Formal Analysis of the Phase Transition.

Define $B(\phi) = BL(\phi) - \alpha \kappa_1$ and $\varepsilon(d|\phi) = (1-\alpha) Pr_G(R(d|\phi), d-1, 1-\alpha)$. By rearranging terms,

$$V(d|\phi) = B(\phi) \times \left( d + \frac{AL(\phi)}{B(\phi)} \right) \times \left( G(R(d|\phi), d, 1-\alpha) - \frac{\kappa_1}{B(\phi)} \right) + \frac{AL(\phi)\kappa_1}{B(\phi)} + dBL(\phi)\varepsilon(d|\phi)$$

The term with $\varepsilon(d|\phi)$ is negligible compared to the other terms. $\frac{AL(\phi)\kappa_1}{B(\phi)}$ is constant with respect to $d$. As $\phi$ increases, $B$ transits from positive to negative at the threshold $\tilde{\phi} = \frac{BD}{\alpha \kappa_1}$. Around this threshold, the optimal $d$ potentially changes dramatically.

The first problem with a formal proof of the characterization of $d^*(\phi)$ is that it is not obvious if (and how) the change happens as the signs of $A$, $k - 1 - \frac{AL(\phi)}{B(\phi)}$ and $1 - \frac{\kappa_1}{B(\phi)}$ all matter for the full characterization of the solution.

The second problem comes from the discrete structure, which gives rise to the term $\varepsilon(d|\phi)$. Under appropriate assumptions $d^*$ can be characterized using the Chernoff Bounds on tails of the binomial CDF, however, the term with $\varepsilon(d|\phi)$ makes a clean proof cumbersome. Without using Chernoff Bounds, or alike, the discreteness in $R$ also makes analyzing $G$ cumbersome. As $d$ changes, $G$ typically evolves monotonically except at the points that $R$ changes with $d$. 

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The third problem is about the constant term that contains $A$. When $A \neq 0$, the expression for $R(d|\phi)$ does not linearly scale with $d$, making the use of Chernoff bounds intractable.

We manage to overcome the first two problems when $A = 0$. Simulations verify the intuition concerning phase transition extends to $A \neq 0$ as well. Moreover, notice that $A = 0$ is the boundary between isolated banks continuing or not continuing so the result encompasses both cases.

The resilience of a bank can be written as

$$R(d|\phi) = d - 1 - S(d|\phi) = d - [(E^{-1}(T(d|\phi)) - D - W)/W]$$

$$= d - \frac{F^{-1}(T(d|\phi)) - D - W}{W} - \epsilon_1(d|\phi)$$

$$= d - d \frac{P\alpha \kappa_t'}{W\alpha \beta L(\phi)} - \frac{PT_0 - S - W}{W} - \epsilon_1(d|\phi)$$

$$= d \left(1 - \alpha + B(\phi)\frac{P}{W\alpha \beta L(\phi)}\right) + \frac{AP}{\alpha \beta W} - \epsilon_1(d|\phi)$$

$$= d \left(1 - \alpha + \mathcal{B}(\phi)\frac{P}{W\alpha \beta L(\phi)}\right) + \epsilon_1(d|\phi)$$

where $\epsilon_1(d|\phi) \in [0, 1)$ is a ceiling term that guarantees that $R(d|\phi)$ is an integer.

**Proposition 1.** Suppose that $A = 0$. $d^*(\phi) = 0$ for all $\phi > \frac{BD}{\kappa^*}$ where

$$\kappa^* = \max \left\{ \kappa_t, \alpha^2 \kappa_t', \min \left\{ \frac{\kappa_t}{1 - \alpha}, \alpha \kappa_t' \right\} \right\}.$$

**Proof.** The condition is equivalent to $\kappa^* > BL$ (we can ignore $\bar{L}$ since it is sufficiently large). If $BL < \alpha^2 \kappa_t$ then $R < 0$ which implies that the terms with $G$ in $V$ are equal to 1. Then $V = d(\mathcal{B} - \kappa_t) \leq 0$ for all $d$. Hence $V$ is maximized at 0. If $BL < \kappa_t$, then $V \leq d(\kappa_t + BL) \leq 0$ for all $d$. So $V$ is maximized at $d = 0$. If $BL < \frac{\kappa_t}{1 - \alpha}$ and $BL < \alpha \kappa_t'$, then $V \leq d(-\kappa_t + BG(R(d), d, 1 - \alpha) + BL\varepsilon(d|\phi)) \leq d(-\kappa_t + BL(1 - \alpha)) \leq 0$ for all $d$. Thus again, $V$ is maximized at $d = 0$. \qed

**Proposition 2.** Suppose that $A = 0$. $d^*(\phi) = k - 1$ for all $\phi < \frac{BD}{\kappa^*}$ where

$$\kappa^{**} = \max \left\{ \frac{\alpha \kappa_t'}{1 - \alpha (\varepsilon' + \sqrt{\varepsilon'^2 + 8 \varepsilon'})}, \frac{\alpha \kappa_t' + \kappa_t}{1 - (k - 1) \exp\left[-(1 - \alpha)(k - 2)\varepsilon'\right]} \right\}$$
for some arbitrarily chosen $\varepsilon' > 0$.

Proof. $V = d(\kappa_l - \alpha \kappa'_i G(R, d, 1 - \alpha) + BLG(R, d - 1, 1 - \alpha)) \leq d(B - \kappa_l + \alpha \kappa'_i \zeta)$ where $\zeta = 1 - G(R, d, 1 - \alpha)$. The Chernoff bounds imply that $\zeta \leq \chi = \text{Exp}\left(-\frac{\delta_l^2}{2 + \delta_l} \mu_d\right)$ where $\mu_d = d(1 - \alpha)$ and $\delta_L$ is given by $(1 + \delta_L) \mu_d = R$ (i.e. $\delta_L = \frac{\alpha}{1 - \alpha} \left(1 - \frac{\alpha \kappa'_i}{B \mu}ight)$). \footnote{Here $A = 0$ plays a key role. Otherwise, $\delta$ would depend on $d$ and it would make the analysis more complicated.}

We first show that if $\frac{B - \kappa_l}{\alpha \kappa'_i} > 0.15$, then the upper bound for $V$, $d(B - \kappa_l + \alpha \kappa'_i \chi)$, is increasing in $d$. The derivative of the continuum version of the upper bound in $d$ is $B - \kappa_l + \alpha \kappa'_i \chi(1 - m)$, where $m = d \frac{\mu_{d}^2 (1 - \alpha)}{2 + \delta_L}$. Notice that $\chi = \text{Exp}(-m)$. Also note that $\frac{m - 1}{\text{Exp}(m)} < 0.15$ for any value of $m > 0$. Hence $\frac{B - \kappa_l}{\alpha \kappa'_i} > 0.15 > (m - 1)\chi$ meaning that the derivative is positive.

Then for any $d \leq k - 2$. $V(d) \leq d(B - \kappa_l + \alpha \kappa'_i \chi_d) \leq (k - 2)(B - \kappa_l + \alpha \kappa'_i \chi_{k - 2})$ Now we find an appropriate lower bound for $V(k - 1)$ and show that $V(d) < V(k - 1)$ for any $d \leq k - 2$. Since $\varepsilon(d|\phi \geq 0)$, $V(k - 1) \geq (k - 1)(\kappa_l + B - B \chi_{k - 1})$. That means we need to prove $\alpha \kappa'_i (k - 2) \chi_{k - 2} + B (k - 1) \chi_{k - 1} \geq B - \kappa_l$. $\chi_d$ is decreasing since $R_d > \mu_d$. Thus, a sufficient condition is $(k - 1) \chi_{k - 2} (B + \alpha \kappa'_i) \leq B - \kappa_l$. Now we prove that if $\phi < \frac{BD}{\varepsilon''}$, the condition holds.

The sufficient condition is equivalent to

$$\frac{\alpha \kappa'_i + \kappa_l}{BL} + (k - 1) \text{Exp}\left(-\frac{(1 - \alpha)(k - 2)}{2 + \delta_L} \frac{\delta_l^2}{2 + \delta_L}\right) \leq 1.$$ 

Both of the summands are in increasing in $L$. Then if $\frac{\delta_l^2}{2 + \delta_L}$ is less than $\varepsilon'$ and also

$$\frac{\alpha \kappa'_i + \kappa_l}{BL} + (k - 1) \text{Exp}\left(-\frac{(1 - \alpha)(k - 2)}{2 + \delta_L} \varepsilon'\right) \leq 1,$$

the sufficient condition holds. The first bound on $\phi$ implies the first (by solving the quadratic), and the second bound on $\phi$ implies the second. (The third bound on $\phi$ corresponds to the condition guaranteeing that the upper bound on $V(d)$ is increasing.) \qed