Notes on Upper Hemi-continuity †

ECON 201B - Game Theory

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These notes are intended to give a discussion about upper and lower hemi-continuity given the importance of the first notion in the proof about the existence of a mixed-strategy Nash Equilibrium in every finite strategic form game.

1 Continuity

First let me define correspondence in formal terms.

Definition 1 *A* ***correspondence*** $\phi : X \rightrightarrows Y$ *is a mapping which associates each* $x \in X$ *with a subset* $\phi(x)$ *of* Y

Two considerations are important here:
1) A correspondence $\phi(x)$ can be empty
2) A correspondence can be regarded as a **function** if $\phi(x)$ consists of one element for each $x \in X$.

Definition 2 $\phi : X \rightrightarrows Y$ *has a* ***closed graph*** *if 1)* $x_n \in X$ *converges to* $x^* \in X$*, 2)* $y_n \in \phi(x_n)$*, and 3)* y_n *converges to* y^**, then* $y^* \in \phi(x^*)$

The notion of continuity most used in economics is upper hemi-continuity

Definition 3 $\phi : X \rightrightarrows Y$ *is* ***upper hemi-continuous (uhc)*** *if 1) it has a closed graph and 2) the image of* ϕ *is compact.*

Note that closed graph property and upper hemi-continuity are equivalent ONLY if the range of ϕ is contained in some compact set $K \subset Y$

However, there is another notion of continuity, which is lower hemi-continuity.

[0] † These notes were prepared as a back up material for TA session. If you have any questions or comments, or notice any errors or typos, please drop me a line at guilord@ucla.edu

Definition 4 *$\phi : X \rightrightarrows Y$ is **lower hemi-continuous (lhc)** if 1) $x_n \in X$ converges to $x^* \in X$ and 2) $y^* \in \phi(x^*)$, then there exists a sequence $\{y_n\}_n$ and N such that $y_n \in \phi(x_n)$ for all $n \geq N$, and y_n converges to y^*.*

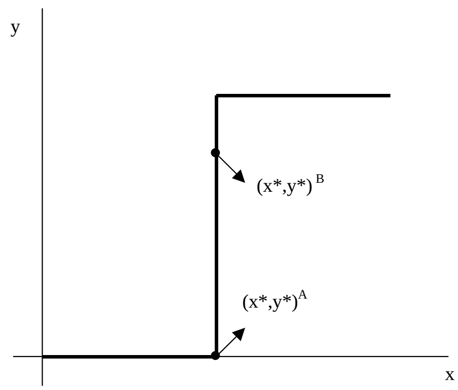
Finally, the concept of continuity considers both concepts

Definition 5 *$\phi : X \rightrightarrows Y$ is **continuous** if it is both upper and lower hemi-continuous*

2 Examples

In order to see how to check if a correspondence is upper hemi-continuous, lower hemi-continuous, both or none, consider the following examples.

The first figure, that certainly will remember you a best response in a typical 2 by 2 game, represents a correspondence that is upper hemi-continuous but not lower hemi-continuous. How can you observe that?.



Let's start following the definition of upper hemi-continuity. First take a sequence $\{x_n\}$ that converges to x^* from the left. Second take the values $y_n \in \phi(x_n)$ (which in this case would be $y_n = 0$ for all x_n). As can be seen $\{y_n\}$ approaches $y^* = 0$ such that $y^* = 0$ belongs to the correspondence of x^* (i.e. $y^* \in \phi(x^*)$ in the point $(x^*, y^*)^A$). Hence the figure corresponds to a closed graph. Furthermore the image of ϕ is compact since it is a closed and bounded set. As a conclusion, this example corresponds to an upper hemi-continuous correspondence.

However this case is not lower hemi-continuous. First, take a point like $(x^*, y^*)^B$ such that $y^* \in \phi(x^*)$. Second, take a sequence $\{x_n\}$ that converges to x^*. Then, it is not possible to find a sequence $y_n \in \phi(x_n)$ such that $\{y_n\}$ approaches to y^* for any $n > N$ big enough.

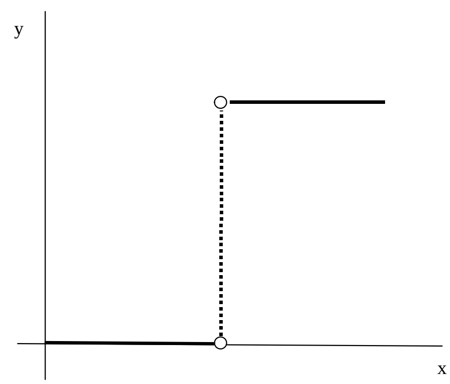
As can be seen, the key to check upper hemi-continuity is to take a sequence of $\{x_n\} \to x^*$ and $\{y_n\} \to y^*$ (such that $y_n \in \phi(x_n)$) and to check whether $y^* \in \phi(x^*)$.

To check lower hemi-continuity it's necessary to take a point $y^* \in \phi(x^*)$ and a sequence of $\{x_n\} \to x^*$ and to check whether it exists a sequence where $y_n \in \phi(x_n)$ and $\{y_n\} \to y^*$

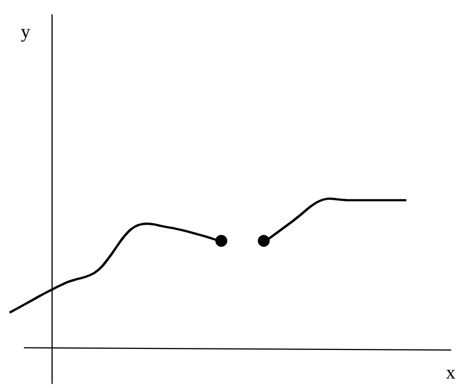
In words, **uhc** shows that any sequence in the correspondence converges to a limit in the correspondence while **lhc** shows that any point in the correspondence can be reached by a sequence in the correspondence. Naturally the combination of these two conditions make a correspondence to be continuous.

In order for you to practice the logic of determining whether a correspondence is uhc, lhc, both or none, here go different cases.

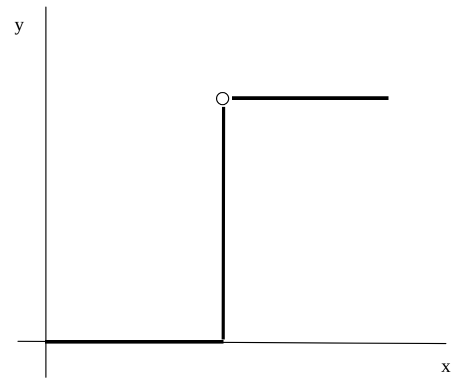
The following example is a case of a correspondence that is lower hemi-continuous but not upper hemi-continuous, Why?



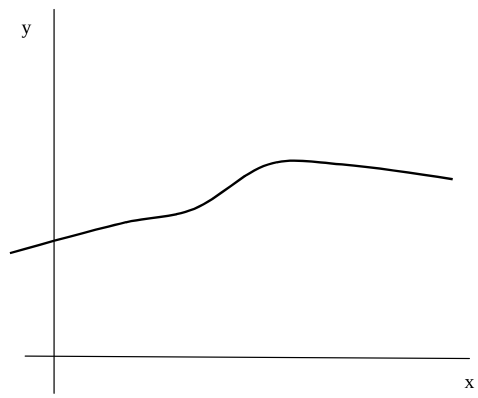
The following example is a case of a correspondence (a function in fact) that is upper hemi-continuous but not lower hemi-continuous, Why?



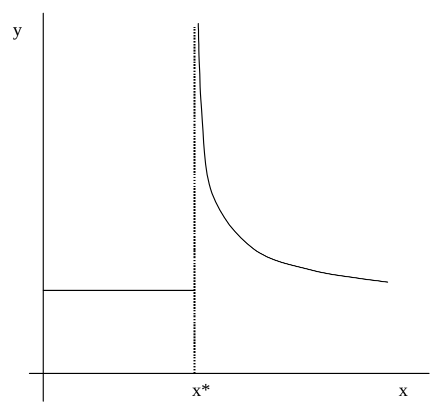
The following example is a case of a correspondence that is not upper hemi-continuous nor lower hemi-continuous, Why?



The following example is a case of a correspondence (a function in fact) that is continuous (i.e. both upper and lower hemi-continuous). Why?



and finally, to see the importance of compactness to define upper hemi-continuity, consider the following example, which is a closed graph but not upper hemi-continuous. Why?



3 Conditions for NE existence

Now we know what is upper hemi-continuity, let's read again the sufficient conditions for a best response correspondence $r : \Sigma \rightrightarrows \Sigma$ to have a fixed point (following Kakutani's theorem) and hence to show the existence of a mixed strategy Nash Equilibrium in every finite strategic form game.

1) Σ is a compact, convex, nonempty subset of a (finite-dimensional) Euclidean space.
2) $r(\sigma)$ is nonempty for all σ
3) $r(\sigma)$ is convex for all σ
4) $r(\cdot)$ is upper hemi-continuous.

As can be seen, even when examples 1, 3 and 5 are upper hemi-continuous, only 1 and 5 satisfy the above conditions for NE existence since example 3 does not fulfil condition 2) of nonemptiness.