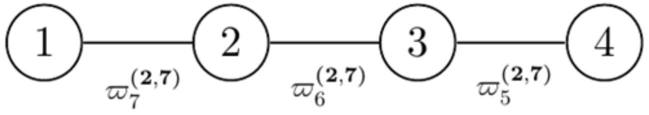
- The 4 chambers identified:

Subchambers	$\varpi_5^{(\mathbf{2},\mathbf{7})}$	$\varpi_6^{(\mathbf{2},\mathbf{7})}$	$\varpi_7^{(\mathbf{2},\mathbf{7})}$	Explicit description
①	+	+	+	$0 < \frac{1}{2}\phi_2 < \phi_1 < \frac{2}{3}\phi_2, \quad \phi_1 < \psi_1$
②	+	+	−	$0 < \frac{1}{2}\phi_2 < \phi_1 < \frac{2}{3}\phi_2, \quad \phi_2 - \phi_1 < \psi_1 < \phi_1$
③	+	−	−	$0 < \frac{3}{2}\phi_1 < \phi_2 < 2\phi_1, \quad 2\phi_1 - \phi_2 < \psi_1 < \phi_2 - \phi_1$
④	−	−	−	$0 < \frac{3}{2}\phi_1 < \phi_2 < 2\phi_1, \quad 0 < \psi_1 < 2\phi_1 - \phi_2$



- The flopping curves between the resolutions are also identified:

Flopping curves									Weight
Resolution I:	η_1^{0A}	$[1;-1,0]$	$(\mathbf{2},\mathbf{7})$	$\leftrightarrow$	Resolution II:	η_{01}^{2}	$[-1;1,0]$	$(\mathbf{2},\mathbf{7})$	$\omega_7^{(\mathbf{2},\mathbf{7})}$
Resolution II:	η_1^{02}	$[1;1,-1]$	$(\mathbf{2},\mathbf{7})$	$\leftrightarrow$	Resolution III:	η_0^{12}	$[-1;-1,1]$	$(\mathbf{2},\mathbf{7})$	$\omega_6^{(\mathbf{2},\mathbf{7})}$
Resolution III:	η_1^{12}	$[1;-2,1]$	$(\mathbf{2},\mathbf{7})$	$\leftrightarrow$	Resolution IV:	η_0^{2B}	$[-1;2,-1]$	$(\mathbf{2},\mathbf{7})$	$\omega_5^{(\mathbf{2},\mathbf{7})}$



$$\varpi_5^{(\mathbf{2},\mathbf{7})} = (1;-2,1), \quad \varpi_6^{(\mathbf{2},\mathbf{7})} = (1;1,-1), \quad \varpi_7^{(\mathbf{2},\mathbf{7})} = (1;-1,0).$$

Step7 of the algorithm

Step 7. Compute the triple intersection polynomial.

- Triple intersection polynomial (for Chamber I):

For D_a^s, D_a^t divisors, ψ_1, ϕ_1, ϕ_2 coroots,

$$\mathscr{F}_{trip} = \int_Y \left[\left(\psi_0 D_0^s + \psi_1 D_1^s + \phi_0 D_0^t + \phi_1 D_1^t + \phi_2 D_2^t\right)^3\right] = \int_B \pi_* f_* \left[\left(\psi_0 D_0^s + \psi_1 D_1^s + \phi_0 D_0^t + \phi_1 D_1^t + \phi_2 D_2^t\right)^3\right]$$
$$= 3T\phi_1\phi_2((-9L+S+6T)\phi_1 - 3(-6L+S+3T)\phi_2) + 6ST\psi_1\phi_1(\phi_1+3\phi_2) - 4T(9L-2S-3T)\phi_2^3$$
$$- 2S(2L+S)\psi_1^3 - 18ST\psi_1\phi_2^2$$
$$- 2S\psi_0\left(\psi_0 - 2\psi_1\right)\left(2(S-L)\psi_0 + (2L+S)\psi_1\right) - 4S(S-L)\psi_0\psi_1^2 + 3T\phi_0^2\left(\phi_1(-2L+S+T) - 2S\psi_1\right)$$
$$+ 2T\phi_0^3(2L-S-2T) + 3T\phi_0\left(\phi_1^2(L-S) + 2S\phi_1\left(\psi_1+\phi_2\right) - 2S\left(\left(\psi_0-\psi_1\right)^2 + \phi_2^2\right)\right),$$

- The triple intersection numbers of the fibral divisors are the coefficients that are pushforwarded to the base B using the pushforward theorems.

5d Prepotential

- In the Coulomb phase of a 5d N=1 supergravity theory
  - The scalar fields of the vector multiplets are restricted to the Cartan sub-algebra of the Lie group. (The Lie group is broken to $U(1)^r$ with r=rank(G).)
  - The charge of an hypermultiplet is simply given by a weight of the representation R.
- 5d prepotential [Intrilligator, Morrison, Seiberg]:
  - The quantum contribution to the prepotential of a 5d gauge theory with the matter fields in the representations R_i of the gauge group.
  - $6\mathscr{F}_{\mathrm{IMS}}(\phi) = \frac{1}{2}\left(\sum_{\alpha} |\langle\alpha,\phi\rangle|^3 - \sum_i \sum_{\varpi\in\mathbf{R}_i} n_{\mathbf{R}_i}|\langle\varpi,\phi\rangle|^3\right).$

    α: fundamental roots, ϕ: elements of Cartan subalgebra of the Lie algebra, ϖ: weights of R
- For SU(2)×G_2-model (Chamber 1):

$$6\mathscr{F}^{(1)}_{\mathrm{IMS}} = -4\phi_2^3(2n_{\mathbf{1},\mathbf{14}} + n_{\mathbf{1},\mathbf{7}} - 2) + 9\phi_1\phi_2^2(-2n_{\mathbf{1},\mathbf{14}} + n_{\mathbf{1},\mathbf{7}} + 2) - 8(n_{\mathbf{1},\mathbf{14}} - 1)\phi_1^3 + 3\phi_1^2\phi_2(8n_{\mathbf{1},\mathbf{14}} - n_{\mathbf{1},\mathbf{7}} - 8)$$
$$+ \psi_1^3(-n_{\mathbf{2},\mathbf{1}} - 7n_{\mathbf{2},\mathbf{7}} - 8n_{\mathbf{3},\mathbf{1}} + 8) + 12\psi_1\left(-3n_{\mathbf{2},\mathbf{7}}\phi_2^2 + 3n_{\mathbf{2},\mathbf{7}}\phi_1\phi_2 - n_{\mathbf{2},\mathbf{7}}\phi_1^2\right)$$

Triple Intersection Polynomial = Prepotetial

- Recall the dictionary: $\mathscr{F}^I_{trip} = 6\mathscr{F}^{(1)}_{\text{IMS}}$.
- $$\mathscr{F}^I_{trip} = 3T\phi_1\phi_2((-9L+S+6T)\phi_1 - 3(-6L+S+3T)\phi_2) + 6ST\psi_1\phi_1(\phi_1+3\phi_2) - 4T(9L-2S-3T)\phi_2^3 - 2S(2L+S)\psi_1^3 - 18ST\psi_1\phi_2^2$$
- $$6\mathscr{F}^{(1)}_{\text{IMS}} = -4\phi_2^3(2n_{\mathbf{1},\mathbf{14}} + n_{\mathbf{1},\mathbf{7}} - 2) + 9\phi_1\phi_2^2(-2n_{\mathbf{1},\mathbf{14}} + n_{\mathbf{1},\mathbf{7}} + 2) - 8(n_{\mathbf{1},\mathbf{14}} - 1)\phi_1^3 + 3\phi_1^2\phi_2(8n_{\mathbf{1},\mathbf{14}} - n_{\mathbf{1},\mathbf{7}} - 8) + \psi_1^3(-n_{\mathbf{2},\mathbf{1}} - 7n_{\mathbf{2},\mathbf{7}} - 8n_{\mathbf{3},\mathbf{1}} + 8) + 12\psi_1\left(-3n_{\mathbf{2},\mathbf{7}}\phi_2^2 + 3n_{\mathbf{2},\mathbf{7}}\phi_1\phi_2 - n_{\mathbf{2},\mathbf{7}}\phi_1^2\right)$$
- Then we get the following linear relations of the number of charged hypers:

$$\left\{\begin{array}{ll} n_{\mathbf{2},\mathbf{1}} + 8n_{\mathbf{3},\mathbf{1}} = S\left(4L + 2S - \frac{7}{2}T\right) + 8, & n_{\mathbf{1},\mathbf{14}} = \frac{1}{2}(-LT + T^2 + 2), \\ n_{\mathbf{2},\mathbf{7}} = \frac{1}{2}ST, & n_{\mathbf{1},\mathbf{7}} = T(5L - S - 2T). \end{array}\right\}$$

- Note: not all number of hypermultiplets are fixed!

6d Anomaly Cancellation via Green-Schwartz

- Number of multiplets are given by:

$$n_T = 9 - K^2, \quad n_V = \dim G = \dim\ \mathrm{SU}(2) + \dim\ \mathrm{G}_2 = 3 + 14 = 17,$$
$$n_H^0 = h^{2,1}(Y) + 1 = 29K^2 + 15KS + 24KT + 3S^2 + 6ST + 6T^2 + 15.$$

F-theory on Y $\downarrow$ 6d $\mathcal{N} = (1,0)$ sugra	M-theory on Y $\downarrow$ 5d $\mathcal{N} = 1$ sugra	F-theory on $Y \times S^1$ $\downarrow$ 5d $\mathcal{N} = 1$ sugra
$n_V^{(6)} = h^{1,1}(Y) - h^{1,1}(B) - 1$ $n_H^0 = h^{2,1}(Y) + 1$ $n_T = h^{1,1}(B) - 1$	$n_V^{(5)} = n_V^{(6)} + n_T + 1 = h^{1,1}(Y) - 1$ $n_H^0 = h^{2,1}(Y) + 1$	

- Gravitational Anomalies are canceled when

$$n_H - n_V^{(6)} + 29n_T - 273 = 0.$$

- For a semi-simple group with two simple components, $G = G_1 + G_2$, the remainder of the anomaly polynomial is given by $I_8 = \frac{K^2}{8}(\mathrm{tr}R^2)^2 + \frac{1}{6}(X_1^{(2)} + X_2^{(2)})\mathrm{tr}R^2 - \frac{2}{3}(X_1^{(4)} + X_2^{(4)}) + 4Y_{\mathbf{27}}$

where
$$\left\{\begin{aligned} X_a^{(2)} &= \left(A_{a,\mathbf{adj}} - \sum_i n_{\mathbf{R_{i,a}}} A_{\mathbf{R_{i,a}}}\right) \mathrm{tr}_{\mathbf{F_a}} F_a^2, \\ X_a^{(4)} &= \left(B_{a,\mathbf{adj}} - \sum_i n_{\mathbf{R_{i,a}}} B_{\mathbf{R_{i,a}}}\right) \mathrm{tr}_{\mathbf{F_a}} F_a^4 + \left(C_{a,\mathbf{adj}} - \sum_i n_{\mathbf{R_{i,a}}} C_{\mathbf{R_{i,a}}}\right) (\mathrm{tr}_{\mathbf{F_a}} F_a^2)^2, \\ Y_{ab} &= \sum_{i,j} n_{\mathbf{R_{i,a},R_{j,b}}} A_{R_{i,a}} A_{\mathbf{R_{j,b}}} \mathrm{tr}_{\mathbf{F_a}} F_a^2 \mathrm{tr}_{\mathbf{F_b}} F_b^2. \end{aligned}\right\}$$

- If the I_8 factors, then the anomalies are all canceled by Green-Schwartz mechanism.
- We check that all the anomalies are canceled once all the number of hypermultiplets in each representation are identified.

Anomaly Cancellation Condition of the $SU(2)\times G_2$-model

- For $SU(2)\times G_2$-model,

$$n_H - n_V^{(6)} + 29 n_T - 273 = 0, \quad I_8 = \frac{1}{2}\left(\frac{1}{2}K\mathrm{tr}R^2 + 2S\mathrm{tr}_{\mathbf{2}}F_1^2 + T\mathrm{tr}_{\mathbf{7}}F_2^2\right)^2,$$

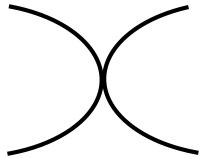
while fixing the number of number of charged hypermultiplets to be

$$\left\{ \begin{array}{l} n_{\mathbf{2},\mathbf{7}} = \frac{1}{2}ST,\ n_{\mathbf{3},\mathbf{1}} = \frac{1}{2}(KS + S^2 + 2),\ n_{\mathbf{2},\mathbf{1}} = -S(8K + 2S + \frac{7}{2}T), \\ n_{\mathbf{1},\mathbf{14}} = \frac{1}{2}(KT + T^2 + 2),\ n_{\mathbf{1},\mathbf{7}} = -T(5K + S + 2T). \end{array} \right\}$$

- This is the unique choice of an anomaly-free theory of a 6d uplift.

Numerical Oddities of $n_{\mathbf{2},\mathbf{1}}$

- The anomaly-free theory carries $n_{\mathbf{2},\mathbf{1}} = -S(8K + 2S + \frac{7}{2}T)$.
- Recall the discriminant of the SU(2)×G_2-model: $\Delta = s^3 t^6 (4f^3 + 27g^2 s)$.
  - The hypermultiplet transforming in (**2**,**1**) is localized at the non-transverse intersection of the divisors S and $\Delta' = (4f^3 + 27g^2 s)$.
  - Then: $n_{\mathbf{2},\mathbf{1}} \overset{?}{=} S \cdot [f] = -S \cdot (4K + S + 2T)$. Does not match.



III

- Look more closely at the curves:
  - The two curves of the fiber III has a projective line and a conic $(y^2 - t^2 s(fx + gts) = 0)$.
  - The discriminant of this conic w.r.t. a projective coordinate [y:s:x] is $\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}t^2 f \\ 0 & -\frac{1}{2}t^2 f & -t^3 g \end{pmatrix} = \frac{1}{4} t^4 f^2$.
  - Then: $n_{\mathbf{2},\mathbf{1}} \overset{?}{=} 2S \cdot [V(f)] = -2S \cdot (4K + S + 2T)$. Does not match.
- Also account for half-hypers in (**2**,**7**) affecting the determinant. So this gives a contribution of $1/2\ ST$.
  - Then: $n_{\mathbf{2},\mathbf{1}} \overset{?}{=} 2S \cdot [V(f)] + \frac{1}{2}ST = -2S \cdot (4K + S + 2T) + \frac{1}{2}ST$. Match!

Thank you for listening! ☺