Derivations for Examples 2.2 and 2.3

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1 Example 2: Present-Value Model

The law of motion for $y_t = [y_{1,t}, y_{2,t}]'$ is of the form

$$y_t = \begin{bmatrix} 0 & 1/r \\ 0 & 1 \end{bmatrix} + u_t, \quad u_t = A' u_{t-1} + \epsilon_t \tag{1}$$

The variables $y_{1,t}$ and $y_{2,t}$ have to satisfy the following relationship:

$$I\!\!E_{t-1}\Big[y_{1,t} + y_{2,t} - (1+r) y_{1,t} \Big] = 0. \tag{2}$$

Note that

$$y_{1,t} = \frac{1}{r}(y_{2,t} - u_{2,t}) + u_{1,t}$$

Thus,

$$\begin{aligned} y_{1,t} + y_{2,t} - (1+r) y_{1,t-1} &= \frac{1}{r} y_{2,t-1} + A_{11} u_{1,t-1} + A_{21} u_{2,t-1} + \epsilon_{1,t} \\ &\quad + y_{2,t-1} + A_{12} u_{1,t-1} + A_{22} u_{2,t-1} + \epsilon_{2,t} \\ &\quad - (1+r) \left[\frac{1}{r}(y_{2,t-1} - u_{2,t-1}) + u_{1,t-1} \right] \qquad (3) \\ &= \Big[A_{11} + A_{12} - (1+r) \Big] u_{1,t-1} \\ &\quad + \left[A_{21} + A_{22} + \frac{1+r}{r} \right] u_{2,t-1} + \epsilon_{1,t} + \epsilon_{2,t} \qquad (4) \end{aligned}$$

Taking conditional expectations with respect to $t-1$ information yields the restrictions

$$A_{11} + A_{12} = (1+r), \quad r(A_{21} + A_{22}) = -(1+r).$$

2 Example 3: Linear-Quadratic Programming Problems

2.1 General Setup

Let $\{\mathcal{F}_t\}_{t=0}^{\infty}$ denote an increasing sequence of σ-algebras defined on an underlying probability space and $\mathbb{E}_t[\cdot]$ the conditional expectation with respect to $\mathcal{F}_t$. Moreover, assume that $\{\eta_t\}$ is a conditionally homoskedastic martingale difference sequence, which obeys $\mathbb{E}_{t-1}[\eta_t] = 0$ and $\mathbb{E}_{t-1}[\eta_t \eta_t'] = I$. The linear quadratic discounted stochastic regulator problem is to choose a $m \times 1$ control process $\{x_t\}$ adapted to $\{\mathcal{F}_t\}$ to maximize

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t [s_t' R s_t + x_t' Q x_t + 2 x_t' H s_t] \right] \tag{5}$$

subject to

$$s_{t+1} = A s_t + B x_t + E \eta_t \tag{6}$$

$$\mathbb{E}_0 \left[\textstyle\sum_{t=0}^{\infty} \beta^t (|s_t|^2 + |x_t|^2) \right] < \infty \quad . \tag{7}$$

Here s_t is a $l \times 1$ state vector with initialization s_0. The transversality condition (7) can be viewed as an infinite horizon counterpart to a terminal condition on wealth in finite horizon economies. The optimal policy is of the form $x_t = -F s_t$. The matrix F is implicitly given by the following system

$$\begin{aligned} P &= R + \beta A' P A - (\beta A' P B + H')(Q + \beta B' P B)^{-1} (\beta B' P A + H), \\ F &= (Q + \beta B' P B)^{-1} (\beta B' P A + H). \end{aligned} \tag{8}$$

2.2 Model Economy

Consider the following stylized model of consumption behavior. A consumer chooses consumption $\{C_t\}_{t=0}^{\infty}$ to maximize the expected utility

$$-\frac{1}{2}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t (C_t - Z_t - \alpha)^2\right] \tag{9}$$

subject to the constraints

$$\begin{aligned} Z_{t+1} &= \psi C_t + \epsilon_{1,t} && (10)\\ W_{t+1} &= (1+r)(W_t - C_t) + I_{1,t+1} + I_{2,t+1} && (11)\\ I_{1,t+1} &= \mu(1-\phi) + \phi I_{1,t} + \epsilon_{2,t} && (12)\\ I_{2,t+1} &= \epsilon_{3,t} && (13)\end{aligned}$$

2.3 Transformation of Model

Define $\tilde{C}_t = C_t - \alpha/(1-\psi)$ and $\tilde{Z}_t = Z_t - \psi\alpha/(1-\psi)$. Thus,

$$-\frac{1}{2}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t (\tilde{C}_t - \tilde{Z}_t)^2\right] \tag{14}$$

and $\tilde{Z}_{t+1} = \psi\tilde{C}_t + \epsilon_{1,t+1}$. Define $\tilde{I}_{1,t} = I_{1,t} - \mu$, and $\tilde{I}_{2,t} = I_{2,t}$. Thus, $\tilde{I}_{1,t+1} = \phi\tilde{I}_{1,t+1} + \epsilon_{2,t+1}$. The wealth accumulation can be written as

$$\begin{aligned} W_{t+1} &= (1+r)\left(W_t - \left[C_t - \frac{\alpha}{1-\psi}\right] - \frac{\alpha}{1-\psi}\right) + \mu_1 + \mu_2 + \phi\tilde{I}_{2,t} + \epsilon_{2,t+1} + \epsilon_{3,t+1}\\ &= (1+r)W_t - (1+r)\tilde{C}_t + \mu_1 + \mu_2 - \frac{\alpha(1+r)}{1-\psi} + \phi\tilde{I}_{2,t} + \epsilon_{2,t+1} + \epsilon_{3,t+1}. && (15)\end{aligned}$$

Now define

$$\tilde{W}_t = W_t - \frac{1}{r}\left(\frac{\alpha(1+r)}{1-\psi} - \mu_1 - \mu_2\right), \tag{16}$$

and rewrite the wealth accumulation as

$$\tilde{W}_{t+1} = (1+r)\tilde{W}_t - (1+r)\tilde{C}_t + \phi\tilde{I}_{1,t} + \epsilon_{2,t+1} + \epsilon_{3,t+1}. \tag{17}$$

2.4 Solution of the Model

For the remainder of this section we will assume $\psi = 0$. Let $\beta = 1/(1+r)$. Define $x_t = \tilde{C}_t$ and $s_t = [\tilde{W}_t, \tilde{I}_{1,t}, \tilde{Z}_t]'$. The system matrices of the objective function are

$$R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}, \quad Q = -\frac{1}{2}, \quad H = [\ 0 \quad 0 \quad 1/2\ ]. \tag{18}$$

The system matrices of the transition equation are

$$A = \begin{bmatrix} 1/\beta & \phi & 0 \\ 0 & \phi & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1/\beta \\ 0 \\ 0 \end{bmatrix}. \tag{19}$$

We will now solve the Riccati Equation. Note that

$$(Q + \beta B'PB)^{-1} = \frac{2\beta}{2p_{11} - \beta} \tag{20}$$

Moreover

$$\begin{aligned}
\beta A'PA &= \beta \begin{bmatrix} 1/\beta & 0 & 0 \\ \phi & \phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} 1/\beta & \phi & 0 \\ 0 & \phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \beta \begin{bmatrix} p_{11}/\beta & p_{12}/\beta & 0 \\ (p_{11}+p_{21})\phi & (p_{12}+p_{22})\phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\beta & \phi & 0 \\ 0 & \phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} p_{11}/\beta & (p_{11}+p_{12})\phi & 0 \\ (p_{11}+p_{21})\phi & (p_{11}+p_{12}+p_{21}+p_{22})\beta\phi^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

and

$$\beta A'PB + H' = \beta \begin{bmatrix} 1/\beta & 0 & 0 \\ \phi & \phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} -1/\beta \\ 0 \\ 0 \end{bmatrix} + H'$$

$$= \beta \begin{bmatrix} 1/\beta & 0 & 0 \\ \phi & \phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -p_{11}/\beta \\ -p_{21}/\beta \\ -p_{31}/\beta \end{bmatrix} + H'$$

$$= \begin{bmatrix} -p_{11}/\beta \\ -(p_{11}+p_{21})\phi \\ 1/2 \end{bmatrix}$$

Therefore,

$$(\beta A'PB + H')(Q + \beta B'PB)^{-1}(\beta B'PA + H)$$

$$= \frac{2\beta}{2p_{11}-\beta} \begin{bmatrix} -p_{11}/\beta \\ -(p_{11}+p_{21})\phi/ \\ 1 \end{bmatrix} \begin{bmatrix} -p_{11}/\beta & -(p_{11}+p_{21})\phi & 1 \end{bmatrix}$$

$$= \frac{2\beta}{2p_{11}-\beta} \begin{bmatrix} p_{11}^2/\beta^2 & p_{11}(p_{11}+p_{12})\phi/\beta & -p_{11}/\beta \\ p_{11}(p_{11}+p_{21})\phi/\beta & (p_{11}+p_{12})(p_{11}+p_{21})\phi^2 & -(p_{11}+p_{21})\phi \\ -p_{11}\beta & -(p_{11}+p_{21})\phi & 1 \end{bmatrix}$$

Note that F does not depend on p_{13}, p_{22}, p_{23}, and p_{33}. Moreover P is symmetric. Hence, we only have to solve for p_{11} and p_{12}. p_{11} has to satisfy the following relationship:

$$p_{11} = p_{11}/\beta - \frac{2\beta}{2p_{11}-\beta} p_{11}^2/\beta^2 \tag{21}$$

Thus,

$$p_{11}\beta(2p_{11}-\beta) = p_{11}(2p_{11}-\beta) - 2p_{11}^2.$$

Hence,

$$2p_{11}\beta - \beta^2 = -\beta$$

and

$$p_{11} = \frac{1}{2}(\beta - 1) \tag{22}$$

The off-diagonal elements are given by the relationship

$$p_{12} = \phi(p_{11}+p_{12}) - \frac{2\beta}{2p_{11}-\beta} p_{11}(p_{11}+p_{12})\phi/\beta \tag{23}$$

which implies that

$$p_{12} = \frac{1}{2}\frac{\phi\beta(\beta-1)}{1-\phi\beta}. \tag{24}$$

We can now proceed with the calculation of the policy function:

$$\begin{aligned}
F_{11} &= \beta(-2\beta)(-1/\beta^2)\frac{1}{2}(\beta-1) \\
&= (\beta-1) \\
F_{12} &= \beta(-2\beta)(-\phi/\beta)\frac{1}{2}\left[(\beta-1)+\frac{\phi\beta(\beta-1)}{1-\phi\beta}\right] \\
&= \frac{\phi\beta(\beta-1)}{1-\phi\beta} \\
F_{13} &= (-2\beta)\frac{1}{2}
\end{aligned}$$

This leads to

$$\tilde{C}_t = (1-\beta)\tilde{W}_t + \frac{\phi\beta(1-\beta)}{1-\phi\beta}\tilde{I}_{1,t} + \beta\tilde{Z}_t \tag{25}$$

$$= \frac{r}{1+r}\tilde{W}_t + \frac{\phi}{(1+r-\phi)(1+r)}\tilde{I}_{1,t} + \frac{r}{1+r}\tilde{Z}_t \tag{26}$$

2.5 Inverse Transformation

The optimal decision rule for $\psi = 0$ in terms of the original variables is

$$C_t = \frac{1}{1+r}\mu + \frac{r}{1+r}W_t + \frac{\phi r}{(1+r-\phi)(1+r)}(I_{1,t}-\mu_1) + \frac{r}{1+r}\epsilon_{1,t} \tag{27}$$

The law of motion for wealth is

$$\begin{aligned}
W_{t+1} &= \mu + (1+r)W_t - (1+r)C_t + \phi(I_{1,t}-\mu_1) + \epsilon_{2,t+1} + \epsilon_{3,t+1} \\
&= W_t + \frac{\phi(1-\phi)}{1+r-\phi}(I_{1,t}-\mu) - r\epsilon_{1,t} + \epsilon_{2,t+1} + \epsilon_{3,t+1} \qquad (28)
\end{aligned}$$

In terms of lagged state-variables consumption evolves according to

$$\begin{aligned}
C_t &= \frac{1}{1+r}\mu + \frac{r}{1+r}W_{t-1} + \frac{\phi r(2-\phi)}{(1+r-\phi)(1+r)}(I_{1,t-1}-\mu_1) \\
&\quad + \frac{r}{1+r}(\epsilon_{1,t}+\epsilon_{2,t}+\epsilon_{3,t}) - \frac{r^2}{1+r}\epsilon_{1,t-1} \qquad (29)
\end{aligned}$$

We obtain the following system:

$$\begin{bmatrix} C_t \\ \Delta W_t \\ I_{1,t} \end{bmatrix} = \begin{bmatrix} \frac{r}{1+r} & \frac{\mu}{1+r} & \frac{\phi r(2-\phi)}{(1+r-\phi)(1+r)} \\ 0 & 0 & \frac{\phi(1-\phi)}{1+r-\phi} \\ 0 & \mu_1 & \phi \end{bmatrix} \begin{bmatrix} W_{t-1} \\ 1 \\ (I_{1,t-1}-\mu_1) \end{bmatrix} \tag{30}$$
$$+ \begin{bmatrix} \frac{r}{1+r} & \frac{r}{1+r} & \frac{r}{1+r} \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \end{bmatrix} + \begin{bmatrix} -\frac{r^2}{1+r} & 0 & 0 \\ -r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_{1,t-1} \\ \epsilon_{2,t-1} \\ \epsilon_{3,t-1} \end{bmatrix}.$$

2.6 The Unrestricted Model

Define $y_{1,t} = C_t$, $y_{2,t} = [\Delta W_t, I_{1,t}]'$, $x_{1,t} = W_{t-1}$, $x_{2,t} = [1, I_{1,t-1}]'$, $y_t = [y'_{1,t}, y'_{2,t}]'$, and $x_t = [x'_{1,t}, x'_{2,t}]$. The regressor $x_{1,t}$ is I(1), the regressor $x_{2,t}$ is $I(0)$. We consider the an unrestricted model of the form

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} A'_{11} & A'_{21} \\ 0 & A'_{22} \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix} \tag{31}$$

or in matrix form

$$Y = XA + U \tag{32}$$

Define the rotation matrix (and its inverse)

$$R = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & \mu \\ 0 & 0 & 1 \end{array}\right], \quad R^{-1} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & -\mu \\ 0 & 0 & 1 \end{array}\right] \tag{33}$$

The model can be expressed in terms of rotated regressors $\tilde{X}$ and parameters $\tilde{A}$ as follows:

$$Y = XR^{-1}RA + U = \tilde{X}\tilde{A} + U \tag{34}$$

Notice that the rotation preserves the zero restriction $\tilde{A}_{12} = 0$. Define $\nu_t = (X_{t-1} - \mu_1) + \epsilon_{2,t} + \epsilon_{3,t} - \epsilon_{1,t-1}$. Assume that $\frac{1}{\sqrt{T}} \sum_{t=1}^{[Tr]} \nu_t \Longrightarrow B(r) \equiv BM(\Omega)$, where Ω is the long-run variance of ν_t.

2.7 Likelihood Estimation

For notational simplicity we drop the ˜. Suppose that $u_t \sim iid\mathcal{N}(0, \Sigma)$ and that Σ is known. Define $a = vec(A)$,

$$a_1 = vec\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix}, \quad a_2 = vec\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix}, \quad a_{ij} = vec(A_{ij})$$

and $P_* = M \otimes P$, where

$$M = \begin{bmatrix} 0_{y2 \times y1} & I_{y2} \end{bmatrix}, \quad P = [1_{x1}, 0_{x2}]$$

M and P_* are partitioned according to y_t and x_t, respectively. The zero-restriction of the A-matrix can be expressed as $P_* a = 0$. The likelihood function is of the form

$$p(Y|X, A) \propto \exp\left\{ -\frac{1}{2}(a - \tilde{a})'(\Sigma^{-1} \otimes X'X)(a - \tilde{a}) \right\}, \tag{35}$$

where

$$\tilde{a} = (I \otimes (X'X)^{-1}X')vec(Y)$$

subject to the constraint $P_* a = 0$. The maximum likelihood estimator of a is

$$\hat{a} = \tilde{a} - [\Sigma \otimes (X'X)^{-1}]P_*' \Big(P_*[\Sigma \otimes (X'X)^{-1}]P_*' \Big)^{-1} P_* \tilde{a} \tag{36}$$

We will subsequently manipulate the maximum likelihood estimator by exploiting the Kronecker structure of the estimator:

$$\begin{aligned}
\hat{a} &= \tilde{a} - [\Sigma \otimes (X'X)^{-1}][M' \otimes P'] \Big([M \otimes P][\Sigma \otimes (X'X)^{-1}][M' \otimes P'] \Big)^{-1} [M \otimes P]\tilde{a} \\
&= \tilde{a} - [\Sigma \otimes (X'X)^{-1}][M' \otimes P'] \Big(M\Sigma M' \otimes P(X'X)^{-1}P' \Big)^{-1} [M \otimes P]\tilde{a} \\
&= \tilde{a} - [\Sigma \otimes (X'X)^{-1}][M' \otimes P'] \Big(\Sigma_{22}^{-1} \otimes [P(X'X)^{-1}P']^{-1} \Big) [M \otimes P]\tilde{a} \\
&= \tilde{a} - [\Sigma \otimes (X'X)^{-1}] \left(\begin{bmatrix} 0 & 0 \\ 0 & \Sigma_{22}^{-1} \end{bmatrix} \otimes P'[P(X'X)^{-1}P']^{-1}P \right) \tilde{a} \\
&= \tilde{a} - \left(\begin{bmatrix} 0 & \Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_2 \end{bmatrix} \otimes (X'X)^{-1}P'[P(X'X)^{-1}P']^{-1}P \right) \tilde{a}
\end{aligned}$$

Thus,

$$\hat{a}_1 = \tilde{a}_1 - \left(\Sigma_{12}\Sigma_{22}^{-1} \otimes (X'X)^{-1}P'[P(X'X)^{-1}P']P\right)\tilde{a}_2 \quad (37)$$

$$\hat{a}_2 = \tilde{a}_2 - \left(I_2 \otimes (X'X)^{-1}P'[P(X'X)^{-1}P']P\right)\tilde{a}_2 \quad (38)$$

Now we analyze the partitions of $\hat{a}$ seperately. Notice that

$$\tilde{a}_i = [I_i \otimes (X'X)^{-1}X']vec(Y_i) \quad (39)$$

Consider the estimator $\hat{a}_2$. It can be easily verified that the elements of $\hat{a}_2$ that correspond to A_{12} are zero. The elements that correspond to A_{22} are obtained by OLS estimation of $Y_2 = X_2A_{22} + U_2$, that is

$$\hat{a}_{22} = [I_2 \otimes (X_2'X_2)^{-1}X_2']vec(Y_2) \quad (40)$$

Let $\hat{U}_2$ be the OLS residuals of Y_2. We will show that the likelihood estimator of a_1 corresponds to the OLS estimator of the following model

$$Y_1 = X\begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} + \hat{U}_2\Sigma_{22}^{-1}\Sigma_{21} + V \quad (41)$$

The term $\hat{U}_2\Sigma_{22}^{-1}\Sigma_{21}$ reflects the conditional expectation of U_1 given $\hat{U}_2$. Vectorizing the equation yields

$$\begin{aligned} vec(Y_1) &= (I_1 \otimes X)a_1 + (\Sigma_{12}\Sigma_{22}^{-1} \otimes I_T)vec(\hat{U}_2) + vec(V) \quad (42) \\ &= (I_1 \otimes X)a_1 + (\Sigma_{12}\Sigma_{22}^{-1} \otimes I_T)vec(Y_2) - (\Sigma_{12}\Sigma_{22}^{-1} \otimes I_T)(I_2 \otimes X)\hat{a}_2 + vec(V) \end{aligned}$$

Thus,

$$\begin{aligned} \hat{a}_1 &= [I_1 \otimes (X'X)^{-1}X']vec(Y_1) - [I_1 \otimes (X'X)^{-1}X']\Big([\Sigma_{12}\Sigma_{22}^{-1} \otimes I_T]vec(Y_2)\Big) \\ &\quad +[I_1 \otimes (X'X)^{-1}X']\Big([\Sigma_{12}\Sigma_{22}^{-1} \otimes I_T](I_2 \otimes X)\hat{a}_2\Big) \\ &= [I_1 \otimes (X'X)^{-1}X']vec(Y_1) - [\Sigma_{12}\Sigma_{22}^{-1} \otimes (X'X)^{-1}X']vec(Y_2) \\ &\quad +[\Sigma_{12}\Sigma_{22}^{-1} \otimes (X'X)^{-1}X'](I_2 \otimes X)\hat{a}_2 \\ &= \tilde{a}_1 - [\Sigma_{12}\Sigma_{22}^{-1} \otimes (X'X)^{-1}]vec(X'Y_2) + [\Sigma_{12}\Sigma_{22}^{-1} \otimes (X'X)^{-1}X'](I_2 \otimes X)\tilde{a}_2 \end{aligned}$$

$$
\begin{aligned}
& -[\Sigma_{12}\Sigma_{22}^{-1} \otimes (X'X)^{-1}X'](I_2 \otimes X)\Big(I_2 \otimes (X'X)^{-1}P'[P(X'X)^{-1}P']P\Big)\tilde{a}_2 \\
= \quad & \tilde{a}_1 - [\Sigma_{12}\Sigma_{22}^{-1} \otimes (X'X)^{-1}]vec(X'Y_2) \\
& +[\Sigma_{12}\Sigma_{22}^{-1} \otimes (X'X)^{-1}X'](I_2 \otimes X)[I_2 \otimes (X'X)^{-1}]vec(X'Y_2) \\
& -\Big(\Sigma_{12}\Sigma_{22}^{-1} \otimes (X'X)^{-1}P'[P(X'X)^{-1}P']P\Big)\tilde{a}_2 \\
= \quad & \tilde{a}_1 - \Big(\Sigma_{12}\Sigma_{22}^{-1} \otimes (X'X)^{-1}P'[P(X'X)^{-1}P']P\Big)\tilde{a}_2
\end{aligned}
$$

which is identical to the expression in Equation (37).

2.8 Limit Distribution

The limit distribution of $\hat{a}_{22}$ is determined by

$$
\hat{A}_{22} = A_{22} + (X_2'X_2)^{-1}X_2'U_2 \tag{43}
$$

Notice that

$$
\hat{U}_2 = X_2(A_{22} - \hat{A}_{22}) + U_2 \tag{44}
$$

Define the matrix $M_i = I - X_i(X_i'X_i)^{-1}X_i'$. Thus,

$$
\begin{aligned}
\hat{A}_{11} &= (X_1'M_2X_1)^{-1}X_1'M_2[Y_1 - X_2(A_{22} - \hat{A}_{22})\Sigma_{22}^{-1}\Sigma_{21} - U_2\Sigma_{22}^{-1}\Sigma_{21}] \\
&= A_{11} + (X_1'M_2X_1)^{-1}X_1'M_2(U_1 - U_2\Sigma_{22}^{-1}\Sigma_{21})
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\hat{A}_{21} &= (X_2'M_1X_2)^{-1}X_2'M_1[[Y_1 - X_2(A_{22} - \hat{A}_{22})\Sigma_{22}^{-1}\Sigma_{21} - U_2\Sigma_{22}^{-1}\Sigma_{21}] \\
&= A_{21} + (X_2'M_1X_2)^{-1}X_2'M_1(U_1 - U_2\Sigma_{22}^{-1}\Sigma_{21}) + (\hat{A}_{22} - A_{22})\Sigma_{22}^{-1}\Sigma_{21}
\end{aligned}
$$

We now examine the behavior of the sample moments. We are again omitting the ˜ to denote rotated regressor.

$$
\begin{aligned}
\frac{1}{T}X_2'X_2 &\Longrightarrow \mathbb{E}[x_2x_2'] \equiv Q_{22} \\
\frac{1}{T^2}X_1'X_1 &\Longrightarrow \int B_1B_1' \equiv Q_{11} \\
\frac{1}{T^{3/2}}X_2'X_1 &= \frac{1}{T^{3/2}}\sum_{t=1}^{T}\begin{bmatrix} W_t \\ \tilde{X}_{2,t}W_t \end{bmatrix} \Longrightarrow \begin{bmatrix} \int B_1 \\ 0 \end{bmatrix} \equiv Q_{21}
\end{aligned}
$$

$$\begin{aligned}
\frac{1}{T}X_2'M_1X_2 &= \frac{1}{T}X_2'X_2 - \left(\frac{1}{T^{3/2}}X_2'X_1\right)\left(\frac{1}{T^2}X_1'X_1\right)^{-1}\left(\frac{1}{T^{3/2}}X_1'X_2\right) \\
&\Longrightarrow Q_{11} - Q_{21}Q_{11}^{-1}Q_{12} \equiv Q_{11.2} \\
\frac{1}{T^2}X_1'M_2X_1 &= \frac{1}{T^2}X_1'X_1 - \left(\frac{1}{T^{3/2}}X_1'X_2\right)\left(\frac{1}{T}X_2'X_2\right)^{-1}\left(\frac{1}{T^{3/2}}X_2'X_1\right) \\
&\Longrightarrow Q_{22} - Q_{12}Q_{22}^{-1}Q_{21} \equiv Q_{22.1}
\end{aligned}$$

Define $U_{1.2} = U_1 - U_2\Sigma_{22}^{-1}\Sigma_{21}$ with covariance matrix $\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$. Now consider

$$\begin{aligned}
\frac{1}{T^{1/2}}vec(X_2'U_2) &= \frac{1}{T^{1/2}}(I_2 \otimes X_2')vec(U_2) \Longrightarrow \mathcal{MN}\left(0, \Sigma_{22} \otimes Q_{22}^{-1}\right) \\
\frac{1}{T}vec(X_1'M_2U_{1.2}) &= \frac{1}{T}vec(X_1'U_{1.2}) - \frac{1}{T}vec(X_1'X_2(X_2'X_2)^{-1}X_2'U_{1.2}) \\
&= \frac{1}{T}(I_1 \otimes X_1')vec(U_{1.2}) \\
&\quad -\frac{1}{T^{1/2}}\left[I_1 \otimes \left(\frac{1}{T^{3/2}}X_1'X_2\right)\left(\frac{1}{T}X_2'X_2\right)^{-1}X_2'\right]vec(U_{1.2}) \\
&\Longrightarrow \mathcal{MN}\left(0, \Sigma_{11.2} \otimes Q_{11.2}\right) \\
\frac{1}{T^{1/2}}vec(X_2'M_1U_{1.2}) &= \frac{1}{T^{1/2}}vec(X_2'U_{1.2}) - \frac{1}{T^{1/2}}vec(X_2'X_1(X_1'X_1)^{-1}X_1'U_{1.2}) \\
&= \frac{1}{T^{1/2}}(I_1 \otimes X_2')vec(U_{1.2}) \\
&\quad -\frac{1}{T}\left[I_1 \otimes \left(\frac{1}{T^{3/2}}X_2'X_1\right)\left(\frac{1}{T^2}X_1'X_1\right)^{-1}X_1'\right]vec(U_{1.2}) \\
&\Longrightarrow \mathcal{MN}\left(0, \Sigma_{11.2} \otimes Q_{22.1}\right)
\end{aligned}$$

Notice that $\frac{1}{T^{1/2}}vec(X_2'U_2)$ is asymptotically independent of $\frac{1}{T}vec(X_1'M_2U_{1.2})$ and $\frac{1}{T^{1/2}}vec(X_2'M_1U_{1.2})$ because the covariance of $U_{1.2}$ and U_2 is zero by construction. Moreover, $U_{1.2}$ is independent of the random matrices $Q_{11.2}$ and $Q_{22.1}$.

Overall we obtain the following limit distribution for the (rotated) parameters

$$\begin{bmatrix} T(\hat{a}_{11} - a_{11}) \\ \sqrt{T}(\hat{a}_{21} - a_{21}) \\ \sqrt{T}(\hat{a}_{22} - a_{22}) \end{bmatrix} \Longrightarrow \eta^{1/2}\mathcal{N}(0, I), \tag{45}$$

where

$$\eta = \begin{bmatrix} (\Sigma_{11.2} \otimes Q_{11.2}^{-1}) & -(\Sigma_{11.2} \otimes Q_{11.2}^{-1} Q_{12} Q_{22}^{-1}) & 0 \\ -(\Sigma_{11.2} \otimes Q_{22}^{-1} Q_{21} Q_{11.2}^{-1}) & (\Sigma_{11.2} \otimes Q_{22.1}^{-1}) + (\Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \otimes Q_{22}^{-1}) & (\Sigma_{12} \otimes Q_{22}^{-1}) \\ 0 & (\Sigma_{21} \otimes Q_{22}^{-1}) & (\Sigma_{22} \otimes Q_{22}^{-1}) \end{bmatrix}$$

2.9 Time Series Extension

In our application the u_t process is not *iid*. It can be expressed as a moving-average of past ϵ_t's

$$u_t = C(L)\epsilon_t, \quad \epsilon_t \sim iid(0\Sigma_\epsilon) \tag{46}$$

Using the approach of Phillips and Solo (1992), we decompose the MA-process as follows

$$u_t = C(1)\epsilon_t + \tilde{C}(L)(L-1)\epsilon_t \tag{47}$$

In our model u_t is an MA(1) process and the expansion is of the form

$$u_t = (C_0 + C_1)\epsilon_t + C_1(\epsilon_{t-1} - \epsilon_t) \tag{48}$$

The OLS-estimator of a can be interpreted as a pseudo-maximum likelihood estimator if the u_t's follow a linear process. For the estimator to be consistent we have to assume that $\mathbb{E}_{t-1}[x_t u_t'] = 0$, which is the case for our structural model. The limit distribution of the estimator is obtained by replacing Σ with $C(1)\Sigma_\epsilon C(1)'$. Since the stationary component of the model has serially correlated errors, the quasi-maximum likelihood estimator is not fully efficient.