An Estimated Search-Based Monetary DSGE Model with Liquid Capital

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1 Introduction

In this note we extend the search-based monetary DSGE model studied in Aruoba and Schorfheide (2010) and introduce liquid capital claims. More specifically, buyers in the decentralized market can use a fraction of their capital stock holdings in addition to money to acquire goods from sellers. We show that if liquid capital is a small fraction of the overall capital then money and capital claims can co-exist as a medium of exchange in the decentralized market. Our analysis extends earlier work by Lagos and Rochetau to an environment in which capital is used as a factor of production in the decentralized market. We then estimate our model using Bayesian methods. We find that the estimated fraction of capital that is liquid is close to zero, which means that the data favor the specification without liquid capital, studied in Aruoba and Schorfheide (2010). The model is described in

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Section 2 and estimation results are summarized in Section 3. Further details on the solution of the model as well as a summary of equilibrium conditions and their log-linearizations are provided in the Appendix.

2 The Model

In every period, households first trade in the decentralized market (DM). According to an idiosyncratic taste shock that is realized at the beginning of the period, households become buyers with probability σ, sellers with probability σ, or non-participants with probability $1-2\sigma$. These shocks are independent across time and households. Given that there are equal measures of seller and buyers, we assume there is an efficient matching technology that matches exactly one buyer with one seller. The taste shocks create a double-coincidence problem where frictionless barter cannot occur. All households are anonymous in this market which means that IOUs will not be accepted in trade. In Aruoba and Schorfheide (2010) we assumed that money is the only medium of exchange. Our extended model will relax this assumption.

Let $\tilde{p}_t$ be the price of a unit of goods in the bilateral exchange and q_t the quantity of goods. We assume that in addition to money, up a fraction $0 \leq a \leq 1$ of the capital stock owned by a buyer can potentially be used to acquire goods from the seller. Suppose that the physical capital is installed in factories and the households can issue claims to the installed capital. In bilateral meetings the buyer can exchange up to a fraction a of her claims against the good produced by the seller and faces the constraint

$$\tilde{p}_t q_t = d_t^m + P_t \zeta_t d_t^k, \quad d_t^m \leq m_t^b, \quad d_t^k \leq a k_t^b. \tag{1}$$

We shall refer to ak_t^b as the *liquid* capital claims. Here d_t^m and d_t^k are the amount of money and capital transferred from buyer to seller, whereas m_t^b and k_t^b denote her total holdings of money and capital. Moreover, P_t is the price level in the CM, and ζ_t is the price (in terms of the CM good) at which the transfer of capital in this transaction is valued. It turns out that in equilibrium ζ_t is equal to

$$\zeta_t = R_t^k + \mu_t(1-\delta). \tag{2}$$

and corresponds to the gross return that the buyer receives from holding capital while the CM is open. At this price, both seller and buyer will be indifferent between the mix of assets used to purchase the goods. Here μ_t is the shadow price of installed capital in terms of the CM good, R_t^k is the rental rate of capital, and δ is the depreciation rate. Since only at the end of the CM the buyer is able to re-balance her portfolio of money, bond, and capital claim holdings it turns out that ζ_t is different from μ_t. Throughout this note we assume that the terms of trade $\tilde{p}_t$ and ζ_t in the bilateral exchange are determined through price taking.

A few remarks are in order. First, consider a household that turns out to be a buyer in the DM and enters the period with money holdings m_t^b and capital holdings k_t^b. Both money and a fraction of capital holdings can yield transaction services in the DM. In the subsequent CM the real return of holding money is $1/\pi_t$ and the real rate is R_t^r. Thus, whenever the economy is not at the "Friedman rule" and $R_t^r > 1/\pi_t$, everything else equal, it is preferable to hold claims to capital instead of money. This in turn suggests that for large values of a the economy will be in a non-monetary equilibrium and all the transactions in the DM will be conducted by exchanging goods against claims to the capital stock.

Second, if the fraction of a is small, then a monetary equilibrium will emerge even if money is a return-dominated asset. While in principle the households could accumulate a capital stock that is sufficiently large enough such that the transactions in the DM could be conducted solely with claims to capital even for small values of a, doing so is costly because our production function is such that the marginal product of capital is a decreasing function of capital. In our subsequent empirical analysis we focus on a monetary equilibrium that arises for small values of a, in which the constraint (1) holds with equality.

The main differences between our model and the one analyzed in Lagos and Rochetau (2008) are the following: (i) in our model capital is a factor of production in the DM; (ii) the capital transferred in the bilateral exchange earns a return while the CM is open and before buyers and sellers can re-balance their portfolio of money, bond, and capital holdings; (iii) we introduce New Keynesian frictions in the DM and add stochastic shocks that make the model amenable to likelihood-based estimation.

The remainder of this section is organized as follows. We study the households' decision problem in Section 2.1 under the assumption that money is valued. We demonstrate in Section 2.2 that in a monetary equilibrium in which monetary policy does not follow the Friedman rule it has to be the case that buyers spend all their money, $d_t^m = m_t^b$, and use all their liquid capital claims $d_t^k = ak_t^b$. Section 2.3 describes a non-monetary equilibrium (NME) that always co-exists with the monetary equilibrium (ME). Finally, in Section 2.4 we use a calibrated version of our model to illustrate that for sufficiently large values of a the monetary equilibrium vanishes.

2.1 Households' Decision Problems

The specification and analysis of our extended search-based DSGE model closely follows Aruoba and Schorfheide (2010). We shall focus in our exposition on those parts of the model that are affected by allowing claims to capital to be used in the bilateral exchange. Throughout this note we assume that buyers and sellers in the decentralized market act as price takers.

We use $V_t^{CM}(\hat{m}_t, k_t, i_{t-1}, b_t, \mathcal{S}_t)$ and $V_t^{DM}(m_t, k_t, i_{t-1}, b_t, \mathcal{S}_t)$ to denote the value functions for households in the CM and DM, respectively. The value of being a buyer (seller) in the

DM is denoted by $V_t^b(\cdot)$ ($V_t^s(\cdot)$). The value function for households entering the decentralized market is given by

$$\begin{aligned} V_t^{DM}(m_t, k_t, i_{t-1}, b_t, \mathcal{S}_t) &= \sigma V_t^b(m_t, k_t, i_{t-1}, b_t, \mathcal{S}_t) + \sigma V_t^s(m_t, k_t, i_{t-1}, b_t, \mathcal{S}_t) \\ &\quad +(1-2\sigma)V_t^{CM}(m_t, k_t, i_{t-1}, b_t, \mathcal{S}_t). \end{aligned} \tag{3}$$

The use of claims to capital in the DM leaves the CM problem unchanged. For convenience, we reproduce the first-order conditions associated with the households' CM problem from Aruoba and Schorfheide (2010):

$$U'(x_t) = A/W_t \tag{4}$$

$$U'(x_t)/P_t = \beta \mathbb{E}_t[V_{t+1,m}^{DM}(m_{t+1}, k_{t+1}, i_t, b_{t+1}, \mathcal{S}_{t+1})] \tag{5}$$

$$\begin{aligned} U'(x_t) &= \mu_t U'(x_t)\left[1 - S\left(\frac{i_t}{i_{t-1}}\right) + \frac{i_t}{i_{t-1}} S'\left(\frac{i_t}{i_{t-1}}\right)\right] \\ &\quad +\beta \mathbb{E}_t[V_{t+1,i}^{DM}(m_{t+1}, k_{t+1}, i_t, b_{t+1}, \mathcal{S}_{t+1})] \end{aligned} \tag{6}$$

$$U'(x_t)\mu_t = \beta \mathbb{E}_t[V_{t+1,k}^{DM}(m_{t+1}, k_{t+1}, i_t, b_{t+1}, \mathcal{S}_{t+1})] \tag{7}$$

$$U'(x_t)/P_t = \beta \mathbb{E}_t[V_{t+1,b}^{DM}(m_{t+1}, k_{t+1}, i_t, b_{t+1}, \mathcal{S}_{t+1})] \tag{8}$$

assuming that an interior solution exists. In addition, we have the following envelope conditions,

$$V_{t,m}^{CM}(\hat{m}_t, k_t, i_{t-1}, b_t, \mathcal{S}_t) = U'(x_t)/P_t \tag{9}$$

$$V_{t,k}^{CM}(\hat{m}_t, k_t, i_{t-1}, b_t, \mathcal{S}_t) = U'(x_t)[R_t^k + \mu_t(1-\delta)] \tag{10}$$

$$V_{t,i}^{CM}(\hat{m}_t, k_t, i_{t-1}, b_t, \mathcal{S}_t) = U'(x_t)\mu_t \left(\frac{i_t}{i_{t-1}}\right)^2 S'(i_t/i_{t-1}) \tag{11}$$

$$V_{t,b}^{CM}(\hat{m}_t, k_t, i_{t-1}, b_t, \mathcal{S}_t) = U'(x_t)R_{t-1}. \tag{12}$$

Here R_t is the nominal interest rate and R_t^k is the rental rate of capital.

We now discuss the decision problems of buyers and sellers in the DM in more detail. A buyer spends d_t^m units of money and d_t^k units of capital to acquire q_t units of goods at the price $\tilde{p}_t$ from the seller subject to the following constraints:

$$0 \leq d_t^m \leq m_t^b, \quad 0 \leq d_t^k \leq a k_t^b, \quad q_t \tilde{p}_t = d_t^m + P_t \zeta_t d_t^k. \tag{13}$$

These constraints ensure that the money and capital claims used in the transactions do not exceed the holdings m_t^b and ak_t^b. The inequalities also impose a non-negativity constraint on d_t^m and d_t^k which prevents the buyer from using the transaction to restructure his money and stock portfolio in addition to acquiring goods from the seller. Moreover, the value of money and capital claims exchanged in the transaction equals the value of the goods purchased. We can use the last equality in (13) to express d_t^m as

$$d_t^m = q_t \tilde{p}_t - P_t \zeta_t d_t^k \tag{14}$$

and in turn eliminate d_t^m from the other model equations. The two inequalities in (13) can be rewritten as:

$$P_t \zeta_t d_t^k \leq q_t \tilde{p}_t \leq m_t^b + P_t \zeta_t d_t^k, \quad 0 \leq d_t^k \leq a k_t^b. \tag{15}$$

Using (14) and (15) the value of being a buyer can be written as

$$\begin{aligned} & V_t^b(m_t^b, k_t^b, i_{t-1}, b_t, \mathcal{S}_t) \\ = \quad & \max_{q_t, d_t^k} \left\{ \chi_t u(q_t) + V_t^{CM} \left(m_t^b - \tilde{p}_t q_t + P_t \zeta_t d_t^k, k_t^b - d_t^k, i_{t-1}, b_t, \mathcal{S}_t \right) \right\} \qquad (16) \\ \text{s.t.} \quad & P_t \zeta_t d_t^k \leq \tilde{p}_t q_t \leq m_t^b + P_t \zeta_t d_t^k \quad \text{with multipliers } \tau_{l,t}^m, \tau_{h,t}^m \\ & 0 \leq P_t \zeta_t d_t^k \leq a P_t \zeta_t k_t^b \quad \text{with multipliers } \tau_{l,t}^k, \tau_{h,t}^k. \end{aligned}$$

The first-order conditions for the buyer take the form

$$FOC(q_t) \;:\; 0 = \chi_t u'(q_t) - \tilde{p}_t (V_{t,m}^{CM} + \tau_{h,t}^m - \tau_{l,t}^m) \tag{17}$$

$$FOC(d_t^k) \;:\; 0 = P_t \zeta_t [V_{t,m}^{CM} + (\tau_{h,t}^m - \tau_{l,t}^m) - (\tau_{h,t}^k - \tau_{l,t}^k)] - V_{t,k}^{CM} \tag{18}$$

and the remaining Kuhn-Tucker conditions (assuming $P_t\zeta_t > 0$) are

$$\begin{aligned} 0 &= \tau^m_{h,t}(\tilde{p}_t q_t - m^b_t - P_t\mu_t d^k_t) \\ 0 &= \tau^m_{l,t}(P_t\mu_t d^k_t - \tilde{p}_t q_t) \\ 0 &= \tau^k_{h,t}(d^k_t - ak^b_t) \\ 0 &= \tau_{l,t} d^k_t \end{aligned}$$

and

$$\tau^m_{h,t} \geq 0, \quad \tau^m_{l,t} \geq 0, \quad \tau^k_{h,t} \geq 0, \quad \tau^k_{l,t} \geq 0.$$

Using the expression for $V^{CM}_{t,m}$ in (9) we can rewrite the first-order condition (17) as

$$\chi_t u'(q_t) = \tilde{p}_t \left[U'(x_t)/P_t + \tau^m_{h,t} - \tau^m_{l,t}\right], \tag{19}$$

which can be interpreted as demand function for the goods produced in the DM.

The value function for the seller is given by

$$\begin{aligned} &V^s_t(m^s_t, k^s_t, i_{t-1}, b_t, \mathcal{S}_t) \\ &= \max_{q_t, d^k_t} \left\{-c(q, k^s_t, Z_t) + V^{CM}_t \left(m^s_t + \tilde{p}_t q_t - P_t\zeta_t d^k_t, k^s_t + d^k_t, i_{t-1}, b_t, \mathcal{S}_t\right)\right\} \end{aligned} \tag{20}$$

and the first-order condition takes the form

$$FOC(q_t) \;:\; 0 = -c_q(q_t, k^s_t, Z_t) + \tilde{p}_t V^{CM}_{t,m} \tag{21}$$

$$FOC(d^k_t) \;:\; 0 = -P_t\zeta_t V^{CM}_{t,m} + V^{CM}_{t,k}. \tag{22}$$

Using again the result that $V^{CM}_{t,m} = U'(x_t)/P_t$ we can express the equilibrium price as

$$\tilde{p}_t = c_q(q_t, k^s_t, Z_t)\frac{P_t}{U'(x_t)}, \tag{23}$$

which can be interpreted as the supply function for the DM good.

In order for the seller's first-order condition (22) to be satisfied it has to be the case that

$$\zeta_t = \frac{V^{DM}_{t,k}}{P_t V^{CM}_{t,m}} = R^k_t + (1-\delta)\mu_t. \tag{24}$$

Thus, if the capital transferred in the transaction is valued at ζ_t, the seller is indifferent between the mix of assets used to purchase her goods. Moreover, at this particular ζ_t the first-order condition (18) of the seller simplifies to:

$$\tau^k_{h,t} - \tau^k_{l,t} = \tau^m_{h,t} - \tau^m_{l,t}, \tag{25}$$

equating the shadow price of money and capital from the buyers perspective. Based on this equation we can rule out equilibria in which the buyer spends all her money but not all of her liquid capital, or vice versa, spends all her liquid capital and only a fraction of her money holdings.

Combining the demand equation (19) and supply equation (23) by eliminating $\tilde{p}_t$ yields

$$\chi_t u'(q_t) = c_q(q_t, k^s_t, Z_t)\left[1 + \frac{P_t}{U'(x_t)}(\tau^m_{h,t} - \tau^m_{l,t})\right]. \tag{26}$$

Thus, in the absence of binding constraints, that is $\tau^m_{h,t} = \tau^m_{l,t} = 0$, the quantity exchanged in the bilateral meeting is given by the solution q^*_t of

$$\chi_t u'(q^*_t) = c_q(q^*_t, k^s_t, Z_t).$$

Since the marginal utility derived from the DM good is a decreasing function and the marginal cost is an increasing function of q, we deduce that $q_t < q^*_t$ whenever the buyer is constrained by her money holdings ($\tau^m_{h,t} \geq 0$).[1]

The marginal values of money for buyers and sellers in the DM are

$$V^b_{t,m} = V^{CM}_{t,m} + \tau^m_{h,t}, \quad V^s_{t,m} = V^{CM}_{t,m}$$

and can be combined using (9) to obtain

$$V^{DM}_{t,m} = U'(x_t)/P_t + \sigma\tau^m_{h,t}. \tag{27}$$

[1] Since the capital holdings might be sub-optimal q^*_t is not necessarily the "first-best" amount of goods produced in the DM.

Similarly, the marginal values of capital for buyers and sellers in the DM are

$$V^b_{t,k} = V^{CM}_{t,k} + aP_t\zeta_t\tau^k_{h,t}, \quad V^s_{t,k} = V^{CM}_{t,k} - c_k(q_t, k^s_t, Z_t).$$

Combining these terms and replacing $V^{CM}_{k,t}$ by (10) and ζ_t by(2) leads to

$$V^{DM}_{t,k} = [R^k_t + (1-\delta)\mu_t](U'(x_t) + \sigma aP_t\tau^k_{h,t}) - \sigma c_k(q_t, k^s_t, Z_t). \tag{28}$$

The remaining envelope conditions are directly given by (11) and (12) because

$$V^{DM}_{t,i} = V^{CM}_{t,i}, \quad V^{DM}_{t,b} = V^{CM}_{t,b}. \tag{29}$$

2.2 Binding and Non-Binding Constraints

We now shall discuss the determination of the Lagrange multipliers. First notice that it is not possible that the upper bounds and lower bounds for d^m_t and d^k_t are binding simultaneously. Thus, either $\tau_{l,t}$ or $\tau_{h,t}$ has to be zero. Moreover, due to the non-negativity constraints on the Lagrange multipliers, it is not possible that the lower bound on d^k_t (d^m_t)and the upper bound on d^m_t (d^k_t) are binding simultaneously. Finally, as long as buyers value the goods produced in the DM, it is not possible for both lower bounds to bind simultaneously. Thus, we deduce that $\tau^m_{l,t} = \tau^k_{l,t} = 0$ and therefore (25) implies that

$$\tau^m_{h,t} = \tau^k_{h,t}.$$

With this restriction in mind, we can consider the following two cases. First, suppose that $\tau^m_{h,t} = \tau^k_{h,t} = 0$. Consider the steady state. We deduce from (5) that $\pi_* = \beta$, that is, this equilibrium is only sustainable at the Friedman rule, when both money and claims to capital generate a real return that is equal to the reciprocal of the discount factor.

Second, suppose that $\tau^m_{h,t} = \tau^k_{h,t} > 0$, meaning that the buyer spends all her money and liquid capital: $d^m_t = m^b_t$ and $d^k_t = ak^b_t$. We can use (26) to solve for the Lagrange multipliers

$\tau_{h,t}^m$:

$$\tau_{h,t}^k = \tau_{h,t}^m = \frac{U'(x_t)}{P_t}\left[\frac{\chi_t u'(q_t)}{c_q(q_t, k_t^s, Z_t)} - 1\right].$$

In turn, the marginal values of money and capital take the form

$$\begin{aligned} V_{t,m}^{DM} &= \frac{U'(x_t)}{P_t}\left[\frac{\sigma\chi_t u'(q_t)}{c_q(q_t, k_t^s, Z_t)} + (1-\sigma)\right] \\ V_{t,k}^{DM} &= U'(x_t)[R_t^k + (1-\delta)\mu_t]\left(1 - \sigma a + \sigma a\frac{\chi_t u'(q_t)}{c_q(q_t, k_t^s, Z_t)}\right) - \sigma c_k(q_t, k_t^s, Z_t). \end{aligned}$$

The amount of money needed to conduct the exchange in the DM is given by

$$m_t^b = q_t c_q(q_t, k_t^s, Z_t)\frac{P_t}{U'(x_t)} - aP_t[R_t^k + (1-\delta)\mu_t]k_t^b.$$

Again, consider the steady state. For a monetary equilibrium to exist it is necessary that $m_*^b > 0$. We shall show below that the monetary equilibrium ceases to exist if $a > \bar{a}_{ME}$.

2.3 Non-Monetary Equilibria

In addition to the monetary equilibrium (if it exists) there always is a non-monetary equilibrium in which all DM transactions are conducted with liquid capital claims. In this case

$$q_t \tilde{p}_t = \zeta_t d_t^k,$$

where ζ_t is the value of a claim to a unit of capital. While we could directly absorb ζ_t into the definition of the price $\tilde{p}_t$, we decided to keep ζ_t for now to make the analysis easily comparable to the analysis of the monetary equilibrium. Solving for d_t^k, the value of being a buyer can be written as

$$\begin{aligned} & V_t^b(0, k_t^b, i_{t-1}, b_t, \mathcal{S}_t) \\ = \quad & \max_{q_t} \left\{\chi_t u(q_t) + V_t^{CM}\left(0, k_t^b - q_t(\tilde{p}_t/\zeta_t), i_{t-1}, b_t, \mathcal{S}_t\right)\right\} \qquad (30) \\ \text{s.t.} \quad & 0 \le q_t(\tilde{p}_t/\zeta_t) \le a k_t^b \text{ with multipliers } \tau_{l,t}, \tau_{h,t}. \end{aligned}$$

The first-order conditions for the buyer take the form

$$FOC(q_t): \quad 0 = \chi_t u'(q_t) - \frac{\tilde{p}_t}{\zeta_t} V_{t,k}^{CM} - \frac{\tilde{p}_t}{\zeta_t}(\tau_{h,t} - \tau_{l,t}). \tag{31}$$

Using the expression for $V_{t,k}^{CM}$ in (9) we can rewrite the first-order condition (31) as

$$\chi_t u'(q_t) = \frac{\tilde{p}_t}{\zeta_t}\left(U'(x_t)[R_t^k + \mu_t(1-\delta)] + \tau_{h,t}^m - \tau_{l,t}^m \right), \tag{32}$$

which represents the demand equation for the DM goods in the non-monetary equilibrium.

The value function for the seller is given by

$$\begin{aligned} & V_t^s(0, k_t^s, i_{t-1}, b_t, \mathcal{S}_t) \\ = \; & \max_{q_t} \left\{ -c(q_t, k_t^s, Z_t) + V_t^{CM}\left(0, k_t^s + q_t(\tilde{p}_t/\zeta_t), i_{t-1}, b_t, \mathcal{S}_t\right) \right\} \end{aligned} \tag{33}$$

and the first-order condition takes the form

$$FOC(q_t) \quad : \quad 0 = -c_q(q_t, k_t^s, Z_t) + \frac{\tilde{p}_t}{\zeta_t} V_{t,k}^{CM} \tag{34}$$

We can solve (34) for the equilibrium price, which yields the supply function of the seller

$$\frac{\tilde{p}_t}{\zeta_t} = \frac{c_q(q_t, k_t^s, Z_t)}{U'(x_t)[R_t^k + (1-\delta)\mu_t]}. \tag{35}$$

Equating demand (32) and supply (35) and eliminating the price $\tilde{p}_t/\zeta_t$ yields

$$\chi_t u'(q_t) = c_q(q_t, k_t^s, Z_t)\left[1 + \frac{\tau_{h,t} - \tau_{l,t}}{U'(x_t)[R_t^k + (1-\delta)\mu_t]}\right]. \tag{36}$$

The marginal values of capital for buyers and sellers in the DM are

$$V_{t,k}^b = V_{t,k}^{CM} + a\tau_{h,t}, \quad V_{t,k}^s = V_{t,k}^{CM} - c_k(q_t, k_t^s, Z_t).$$

Combining these terms leads to

$$V_{t,k}^{DM} = U'(x_t)[R_t^k + (1-\delta)\mu_t] + \sigma a \tau_{h,t} - \sigma c_k(q_t, k_t^s, Z_t). \tag{37}$$

Since households do value the consumption of DM goods for the parameterizations that we consider, the constraint $0 \leq d_t^k$, is not binding and therefore $\tau_{l,t} = 0$. Thus, the Lagrange multiplier for the upper bound on d_t^k is given by

$$\tau_{h,t} = U'(x_t)[R_t^k + (1-\delta)\mu_t] \left(\frac{\chi_t u'(q_t)}{c_q(q_t, k_t^s, Z_t)} - 1 \right) \geq 0. \tag{38}$$

If the Lagrange multiplier $\tau_{h,t}$ is strictly positive then the constraint $d_t^k \leq a k_t^b$ is binding. Thus, we can express the transaction price as

$$\frac{\tilde{p}_t}{\zeta_t} = a k_t^b / q_t$$

implying that

$$c_q(q_t, k_t^s, Z_t) = \frac{a k_t^b}{q_t} U'(x_t)[R_t^k + (1-\delta)\mu_t]. \tag{39}$$

The left-hand-side reflects the seller's marginal cost of producing a DM good and the right-hand-side captures her marginal benefit (in terms of utility units). In turn, the marginal value of capital is given by

$$V_{t,k}^{DM} = U'(x_t)[R_t^k + (1-\delta)\mu_t] \left(1 - \sigma a + \sigma a \frac{\chi_t u'(q_t)}{c_q(q_t, k_t^s, Z_t)} \right) - \sigma c_k(q_t, k_t^s, Z_t). \tag{40}$$

If a is large not all the liquid capital is needed to conduct the DM transactions. In this case the non-negativity constraint on the Lagrange multiplier becomes binding, that is, $\tau_{h,t} = 0$ and q_t is to determined by equating the marginal benefit of consuming a unit of the DM good with the marginal cost of producing it:

$$\chi_t u'(q_t) = c_q(q_t, k_t^s, Z_t) = 0. \tag{41}$$

The marginal value of capital in the DM simplifies to

$$V_{t,k}^{DM} = U'(x_t)[R_t^k + (1-\delta)\mu_t] - \sigma c_k(q_t, k_t^s, Z_t). \tag{42}$$

2.4 Calibration

We calibrate the model with liquid capital claims using the posterior mean parameter estimates for the price taking version of the search-based DSGE model reported in Aruoba and Schorfheide (2010). We then compute the steady states for the monetary and the non-monetary equilibrium as a function of the parameter a. The results are summarized in Figure 1. For values of a below 0.057 (to the left of the dotted vertical line) real money balances in the ME are non-negative, meaning that a monetary equilibrium exists. As discussed below, in this case buyers spend all their liquid assets in the DM transaction. At the same time there also exists a non-monetary equilibrium. The DM output in the monetary equilibrium exceeds the output in the non-monetary equilibrium because more liquid assets are available to facilitate the bilateral exchange. To sustain a non-monetary equilibrium, the economy has to accumulate a large capital stock, which is decreasing in the fraction of capital claims that are liquid.

For values of a greater than 0.057 the monetary equilibrium ceases to exist and all transactions in the DM are conducted with claims to capital. For values of a between 0.057 and 0.07 the liquidity constraint $d_t^k \leq a k_t^b$ is binding and the associated Lagrange multiplier is positive. For values of $a > 0.07$ the liquidity constraint is not binding anymore, the Lagrange multiplier is zero, and the steady states as a function of a are constant.

3 Estimation Results

We now use Bayesian methods to estimate the previously described model with liquid capital. Since real money balances in the U.S. are positive, we assume that the observed data are obtained from a monetary equilibrium treating a as parameter. We restrict the domain of a to values that are consistent with a monetary equilibrium in the steady state. We then loglinearize the equilibrium conditions around this steady state assuming that the constraints $d_t^k = ak_t^b$ and $d_t^k = m_t^b$ are binding in every period. As a prior distribution for a we use a Gamma distribution centered at 0.05 with standard deviation 0.03. The joint prior for all model parameters is truncated to ensure the existence of a unique rational expectations equilibrium in which money is valued. The resulting (truncated) marginal prior for a has a mean of 0.033 and a standard deviation of 0.02.

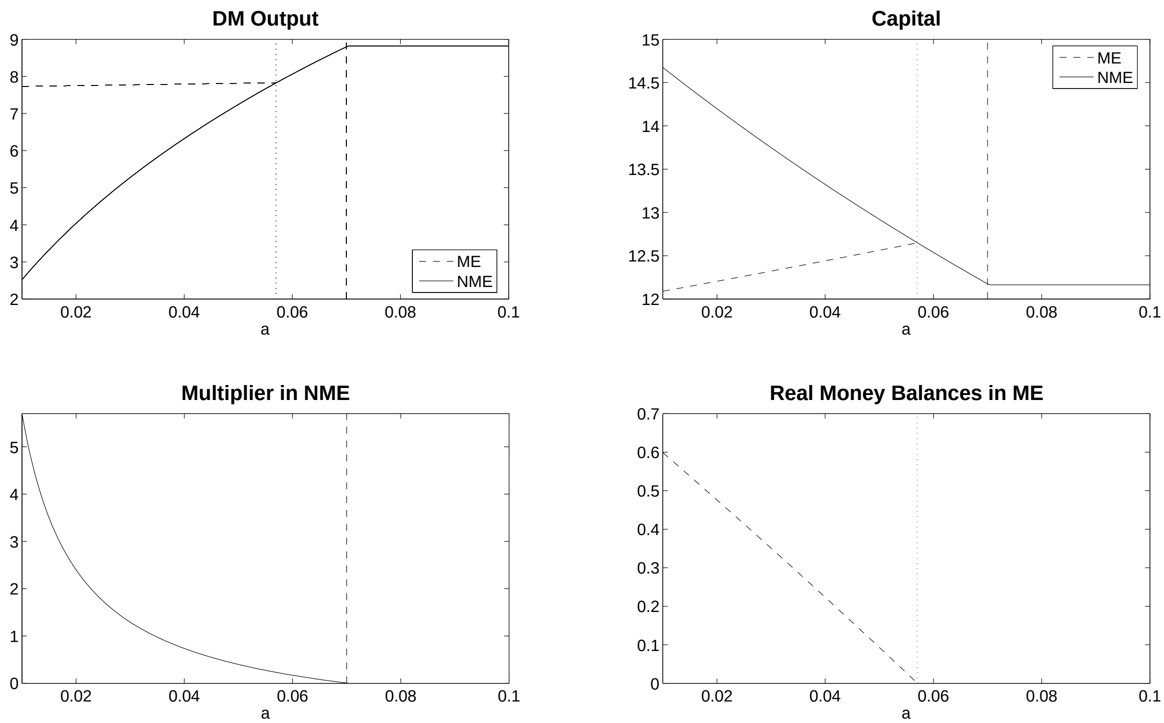
The estimation results are summarized in Table 1. We report posterior mean estimates and 90% credible sets. The columns labeled SBM(LC) contain the parameter estimates of the liquid capital model, whereas the columns SBM(PT) reproduce the estimates of the model without liquid capital reported in Aruoba and Schorfheide (2010). The main result is that the estimate of a is essentially zero and the log Bayes factor, given by the difference of the log marginal data densities reported in the last row of the table, favor the model specification without liquid capital. Given that the posterior of a peaks near zero, the estimates of the remaining parameters are essentially identical.

References

Aruoba, S. Borağan and Frank Schorfheide (2010): "Sticky Prices versus Monetary Frictions: An Estimation of Policy Trade-offs," *Manuscript*, University of Maryland and University of Pennsylvania, Department of Economics.

Lagos, Ricardo and Guillaume Rochetau (2008): "Money and Capital as Competing Media of Exchange," *Journal of Economic Theory*, **142**, 247-258.

Figure 1: Steady States in the Liquidity Model



Notes: The four panels depict the steady states as a function of the fraction a of capital that can be used for DM transactions. For values of a to the left of the dotted vertical line both a monetary (ME) and a non-monetary (NME) equilibrium exist. To the right of the dotted vertical line only the NME exists. To the right of the dashed vertical line the constraint $d^k \leq ak^s$ is not binding anymore.

Table 1: Liquid Capital Model SBM(LC) vs. SBM(PT) Estimates

	SBM(LC)		SBM(PT)	
Name	Mean	90% Intv	Mean	90% Intv
		Household		
a	.001	[.000, .002]	0.00	
θ	0.00	[0.00, 0.00]	0.00	[0.00, 0.00]
$\tilde{\sigma}$	0.59	[0.52, 0.66]	0.59	[0.52, 0.66]
		Firms		
α	0.27	[0.26, 0.28]	0.27	[0.26, 0.28]
λ	0.19	[0.18, 0.21]	0.19	[0.18, 0.21]
ζ	0.83	[0.79, 0.88]	0.84	[0.80, 0.88]
ι	0.54	[0.28, 0.77]	0.57	[0.31, 0.82]
S''	5.50	[2.49, 8.25]	5.08	[2.42, 7.71]
		Central Bank		
ψ_2	0.80	[0.61, 0.98]	0.83	[0.64, 1.02]
ρ_R	0.59	[0.54, 0.65]	0.60	[0.55, 0.65]
σ_R	0.37	[0.31, 0.42]	0.37	[0.31, 0.42]
$\sigma_{R,2}$	0.86	[0.62, 1.09]	0.85	[0.62, 1.08]
$\tilde{\pi}^*_{0,A}$	-0.38	[-4.00, 3.03]	0.02	[-3.22, 3.28]
σ_π	0.05	[0.04, 0.05]	0.05	[0.04, 0.05]
		Shocks		
ρ_g	0.87	[0.83, 0.90]	0.87	[0.83, 0.90]
σ_g	1.06	[0.95, 1.17]	1.06	[0.94, 1.16]
ρ_χ	0.96	[0.95, 0.97]	0.96	[0.95, 0.97]
σ_χ	1.87	[1.67, 2.05]	1.88	[1.70, 2.05]
ρ_z	0.82	[0.75, 0.88]	0.83	[0.77, 0.89]
σ_z	1.05	[0.90, 1.20]	1.06	[0.91, 1.21]
$\ln p(Y)$	-1017.38		-1,007.26	

A Further Details on the Model

A.1 Households' Optimality Conditions

Given exogenous states, policy and prices, $\left\{q_t, X_t, H_t, K_t, I_t, \mu_t, \mathcal{M}_t, \Xi^p_{t+1|t}\right\}_{t=0}^{\infty}$ satisfy

$$W_t = \frac{A}{U'(X_t)} \tag{A.1}$$

$$1 = \beta E_t\left[\frac{U'(X_{t+1})}{U'(X_t)}\frac{R_t}{\pi_{t+1}}\right] \tag{A.2}$$

$$\begin{aligned} 1 &= \mu_t\left[1 - S\left(\frac{I_t}{I_{t-1}}\right) + \frac{I_t}{I_{t-1}}S'\left(\frac{I_t}{I_{t-1}}\right)\right] \\ &\quad + \beta E_t\left\{\mu_{t+1}\frac{U'(X_{t+1})}{U'(X_t)}\left(\frac{I_{t+1}}{I_t}\right)^2 S'\left(\frac{I_{t+1}}{I_t}\right)\right\} \end{aligned} \tag{A.3}$$

$$K_{t+1} = (1-\delta)K_t + \left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_t \tag{A.4}$$

$$\mu_t = \beta E_t\left\{\begin{array}{c} \frac{U'(X_{t+1})}{U'(x_t)}\left(1 - \sigma a + \sigma a\frac{\chi_{t+1}u'\left(q_{t+1}\right)}{c_q(q_{t+1}, K_{t+1}, Z_{t+1})}\right)\left[R^k_{t+1} + (1-\delta)\mu_{t+1}\right] \\ -\frac{\sigma}{U'(X_t)}c_k(q_{t+1}, K_{t+1}, Z_{t+1}) \end{array}\right\} \tag{A.5}$$

$$q_t = \frac{A}{W_t c_q(q_t, K_t, Z_t)}\left[\frac{\mathcal{M}_t}{\pi_t} + a[R^k_t + (1-\delta)\mu_t]K_t\right] \tag{A.6}$$

$$U'(X_t) = \beta E_t\left\{\frac{U'(X_{t+1})}{\pi_{t+1}}\left[\frac{\sigma\chi_{t+1}u'(q_{t+1})}{c_q(q_{t+1}, K_{t+1}, Z_{t+1})} + (1-\sigma)\right]\right\} \tag{A.7}$$

$$\Xi^p_{t+1|t} = \frac{U'(X_{t+1})}{U'(X_t)\pi_{t+1}} \tag{A.8}$$

Note that total GDP can be calculated as

$$\begin{aligned} \mathcal{Y}_t &= Y_t + \frac{\sigma\tilde{p}_t q_t}{P_t} \\ &= Y_t + \frac{\sigma(M_t + aP_t[R^k_t + \mu_t(1-\delta)]K_t)}{P_t} \\ &= Y_t + \frac{\sigma\mathcal{M}_t}{\pi_t} + \sigma a[R^k_t + \mu_t(1-\delta)]K_t \end{aligned} \tag{A.9}$$

A.2 Steady States

First, we solve for the steady state conditional on the parameters A, B and Z_*. Suppose q_*, K_* and H_* are given then we can solve for the following steady states recursively:

$$
\begin{aligned}
R_* &= \pi_*/\beta \\
p^o_* &= \left[\frac{1}{1-\zeta} - \frac{\zeta}{1-\zeta}\left(\frac{\pi_{**}}{\pi_*}\right)^{-\frac{1-\iota}{\lambda}}\right]^{-\lambda} \\
D_* &= \frac{(1-\zeta)(p^o_*)^{-\frac{1+\lambda}{\lambda}}}{1-\zeta\left(\frac{\pi_{**}}{\pi_*}\right)^{-\frac{(1+\lambda)(1-\iota)}{\lambda}}} \\
R^k_* &= \frac{\alpha Z_* p^o_*}{1+\lambda}\left[\frac{1-\zeta\beta\left(\frac{\pi_{**}}{\pi_*}\right)^{-(1-\iota)/\lambda}}{1-\zeta\beta\left(\frac{\pi_{**}}{\pi_*}\right)^{-(1-\iota)(1+\lambda)/\lambda}}\right]^{-1}\left(\frac{H_*}{K_*}\right)^{1-\alpha} \\
W_* &= \frac{1-\alpha}{\alpha}\frac{K_*}{H_*}R^k_* \\
X_* &= \left(\frac{BW_*}{A}\right)^{\frac{1}{\gamma}} \\
\bar{Y}_* &= \left(Z_* K^\alpha_* H^{1-\alpha}_* - \mathcal{F}\right) \\
Y_* &= \frac{1}{D_*}\bar{Y}_* \\
\mathcal{M}_* &= \pi_*\left[\frac{q_* c_q(q_*, K_*, Z_*)W_*}{A} - a[R^k_* + (1-\delta)]K_*\right] \\
\mathcal{Y}_* &= Y_* + \sigma\mathcal{M}_*/\pi_* + \sigma a[R^k_* + (1-\delta)]K \\
\pi^{DM}_* &= \pi^{GDP}_* = \pi_* \\
I_* &= \delta K_*
\end{aligned}
$$

To determine q_*, K_* and H_* we solve the following equations jointly:

$$
\begin{aligned}
R_* &= 1 + \sigma\left[\frac{\chi_* u'(q_*)}{c_q(q_*, K_*, Z_*)} - 1\right] \\
1 &= \beta\left(1 - \sigma a + \sigma a\frac{\chi_* u'(q_*)}{c_q(q_*, K_*, Z_*)}\right)\left[1 + R^k_* - \delta\right] - \sigma\beta\frac{c_k(q_*, K_*, Z_*)}{U'_*} \\
Y_* &= X_* + I_* + G_*
\end{aligned}
$$

In addition, we need to verify the non-negativity of the Lagrange multipliers, which can be done by evaluating the sign of

$$\tilde{\tau}^m_{h,*} = \tilde{\tau}^k_{h,*} = \frac{\chi_* u'(q_*)}{c_q(q_*, k_*, Z_*)} - 1 = \frac{1}{\sigma}(R^* - 1) \geq 0.$$

For a monetary equilibrium to exist we require $\mathcal{M}_* > 0$.

For estimation purposes it is useful to parameterize the model in terms of $\mathcal{Y}_*$, H_*/Y_*, and $\mathcal{M}_*$ and solve the steady state conditions for A, B, and Z_*. Suppose q_*, H_*, and K_* are given then we can solve for the following steady states recursively:

$$\begin{aligned}
R_* &= \pi_*/\beta \\
p^o_* &= \left[\frac{1}{1-\zeta} - \frac{\zeta}{1-\zeta} \left(\frac{\pi_{**}}{\pi_*} \right)^{-\frac{1-\iota}{\lambda}} \right]^{-\lambda} \\
D_* &= \frac{(1-\zeta)(p^o_*)^{-\frac{1+\lambda}{\lambda}}}{1 - \zeta \left(\frac{\pi_{**}}{\pi_*} \right)^{-\frac{(1+\lambda)(1-\iota)}{\lambda}}} \\
Y_* &= (H_*/Y_*)^{-1} H_* \\
\bar{Y}_* &= Y_* D_* \\
Z_* &= (\bar{Y}_* + \mathcal{F})/(K_*^\alpha H_*^{1-\alpha}) \\
R^k_* &= \frac{\alpha Z_* p^o_*}{1+\lambda} \left[\frac{1 - \zeta\beta \left(\frac{\pi_{**}}{\pi_*} \right)^{-(1-\iota)/\lambda}}{1 - \zeta\beta \left(\frac{\pi_{**}}{\pi_*} \right)^{-(1-\iota)(1+\lambda)/\lambda}} \right]^{-1} \left(\frac{H_*}{K_*} \right)^{1-\alpha} \\
W_* &= \frac{1-\alpha}{\alpha} \frac{K_*}{H_*} R^k_* \\
I_* &= \delta K_* \\
X_* &= Y_* - I_* - G_* \\
A &= \frac{q_* c_q(q_*, K_*, Z_*) W_*}{\mathcal{M}_*/\pi_* + a[R^k_* + (1-\delta)]K_*} \\
U'_* &= A/W_* \\
B &= U'_* X_*^\gamma \\
\pi^{DM}_* &= \pi^{GDP}_* = \pi_*
\end{aligned}$$

To determine q_* and K_* we solve the following three equations jointly:

$$\begin{aligned} R_* &= 1 + \sigma \left[\frac{\chi_* u'(q_*)}{c_q(q_*, K_*, Z_*)} - 1 \right] \\ 1 &= \beta \left(1 - \sigma a + \sigma a \frac{\chi_* u'(q_*)}{c_q(q_*, K_*, Z_*)} \right) \left[1 + R^k_* - \delta \right] - \sigma \beta \frac{c_k(q_*, K_*, Z_*)}{U'_*} \\ (H_*/Y_*)^{-1} H_* &= \mathcal{Y}_* - \sigma \mathcal{M}_* / \pi_* - \sigma a [R^k_* + (1-\delta)] K_* \end{aligned}$$

A.3 Log-Linearizations

In the subsequent presentation of the log-linearized equations we adopt the convention that we abbreviate time t expectations of a $t+1$ variable simply by a time $t+1$ subscript, omitting the expectation operator.

Household's Problem: The optimality conditions for the household can be expressed as

$$\tilde{W}_t = \gamma \tilde{X}_t \tag{A.10}$$

$$\tilde{X}_t = \tilde{X}_{t+1} - \frac{1}{\gamma}(\tilde{R}_t - \tilde{\pi}_{t+1}) \tag{A.11}$$

$$\tilde{i}_t = \frac{1}{1+\beta}\tilde{i}_{t-1} + \frac{\beta}{1+\beta}\tilde{i}_{t+1} + \frac{1}{(1+\beta)S''}\tilde{\mu}_t \tag{A.12}$$

$$\tilde{k}_{t+1} = (1-\delta)\tilde{k}_t + \delta \tilde{i}_t \tag{A.13}$$

$$\begin{aligned} \tilde{q}_t + \tilde{c}_{q,t} + \tilde{W}_t &= \frac{\mathcal{M}_*}{\mathcal{M}_* + a\pi_* K_*(R^k_* + 1 - \delta)}[\tilde{\mathcal{M}}_t - \tilde{\pi}_t] + \frac{a\pi_* K_*(R^k_* + 1 - \delta)}{\mathcal{M}_* + a\pi_* K_*(R^k_* + 1 - \delta)}\tilde{k}_t \\ &\quad + \frac{a\pi_* K_* R^k_*}{\mathcal{M}_* + a\pi_* K_*(R^k_* + 1 - \delta)}\tilde{R}^k_t + \frac{a\pi_* K_*(1-\delta)}{\mathcal{M}_* + a\pi_* K_*(R^k_* + 1 - \delta)}\tilde{\mu}_t \end{aligned} \tag{A.14}$$

$$\tilde{R}_t = \frac{R_* - 1 + \sigma}{R_*}[\tilde{\chi}_{t+1} - \tilde{c}_{q,t+1} - \eta \frac{q_*}{(q_* + \kappa)}\tilde{q}_{t+1}] \tag{A.15}$$

$$\tilde{\Xi}^p_{t|t-1} = -\gamma(\tilde{X}_t - \tilde{X}_{t-1}) - \tilde{\pi}_t \tag{A.16}$$

If $\eta = 1$ and $\kappa \approx 0$ we can write the money demand equation as

$$\begin{aligned} (1 - A_1)\tilde{\mathcal{M}}_{t+1} &= -\frac{R_*}{R_* - 1 + \sigma}\tilde{R}_t + \gamma \tilde{X}_{t+1} + (1 - A_1)\tilde{\pi}_{t+1} + \tilde{\chi}_{t+1} \\ &\quad - A_1 \left(\tilde{k}_{t+1} + \frac{R^k_*}{R^k_* + 1 - \delta}\tilde{R}^k_{t+1} + \frac{1-\delta}{R^k_* + 1 - \delta}\tilde{\mu}_{t+1} \right), \end{aligned} \tag{A.17}$$

where

$$A_1 = \frac{a\pi_* K_*(R^k_* + 1 - \delta)}{\mathcal{M}_* + a\pi_* K_*(R^k_* + 1 - \delta)}.$$

If $a = 0$ then $A_1 = 0$ and the variables $\tilde{k}_{t+1}$, $\tilde{R}^k_{t+1}$, and $\tilde{\mu}_{t+1}$ drop from the money demand equation.

We now turn to the log-linearization of

$$\mu_t = \beta E_t \left\{ \begin{array}{c} \frac{U'(X_{t+1})}{U'(X_t)} \left(1 - \sigma a + \sigma a \frac{\chi_{t+1} u'(q_{t+1})}{c_q(q_{t+1}, K_{t+1}, Z_{t+1})}\right) [R^k_{t+1} + (1-\delta)\mu_{t+1}] \\ - \frac{\sigma c_k(q_{t+1}, K_{t+1}, Z_{t+1})}{U'(X_t)} \end{array} \right\}$$

Define the coefficients

$$\begin{aligned} A_1 &= a\sigma\chi_* u'(q_*)/c_q(q_*, K_*, Z_*) = a(R_* - 1 + \sigma) \\ A_2 &= [\beta(1-\sigma a) + \beta A_1](R^k_* + (1-\delta)) = \beta(1 + a(R_* - 1))(R^k_* + (1-\delta)) \\ A_3 &= \beta\sigma c_k(q_*, K_*, Z_*)/U'_* = A_2 - 1 \end{aligned}$$

The log-linearized equation takes the form

$$\begin{aligned} \tilde{\mu}_t - \gamma\tilde{X}_t &= -\gamma A_2 \tilde{X}_{t+1} + A_2 \frac{A_1}{1 - \sigma a + A_1}\left(\tilde{\chi}_{t+1} - \tilde{c}_{q,t+1} - \eta\frac{q_*}{(q_* + \kappa)}\tilde{q}_{t+1}\right) \quad \text{(A.18)} \\ &\quad + A_2 \frac{R^k_*}{R^k_* + (1-\delta)}\tilde{R}^k_{t+1} + A_2\frac{1-\delta}{R^k_* + (1-\delta)}\tilde{\mu}_{t+1} - A_3 \tilde{c}_{k,t+1} \end{aligned}$$

In terms of the CM goods, aggregate output evolves according to

$$\tilde{\mathcal{Y}}_t = \frac{Y_*}{\mathcal{Y}_*}\tilde{Y}_t + \frac{\sigma\mathcal{M}_*/\pi_*}{\mathcal{Y}_*}[\tilde{M}_t - \tilde{\pi}_t] + \frac{\sigma a(R^k_* + 1 - \delta)K_*}{\mathcal{Y}_*}\tilde{K}_t + \frac{\sigma a(1-\delta)K_*}{\mathcal{Y}_*}\tilde{\mu}_t + \frac{\sigma a R^k_* K_*}{\mathcal{Y}_*}\tilde{R}^k_t.$$

Define the DM share as $s_* = [\sigma\mathcal{M}_*/\pi_* + \sigma a(R^k_* + 1 - \delta)K_*]/\mathcal{Y}_*$. Then aggregate GDP evolves according to

$$\tilde{\mathcal{Y}}^{GDP}_t = (1 - s_*)\tilde{Y}_* + s_*\tilde{q}_t.$$

Finally, we will construct a log-linear approximation for the law of motion of DM inflation. Define the price level in the DM as the ratio of nominal and real output:

$$P^{DM}_t = \frac{M_t + aP_t[R^k_t + (1-\delta)\mu_t]K_t}{q_t}.$$

Thus, DM inflation is given by

$$\begin{aligned}
\pi_t^{DM} &= \frac{M_t + aP_t[R_t^k + (1-\delta)\mu_t]K_t}{M_{t-1} + aP_{t-1}[R_{t-1}^k + (1-\delta)\mu_{t-1}]K_{t-1}} \frac{q_{t-1}}{q_t} \\
&= \frac{M_t/P_{t-1} + a(P_t/P_{t-1})[R_t^k + (1-\delta)\mu_t]K_t}{M_{t-1}/P_{t-1} + a[R_{t-1}^k + (1-\delta)\mu_{t-1}]K_{t-1}} \frac{q_{t-1}}{q_t} \\
&= \frac{\mathcal{M}_t + a\pi_t[R_t^k + (1-\delta)\mu_t]K_t}{\mathcal{M}_{t-1}/\pi_{t-1} + a[R_{t-1}^k + (1-\delta)\mu_{t-1}]K_{t-1}} \frac{q_{t-1}}{q_t}
\end{aligned}$$

Define the constants

$$\begin{aligned}
A_1 &= \frac{\mathcal{M}_*/\pi_*}{\mathcal{M}_*/\pi_* + a(R_*^k + 1 - \delta)K_*} \\
A_2 &= \frac{a(R_*^k + 1 - \delta)K_*}{\mathcal{M}_*/\pi_* + a(R_*^k + 1 - \delta)K_*}.
\end{aligned}$$

The log-linearization yields

$$\begin{aligned}
\tilde{\pi}_t^{DM} &= A_1[\Delta\tilde{\mathcal{M}}_t + \tilde{\pi}_{t-1}] - \Delta\tilde{q}_t \\
&\quad + A_2[\tilde{\pi}_t + \Delta\tilde{K}_t] + A_2 \frac{R_*^k}{R_*^k + 1 - \delta} \Delta\tilde{R}_t^k + A_2 \frac{1-\delta}{R_*^k + 1 - \delta} \Delta\tilde{\mu}_t
\end{aligned}$$