Online Appendix for *Sequential Monte Carlo Sampling for DSGE Models*

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A Proofs For Section 4

A.1 Preliminaries

Throughout this section we will assume that $h(\theta)$ is scalar and we use absolute values $|h|$ instead of a general norm $\|h\|$. Extensions to vector-valued h functions are straightforward.

For the subsequent derivations it is convenient to define the incremental weights

$$v_n(\theta) = \frac{Z_{n-1}}{Z_n}[p(Y|\theta)]^{\phi_n - \phi_{n-1}}.$$

Recall that $Z_n = \int [p(Y|\theta)]^{\phi_n} p(\theta) d\theta$. Thus,

$$\mathbb{E}_{\pi_{n-1}}[h(\theta)v_n(\theta)] = \int h(\theta)\frac{Z_{n-1}}{Z_n}[p(Y|\theta)]^{\phi_n-\phi_{n-1}}\frac{1}{Z_{n-1}}[p(Y|\theta)]^{\phi_{n-1}}d\theta = \mathbb{E}_{\pi_n}[h(\theta)]. \tag{A-1}$$

In turn, we can write

$$\tilde{W}_n^i = \frac{v_n(\theta^i_{n-1})W^i_{n-1}}{\frac{1}{N}\sum_{i=1}^N v_n(\theta^i_{n-1})W^i_{n-1}}$$

and re-express $\tilde{h}_{n,N}$ in (6) as

$$\tilde{h}_{n,N} = \frac{\frac{1}{N}\sum_{i=1}^N h(\theta^i_{n-1})v_n(\theta^i_{n-1})W^i_{n-1}}{\frac{1}{N}\sum_{i=1}^N v_n(\theta^i_{n-1})W^i_{n-1}}. \tag{A-2}$$

We will make repeated use of the following moment bound for $r > 1$

$$\begin{aligned}\mathbb{E}\big[|X - \mathbb{E}[X]|^r\big] &\leq 2^{r-1}\big(\mathbb{E}\big[|X|^r\big] + |\mathbb{E}[X]|^r\big) \\ &\leq 2^r\mathbb{E}\big[|X|^r\big].\end{aligned} \tag{A-3}$$

The first inequality follows from the C_r inequality and the second inequality follows from Jensen's inequality. We will also use the fact that if $h \in \mathcal{H}_2$, then there exists a $\delta^* > 0$ such that $\|h\|^{2+\delta_*} \in$

$\mathcal{H}_1$. Recall that $\mathcal{H}_2$ is defined such that $h \in \mathcal{H}_2$ implies that there exists a $\delta > 0$ such that $\int \|h(\theta)\|^{2+\delta} p(\theta) d\theta < \infty$. Now let $\delta_* = \delta/(1+\epsilon)$ where $\epsilon > 0$. Thus, we can find $\tilde{\delta} > 0$ such that

$$\left(\|h(\theta)\|^{2+\delta/(1+\epsilon)}\right)^{1+\tilde{\delta}} = \|h(\theta)\|^{2+\delta},$$

which shows that $\|h\|^{2+\delta_*} \in \mathcal{H}_1$.

Finally, we will make use of a central limit theorem of the following form:

Theorem 5 *Let $\mathcal{F}_N$ be a sequence of σ-algebras. Suppose the sequence of random variables X_i has the properties*

(i) $\mathbb{E}[X_i|\mathcal{F}_N] = 0$*;*

(ii) $\mathbb{E}[X_i^2|\mathcal{F}_N] = \sigma_i^2$

(iii) $\frac{1}{N}\sum_{i=1}^N \sigma_i^2 \xrightarrow{a.s.} \sigma^2$

(iv) There exists a $\delta > 0$ and a sequence of random variables M_N such that $\frac{1}{N}\sum_{i=1}^N \mathbb{E}[|X_i|^{2+\delta}|\mathcal{F}_N] \leq M_N \xrightarrow{a.s.} \Delta < \infty$.

Let $\mathcal{N}$ denote the set of events for which the almost sure convergence in (iii) and (iv) fails. Then, except for events in $\mathcal{N}$:

$$\frac{1}{\sqrt{N}} \sum_{i=1}^N X_i \,\Big|\, \mathcal{F}_N \Longrightarrow N(0, \sigma^2).$$

It can be verified that the conditions stated in the Theorem 5 imply that the Liapounov conditions are satisfied. One can also verify that the regularity conditions for the CLT presented in Theorem 7.19 of Pollard (2002) are satisfied.

A.2 Correction Step

Proof of Theorem 1: Almost-Sure Convergence. Note that since $p(Y|\theta)$ is bounded and $\phi_n - \phi_{n-1} < 1$

$$\begin{aligned}
\mathbb{E}_{\pi_{n-1}}\left[\left|h(\theta)[p(Y|\theta)]^{\phi_n-\phi_{n-1}}\right|^{2+\delta}\right] &\leq \mathbb{E}_{\pi_{n-1}}\left[\left|h(\theta)\right|^{2+\delta}\left|[p(Y|\theta)]^{\phi_n-\phi_{n-1}}\right|^{2+\delta}\right] \\
&\leq M^{2+\delta}\mathbb{E}_{\pi_{n-1}}\left[\left|h(\theta)\right|^{2+\delta}\right] < \infty.
\end{aligned}$$

Thus, for each $h(\theta) \in \mathcal{H}_1$, $v_n(\theta)h(\theta) \in \mathcal{H}_1$. We deduce from Assumption 2 and (A-1) that

$$\tilde{h}_{n,N} \xrightarrow{a.s.} \frac{\mathbb{E}_{\pi_{n-1}}\big[h(\theta)v_n(\theta)\big]}{\mathbb{E}_{\pi_{n-1}}\big[v_n(\theta)\big]} = \mathbb{E}_{\pi_n}[h(\theta)].$$

Convergence in Distribution. We use a first-order Taylor series expansion of $\tilde{h}_{n,N}$. Notice that the numerator converges to $\mathbb{E}_{\pi_n}[h]$ and the denominator converges to one. Thus,

$$\begin{aligned}
\sqrt{N}\big(\tilde{h}_{n,N} - \mathbb{E}_{\pi_n}[h]\big) &= \sqrt{N}\left(\frac{1}{N}\sum_{i=1}^{N} h(\theta^i_{n-1})v_n(\theta^i_{n-1})W^i_{n-1} - \mathbb{E}_{\pi_n}[h]\right) \\
&\quad -\mathbb{E}_{\pi_n}[h]\sqrt{N}\left(\frac{1}{N}\sum_{i=1}^{N} v_n(\theta^i_{n-1})W^i_{n-1} - 1\right) + o_p(1) \\
&= \sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\big(h(\theta^i_{n-1}) - \mathbb{E}_{\pi_n}[h]\big)v_n(\theta^i_{n-1})W^i_{n-1} + o_p(1) \\
&\Longrightarrow N\big(0, \tilde{\Omega}_n\big),
\end{aligned}$$

where

$$\tilde{\Omega}_n(h) = \Omega_{n-1}\big(v_{n-1}(\theta)\big(h(\theta) - \mathbb{E}_{\pi_n}[h(\theta)]\big)\big).$$

According to Assumption 1, the incremental weights $v_n(\theta)$ are bounded and $v_n(\theta)h(\theta) \in \mathcal{H}_2$. Thus, the convergence follows from Assumption 2. □

A.3 Selection Step

Proof of Theorem 2: Case (ii), $\hat{\rho}_n = 0$ follows directly from Theorem 1. Thus, in the remainder of the proof we focus on Case (i), $\hat{\rho}_n = 1$.

Almost-Sure Convergence. Define $\mathcal{F}_{n-1,N}$ to be the σ-algebra generated by $\{\theta^i_{n-1}, \tilde{W}^i_n\}_{i=1}^N$. Let

$$\mathbb{E}[h(\hat{\theta})|\mathcal{F}_{n-1,N}] = \frac{1}{N}\sum_{i=1}^{N} h(\theta^i_{n-1})\tilde{W}^i_n$$

and write

$$\begin{aligned}
\hat{h}_{n,N} - \mathbb{E}_{\pi_n}[h] &= \frac{1}{N}\sum_{i=1}^{N}\big(h(\hat{\theta}_n^i) - \mathbb{E}[h(\hat{\theta})|\mathcal{F}_{n-1,N}]\big) + \frac{1}{N}\sum_{i=1}^{N}\big(\mathbb{E}[h(\hat{\theta})|\mathcal{F}_{n-1,N}] - \mathbb{E}_{\pi_n}[h]\big) \\
&= \frac{1}{N}\sum_{i=1}^{N}\big(h(\hat{\theta}_n^i) - \mathbb{E}[h(\hat{\theta})|\mathcal{F}_{n-1,N}]\big) + \big(\tilde{h}_{n,N} - \mathbb{E}_{\pi_n}[h]\big) \\
&= I + II, \qquad \text{(A-4)}
\end{aligned}$$

say. Conditional on $\mathcal{F}_{n-1,N}$ the $h(\hat{\theta}_n^i)$'s form a triangular array of random variables that are *iid* within each row with mean $\mathbb{E}[h(\hat{\theta})|\mathcal{F}_{n-1,N}]$. Using a SLLN for triangular arrays of *iid* random variables we obtain $I \stackrel{a.s.}{\longrightarrow} 0$. Moreover, we can deduce from Theorem 1 that $II \stackrel{a.s}{\longrightarrow} 0$ and the statement of the theorem follows.

Convergence in Distribution. Conditional on $\mathcal{F}_{n-1,N}$ term I in (A-4) is an average of random variables with mean zero and centered moments of order r given by

$$\mathbb{E}\left[\big|h(\theta_{n-1}^i - \mathbb{E}[h(\hat{\theta})|\mathcal{F}_{n-1,N}\big|^r\Big|\mathcal{F}_{n-1,N}\right] = \frac{1}{N}\sum_{i=1}^{N}\big|h(\theta_{n-1}^i) - \mathbb{E}[h(\hat{\theta})|\mathcal{F}_{n-1,N}\big|^r\tilde{W}_n^i.$$

Using (A-3), we deduce that for $r = 2+\delta$

$$\begin{aligned}
\mathbb{E}\left[\big|h(\theta_{n-1}^i) - \mathbb{E}[h(\hat{\theta})|\mathcal{F}_{n-1,N}\big|^{2+\delta}\Big|\mathcal{F}_{n-1,N}\right] &\leq 2^{2+\delta}\frac{1}{N}\sum_{i=1}^{N}\big|h(\theta_{n-1}^i)\big|^{2+\delta} \\
&\stackrel{a.s.}{\longrightarrow} \mathbb{E}_{\pi_n}[|h(\theta)|^{2+\delta}] < \infty
\end{aligned}$$

For $h \in \mathcal{H}_2$ there exists a $\delta > 0$ such that $|h|^{2+\delta} \in \mathcal{H}_1$ (see Section A.1). Thus, the almost-sure convergence follows from Theorem 1 and condition (iv) of the CLT in Theorem 5 is satisfied. Therefore,

$$\sqrt{N}\cdot I|\mathcal{F}_{n-1,N} \Longrightarrow N\big(0, \mathbb{V}_\pi[h]\big).$$

Moreover, according to Theorem 1

$$\sqrt{N}\cdot II \Longrightarrow N\big(0, \tilde{\Omega}(h)\big).$$

The two pieces can be spliced together as follows. Consider the characteristic function

$$
\begin{aligned}
&\mathbb{E}\big[\exp\{iu\sqrt{N}(\hat{h}_{n,N} - \mathbb{E}_{\pi_n}[h])\}\big] \\
&\quad= \mathbb{E}\big[\exp\{iu\sqrt{N}\cdot II\}\exp\{iu\sqrt{N}\cdot I\}\big] \\
&\quad= \mathbb{E}\Big[\mathbb{E}\big[\exp\{iu\sqrt{N}\cdot II\}\exp\{iu\sqrt{N}\cdot I\}\big|\mathcal{F}_{n,N}\big]\Big] \\
&\quad= \mathbb{E}\Big[\exp\{iu\sqrt{N}\cdot II\}\mathbb{E}\big[\exp\{iu\sqrt{N}\cdot I\}\big|\mathcal{F}_{n,N}\big]\Big] \\
&\quad= \mathbb{E}\Big[\exp\{iu\sqrt{N}\cdot II\}\exp\{-\mathbb{V}_{\pi_n}[h]u^2/2\}\Big] + \mathcal{R}_{n,N}.
\end{aligned}
$$

The remainder term can be bounded as follows

$$
\begin{aligned}
|\mathcal{R}_{n,N}| &\leq \mathbb{E}\Big[\big|\exp\{iu\sqrt{N}\cdot II\}\big| \\
&\qquad \cdot\Big|\mathbb{E}\big[\exp\{iu\sqrt{N}\cdot I\}\big|\hat{\mathcal{F}}_{n,N}\big] - \exp\{-\mathbb{V}_{\pi_n}[h]u^2/2\}\Big|\Big] \\
&= \mathbb{E}\Big[\Big|\mathbb{E}\big[\exp\{iu\sqrt{N}\cdot I\}\big|\hat{\mathcal{F}}_{n,N}\big] - \exp\{-\mathbb{V}_{\pi_n}[h]u^2/2\}\Big|\Big] \\
&\longrightarrow 0
\end{aligned}
$$

The equality follows from $|\exp\{i\varphi\}| = 1$. Note that $\sqrt{N}\cdot I|\hat{\mathcal{F}}_{n,N} \Longrightarrow N(0, \mathbb{V}_{\pi_n}[h])$ implies that $\mathbb{E}\big[\exp\{iu\sqrt{N}I\}\big|\hat{\mathcal{F}}_{n,N}\big] \longrightarrow \exp\{-\mathbb{V}_{\pi_n}[h]u^2/2\}$. Thus, the convergence to zero follows from the Dominated Convergence Theorem. This leaves us with

$$
\begin{aligned}
&\mathbb{E}\big[\exp\{iu\sqrt{N}(\hat{h}_{n,N} - \mathbb{E}_{\pi_n}[h])\}\big] \\
&\quad= \mathbb{E}\Big[\exp\{iu\sqrt{N}\cdot II\}\Big]\exp\{-\mathbb{V}_{\pi}[h]u^2/2\} + o(1) \\
&\quad\longrightarrow \exp\big\{-\hat{\Omega}_n(h)u^2/2\big\},
\end{aligned}
$$

where

$$
\hat{\Omega}_n(h) = \tilde{\Omega}_n(h) + \mathbb{V}_{\pi_n}[h].
$$

The limit corresponds to the characteristic function of a $N\big(0, \hat{\Omega}_n(h)\big)$ random variable and we obtain the desired convergence in distribution result from the Continuity Theorem. □

A.4 Mutation Step Without Adaption

Proof of Theorem 3. We denote the conditional mean and variance associated with the transition kernel $K_n(\theta|\hat{\theta};\zeta_n)$ by $\mathbb{E}_{K_n(\cdot|\hat{\theta};\zeta_n)}[\cdot]$ and $\mathbb{V}_{K_n(\cdot|\hat{\theta};\zeta_n)}[\cdot]$. Since π_n is the invariant distribution associated with the transition kernel K_n, note that if $\hat{\theta} \sim \pi_n$, then

$$\begin{aligned} \int_{\hat{\theta}} \mathbb{E}_{K_n(\cdot|\hat{\theta};\zeta_n)}[h]\pi_n(\hat{\theta})d\hat{\theta} &= \int_{\hat{\theta}}\int_{\theta} h(\theta)K_n(\theta|\hat{\theta};\zeta_n)d\theta\pi_n(\hat{\theta})d\hat{\theta} \\ &= \int_{\theta} h(\theta)\int_{\hat{\theta}} K_n(\theta|\hat{\theta};\zeta_n)\pi_n(\hat{\theta})d\hat{\theta}d\theta \\ &= \int_{\theta} h(\theta)\pi_n(\theta)d\theta = \mathbb{E}_{\pi_n}[h]. \end{aligned} \tag{A-5}$$

Using the fact that $\frac{1}{N}\sum_{i=1}^N W_n^i = 1$ we can write

$$\begin{aligned} \bar{h}_{n,N} - \mathbb{E}_{\pi_n}[h] &= \frac{1}{N}\sum_{i=1}^N \big(h(\theta_n^i) - \mathbb{E}_{K_n(\cdot|\hat{\theta}_n^i;\zeta_n)}[h]\big)W_n^i + \frac{1}{N}\sum_{i=1}^N \big(\mathbb{E}_{K_n(\cdot|\hat{\theta}_n^i;\zeta_n)}[h] - \mathbb{E}_{\pi_n}[h]\big)W_n^i \\ &= I + II, \end{aligned} \tag{A-6}$$

say. Let $\hat{\mathcal{F}}_{n,N}$ be the σ-algebra generated by $\{\hat{\theta}_n^i, W_n^i\}_{i=1}^N$. Notice that conditional on $\hat{\mathcal{F}}_{n,N}$ the weights W_n^i are known and the summands in term I form a triangular array of random variables that within each row are independently but not identically distributed with mean zero. We will distinguish between Case (i) in which the particles were resampled and Case (ii) in which the particles were not resampled. Throughout the proof we focus on establishing convergence in distribution because it relies on more stringent moment bounds that also suffice to prove the almost-sure convergence.

Case (i), $\hat{\rho}_n = 1$. After the particles have been resampled the weights $W_n^i = 1$ for all i such that

$$I = \frac{1}{N}\sum_{i=1}^N \big(h(\theta_n^i) - \mathbb{E}_{K_n(\cdot|\hat{\theta}_n^i;\zeta_n)}[h]\big).$$

Conditional on $\hat{\mathcal{F}}_{n,N}$ term I is an average of independently and non-identically distributed random variables with distributions given by the transition kernel $K_n(\cdot|\hat{\theta}_n^i;\zeta_n)$. To establish the convergence in distribution of $\sqrt{N}I$ we have to verify condition (iv) of Theorem 5:

$$\frac{1}{N}\sum_{i=1}^N \mathbb{E}_{K_n(\cdot|\hat{\theta}_n^i;\zeta_n)}\big[\big|h(\theta_n^i) - \mathbb{E}_{K_n(\cdot|\hat{\theta}_n^i;\zeta_n)}[h]\big|^{2+\delta}\big] \le 2^{2+\delta}\frac{1}{N}\sum_{i=1}^N \mathbb{E}_{K_n(\cdot|\hat{\theta}_n^i;\zeta_n)}\big[\big|h(\theta)\big|^{2+\delta}\big]. \tag{A-7}$$

In order to deduce the almost-sure convergence of the r.h.s we need to establish that $\psi(\hat{\theta}) = \mathbb{E}_{K_n(\cdot|\hat{\theta};\zeta_n)}\big[|h|^{2+\delta}\big] \in \mathcal{H}_1$:

$$\begin{aligned} \mathbb{E}_{\pi_n}[\psi(\hat{\theta})] &= \mathbb{E}_{\pi_n}\left[\left|\mathbb{E}_{K_n(\cdot|\hat{\theta};\zeta_n)}\big[|h|^{2+\delta}\big]\right|^{1+\eta}\right] \\ &\leq \mathbb{E}_{\pi_n}\left[\mathbb{E}_{K_n(\cdot|\hat{\theta};\zeta_n)}\big[|h|^{(2+\delta)(1+\eta)}\big]\right] \\ &= \mathbb{E}_{\pi_n}\big[|h|^{(2+\delta)(1+\eta)}\big] < \infty \end{aligned} \tag{A-8}$$

for $h \in \mathcal{H}_2$ and suitable choices of δ and η. The final equality follows from the fact that the transition kernel preserves the distribution π_n. Thus, condition (iv) of Theorem 5 is satisfied. We deduce that

$$\sqrt{N} \cdot I \,\big|\, \hat{\mathcal{F}}_{n,N} \implies N\big(0, \mathbb{E}_{\pi_n}\big[\mathbb{V}_{K(\cdot|\theta;\zeta_n)}[h]\big]\big).$$

The convergence of term $\sqrt{N} \cdot II$ follows from Theorem 2 provided that $\big(\mathbb{E}_{K_n(\cdot|\hat{\theta};\zeta_n)}[h] - \mathbb{E}_{\pi_n}[h]\big) \in \mathcal{H}_2$. Using the C_r inequality, Jensen's inequality, and Assumption 3 we find that

$$\mathbb{E}_{\pi_n}\big[|\mathbb{E}_{K_n(\cdot|\hat{\theta};\zeta_n)}[h] - \mathbb{E}_{\pi_n}[h]|^{2+\delta}\big] \leq 2^{1+\delta}\big(\mathbb{E}_{\pi_n}\big[|\mathbb{E}_{K_n(\cdot|\hat{\theta};\zeta_n)}[h]|^{2+\delta}\big] + \mathbb{E}_{\pi_n}\big[|h|^{2+\delta}\big]\big) < \infty.$$

The proof of the theorem can be completed by following the same steps as in the proof of Theorem 2 to obtain the limit distribution of $(\sqrt{N} \cdot I) + (\sqrt{N} \cdot II)$.

Case (ii), $\hat{\rho}_n = 0$. Suppose that the particles were re-sampled for the last time in iteration $n-2$. Thus, the weights in iteration $n-1$ are given by

$$W_{n-1}^i = \tilde{W}_{n-1}^i = \frac{v_{n-1}(\theta_{n-2}^i)}{\frac{1}{N}\sum_{i=1}^N v_{n-1}(\theta_{n-2}^i)}.$$

In the mutation step of iteration $n-1$ the particles values θ_{n-2}^i are turned into $\theta_{n-1}^i \sim K_{n-1}(\theta_{n-1}|\theta_{n-2}^i;\zeta_n)$. The weights in iteration n are given by

$$W_n^i = \tilde{W}_n^i = \frac{v_n(\theta_{n-1}^i)W_{n-1}^i}{\frac{1}{N}\sum_{i=1}^N v_n(\theta_{n-1}^i)W_{n-1}^i} = \frac{v_n(\theta_{n-1}^i)v_{n-1}(\theta_{n-2}^i)}{\frac{1}{N}\sum_{i=1}^N v_n(\theta_{n-1}^i)v_{n-1}(\theta_{n-2}^i)}.$$

Thus, we can write term I as

$$\begin{aligned}\sqrt{N}\cdot I &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\big(h(\theta_n^i)-\mathbb{E}_{K_n(\cdot|\hat{\theta}_n^i;\zeta_n)}[h]\big)v_n(\theta_{n-1}^i)v_{n-1}(\theta_{n-2}^i)\\ &\quad+\left(\frac{1}{\frac{1}{N}\sum_{i=1}^{N}v_n(\theta_{n-1}^i)v_{n-1}(\theta_{n-2}^i)}-1\right)\\ &\quad\times\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\big(h(\theta_n^i)-\mathbb{E}_{K_n(\cdot|\hat{\theta}_n^i;\zeta_n)}[h]\big)v_n(\theta_{n-1}^i)v_{n-1}(\theta_{n-2}^i)\right)\\ &= \sqrt{N}\cdot Ia+\sqrt{N}\cdot Ib,\end{aligned}$$

say. Using Assumption 1, we can bound

$$\begin{aligned}\big|v_n(\theta_{n-1})v_{n-1}(\theta_{n-2})\big| &\leq M^{\phi_n-\phi_{n-2}}\frac{\int[p(Y|\theta)]^{\phi_{n-2}}p(\theta)d\theta}{\int[p(Y|\theta)]^{\phi_n}p(\theta)d\theta}\\ &\leq \frac{M}{\int[p(Y|\theta)]^{\phi_n}p(\theta)d\theta}.\end{aligned} \tag{A-9}$$

Recall that $\phi_n/\phi_2>1$. Using Jensen's inequality, we deduce

$$\int[p(Y|\theta)]^{\phi_n}p(\theta)d\theta=\int\Big([p(Y|\theta)]^{\phi_2}\Big)^{\phi_n/\phi_2}p(\theta)d\theta\geq\left(\int[p(Y|\theta)]^{\phi_2}p(\theta)d\theta\right)^{\phi_n/\phi_2}. \tag{A-10}$$

Thus, combining (A-9) and (A-10) we deduce from Assumption 1(iii)

$$\big|v_n(\theta_{n-1})v_{n-1}(\theta_{n-2})\big|\leq\frac{M}{\big[\int[p(Y|\theta)]^{\phi_2}p(\theta)d\theta\big]^{\phi_n/\phi_2}}<\infty. \tag{A-11}$$

In turn, the moment bound (A-7) derived for *Case (i)* can be adjusted to account for the presence of the non-unity particle weights. Conditioning on $\hat{\mathcal{F}}_{n,N}$:

$$\begin{aligned}&\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}_{K_n(\cdot|\hat{\theta}_n^i;\zeta_n)}\big[\big|\big(h(\theta_n^i)-\mathbb{E}_{K_n(\cdot|\hat{\theta}_n^i;\zeta_n)}[h]\big)v_n(\theta_{n-1}^i)v_{n-1}(\theta_{n-2}^i)\big|^{2+\delta}\big]\\ &\leq 2^{2+\delta}\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}_{K_n(\cdot|\hat{\theta}_n^i;\zeta_n)}\big[\big|h(\theta)\big|^{2+\delta}\big]\cdot\big|v_n(\theta_{n-1}^i)v_{n-1}(\theta_{n-2}^i)\big|^{2+\delta}\big]\\ &\leq 2^{2+\delta}\left(\frac{M}{\big[\int[p(Y|\theta)]^{\phi_2}p(\theta)d\theta\big]^{\phi_n/\phi_2}}\right)^{2+\delta}\left(\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}_{K_n(\cdot|\hat{\theta}_n^i;\zeta_n)}\big[\big|h(\theta)\big|^{2+\delta}\big]\right).\end{aligned}$$

A slight modification of (A-8) establishes that term $\sqrt{N}\cdot Ia$ satisfies condition (iv) of Theorem 5.

We now turn to the analysis of the covariance matrix. Consider

$$\begin{aligned}
& \frac{1}{N}\sum_{i=1}^{N} \mathbb{V}_{K_n(\cdot|\theta^i_{n-1};\zeta_n)}[h] v_n^2(\theta^i_{n-1}) v_{n-1}^2(\theta^i_{n-2}) \\
& \stackrel{a.s.}{\longrightarrow} \int_{\theta_{n-1}}\int_{\theta_{n-2}} \mathbb{V}_{K_n(\cdot|\theta_{n-1};\zeta_n)}[h] v_n^2(\theta_{n-1}) K_{n-1}(\theta_{n-1}|\theta_{n-2};\zeta_n) v_{n-1}^2(\theta_{n-2})\pi_{n-2}(\theta_{n-2}) d\theta_{n-2} d\theta_{n-1} \\
& = \mathbb{E}\big[W^2 \mathbb{V}_{K_n(\cdot|\theta;\zeta_n)}[h]\big].
\end{aligned}$$

The final expression is stated in slight abuse of notation in that the dependence of W on the particles is not spelled out. In sum,

$$\sqrt{N}\cdot Ia \mid \hat{\mathcal{F}}_{n,N} \Longrightarrow N\big(0, \mathbb{E}\big[W\mathbb{V}_{K_n(\cdot|\theta;\zeta_n)}[h]\big]\big)$$

For term $\sqrt{N}\cdot Ib$ note that

$$\begin{aligned}
& \mathbb{E}[v_n(\theta_{n-1})v_{n-1}(\theta_{n-2})] \\
& = \int_{\theta_{n-1}}\int_{\theta_{n-2}} v_n(\theta_{n-1})K_{n-1}(\theta_{n-1}|\theta_{n-2},\zeta_n)v_{n-1}(\theta_{n-2})\pi_{n-2}(\theta_{n-2})d\theta_{n-2}d\theta_{n-1} \\
& = \int_{\theta_{n-1}}\int_{\theta_{n-2}} v_n(\theta_{n-1})K_{n-1}(\theta_{n-1}|\theta_{n-2},\zeta_n)\pi_{n-1}(\theta_{n-2})d\theta_{n-2}d\theta_{n-1} \\
& = \int_{\theta_{n-1}} v_n(\theta_{n-1})\pi_{n-1}(\theta_{n-1})d\theta_{n-1} \\
& = 1.
\end{aligned}$$

Thus, $\sqrt{N}\cdot Ib \stackrel{p}{\longrightarrow} 0$.

The analysis of the term $\sqrt{N}\cdot II$ does not require any additional modification because it is based on Theorem 2. It is straightforward to extend the analysis to the case in which the resampling occurred p iterations ago. □

A.5 Mutation Step With Adaption

Proof of Theorem 4. Using (14), we begin by decomposing $\bar{h}_{n,N}$ as follows:

$$\begin{aligned}
\bar{h}_{n,N} - \mathbb{E}_{\pi_n}[h] &= \frac{1}{N}\sum_{i=1}^{N}\big(h(\theta_n^i) - \mathbb{E}_{K_n(\cdot|\hat{\theta}_n^i;\hat{\zeta}_n)}[h]\big)W_n^i \\
&\quad + \frac{1}{N}\sum_{i=1}^{N}\big(\mathbb{E}_{K_n(\cdot|\hat{\theta}_n^i;\zeta_n)}[h] - \mathbb{E}_{\pi_n}[h]\big)W_n^i + \frac{1}{N}\sum_{i=1}^{N}\big(\Psi_n(\hat{\theta}_n^i,\hat{\zeta}_n;h) - \Psi_n(\hat{\theta}_n^i,\zeta_n;h)\big)W_n^i \\
&= I + II + III,
\end{aligned}$$

say. First, let $\hat{\mathcal{F}}_{n,N}$ be the σ-algebra generated by $\{\hat{\theta}_n^i, W_n^i\}_{i=1}^N$ and note that $\mathcal{F}_{n-1,N} \subseteq \hat{\mathcal{F}}_{n,N}$. Write

$$\begin{aligned}
\sqrt{N}\cdot I &= \frac{\sqrt{\frac{1}{N}\sum_{i=1}^N \mathbb{V}_{K_n(\cdot|\hat{\theta}_n^i;\zeta_n)}[h](W_n^i)^2}}{\sqrt{\frac{1}{N}\sum_{i=1}^N \mathbb{V}_{K_n(\cdot|\hat{\theta}_n^i;\hat{\zeta}_n)}[h](W_n^i)^2}}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\big(h(\theta_n^i) - \mathbb{E}_{K_n(\cdot|\hat{\theta}_n^i;\hat{\zeta}_n)}[h]\big)W_n^i\right) \\
&\quad + \left(1 - \frac{\sqrt{\frac{1}{N}\sum_{i=1}^N \mathbb{V}_{K_n(\cdot|\hat{\theta}_n^i;\zeta_n)}[h](W_n^i)^2}}{\sqrt{\frac{1}{N}\sum_{i=1}^N \mathbb{V}_{K_n(\cdot|\hat{\theta}_n^i;\hat{\zeta}_n)}[h](W_n^i)^2}}\right)\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\big(h(\theta_n^i) - \mathbb{E}_{K_n(\cdot|\hat{\theta}_n^i;\hat{\zeta}_n)}[h]\big)W_n^i\right) \\
&= \sqrt{N}\cdot Ia + \sqrt{N}\cdot Ib.
\end{aligned}$$

Now consider (omitting n subscripts from ζ, θ, and Ψ)

$$\begin{aligned}
\mathbb{V}_{K_n(\cdot|\hat{\theta};\hat{\zeta})}[h] &= \mathbb{E}_{K_n(\cdot|\hat{\theta};\hat{\zeta})}[h^2] - \big(\mathbb{E}_{K_n(\cdot|\hat{\theta};\hat{\zeta})}[h]\big)^2 \\
&= \Psi(\hat{\theta},\hat{\zeta};h^2) - 2h(\hat{\theta})\Psi(\hat{\theta},\hat{\zeta};h) - \Psi^2(\hat{\theta},\hat{\zeta};h).
\end{aligned}$$

Using Assumption 4(ii) we can take a Taylor series approximations of the form

$$\begin{aligned}
\Psi(\hat{\theta},\hat{\zeta};h) &= \Psi(\hat{\theta},\zeta;h) + \Psi_\zeta(\hat{\theta},\zeta;h)(\hat{\zeta}-\zeta) + \frac{1}{2}(\hat{\zeta}-\zeta)'\Psi_{\zeta\zeta}(\hat{\theta},\zeta_*(\hat{\theta});h)(\hat{\zeta}-\zeta) \\
\Psi(\hat{\theta},\hat{\zeta};h^2) &= \Psi(\hat{\theta},\zeta;h^2) + \Psi_\zeta(\hat{\theta},\zeta;h^2)(\hat{\zeta}-\zeta) + \frac{1}{2}(\hat{\zeta}-\zeta)'\Psi_{\zeta\zeta}(\hat{\theta},\zeta_*(\hat{\theta});h^2)(\hat{\zeta}-\zeta).
\end{aligned}$$

Since $\sqrt{N}(\hat{\zeta}_n - \zeta_n) = O_p(1)$ we deduce that

$$\left(1 - \frac{\sqrt{\frac{1}{N}\sum_{i=1}^N \mathbb{V}_{K_n(\cdot|\hat{\theta}_n^i;\zeta_n)}[h](W_n^i)^2}}{\sqrt{\frac{1}{N}\sum_{i=1}^N \mathbb{V}_{K_n(\cdot|\hat{\theta}_n^i;\hat{\zeta}_n)}[h](W_n^i)^2}}\right) = o_p(1).$$

In turn, $\sqrt{N}\cdot Ib = o_p(1)$. The term $\sqrt{N}\cdot Ia$ has the same limit distribution as term I in (A-6).

Second, note that term II above has the same limit distribution as term II in (A-6). Finally, consider term III, which captures $\mathbb{E}_{K_n(\cdot|\hat{\theta}^i_n;\hat{\zeta}_n)}[h] - \mathbb{E}_{K_n(\cdot|\hat{\theta}^i_n;\zeta_n)}[h]$. Using Assumption 4(ii) we can take a Taylor series approximation of the form

$$\Psi(\hat{\theta}, \hat{\zeta}; h) = \Psi(\hat{\theta}, \zeta; h) + \Psi_\zeta(\hat{\theta}, \zeta; h)(\hat{\zeta} - \zeta) + \frac{1}{2}(\hat{\zeta} - \zeta)'\Psi_{\zeta\zeta}(\hat{\theta}, \zeta_*(\hat{\theta}); h)(\hat{\zeta} - \zeta).$$

Thus, re-introducing the n subscripts,

$$\begin{aligned}
\sqrt{N} \cdot III &= \left(\frac{1}{N} \sum_{i=1}^N \Psi_{n,\zeta}(\hat{\theta}^i_n, \zeta_n; h) W^i_n \right) \sqrt{N}(\hat{\zeta}_n - \zeta_n) \\
&\quad + \frac{1}{2\sqrt{N}} \sqrt{N}(\hat{\zeta}_n - \zeta_n) \left(\frac{1}{N} \sum_{i=1}^N \Psi_{n,\zeta\zeta}\big(\hat{\theta}^i_n, \zeta_*(\hat{\zeta}^i_n); h\big) W^i_n \right) \sqrt{N}(\hat{\zeta}_n - \zeta_n) \\
&= o_p(1) O_p(1) + \frac{1}{2\sqrt{N}} O_p(1) O_p(1) O_p(1) = o_p(1).
\end{aligned}$$

The first $o_p(1)$ is obtained from

$$\frac{1}{N} \sum_{i=1}^N \Psi_{n,\zeta}(\hat{\theta}^i_n, \zeta_n; h) W^i_n \xrightarrow{a.s.} \mathbb{E}_{\pi_n}[\Psi_\zeta(\hat{\theta}, \zeta_n; h)] = 0.$$

Here we used

$$0 = \frac{\partial}{\partial \zeta} \mathbb{E}_{\pi_n}[\Psi(\hat{\theta}, \zeta; h)] = \mathbb{E}_{\pi_n}[\Psi_\zeta(\hat{\theta}, \zeta; h)].$$

According to Assumption 4(i) $\sqrt{N}(\hat{\zeta}_n - \zeta_n) = O_p(1)$. This completes the proof. □

Discussion of Assumption 4 for $M = 1$, $N_{blocks} = 1$, and $\alpha = 1$. In this case the mutation consists of a single-block random-walk Metropolis step. The transition density can be expressed as

$$K_n(\theta|\hat{\theta}; \zeta) = \alpha_n(\theta|\hat{\theta}; \zeta) q(\theta|\hat{\theta}; \zeta) + r_n(\hat{\theta}; \zeta)\delta(\theta|\hat{\theta}), \tag{A-12}$$

where $q(\theta|\hat{\theta}; \zeta)$ is the proposal density, $\delta(\theta|\hat{\theta})$ is the dirac function,[17] and

$$\alpha_n(\theta|\hat{\theta}; \zeta) = \min\left\{ 1, \frac{\pi_n(\theta)/q(\theta|\hat{\theta}; \zeta)}{\pi_n(\hat{\theta})/q(\hat{\theta}|\theta; \zeta)} \right\}, \quad r_n(\hat{\theta}; \zeta) = 1 - \int \alpha_n(\theta|\hat{\theta}; \zeta) q(\theta|\hat{\theta}; \zeta) d\theta.$$

The function $r_n(\hat{\theta}; \zeta)$ is the probability that the proposed draw is rejected. For the random-walk Metropolis step $q(\theta|\hat{\theta}; \zeta) = q(\hat{\theta}|\theta; \zeta)$ and the conditional acceptance probability does not depend on

[17] It has the properties that $\delta(\theta|\hat{\theta}) = 0$ for $\theta \neq \hat{\theta}$ and $\int \delta(\theta|\hat{\theta}) d\theta = 1$.

ζ, that is, $\alpha_n(\theta|\hat{\theta};\zeta) = \alpha_n(\theta|\hat{\theta})$. Thus, we can write

$$\alpha_n(\theta|\hat{\theta}) = \min\left\{1, \frac{\pi_n(\theta)}{\pi_n(\hat{\theta})}\right\}, \quad r_n(\hat{\theta};\zeta) = 1 - \int \alpha_n(\theta|\hat{\theta})q(\theta|\hat{\theta};\zeta)d\theta.$$

The function $\Psi(\hat{\theta},\zeta;h)$ is given by

$$\Psi(\hat{\theta},\zeta;h) = \int h(\theta)\alpha_n(\theta|\hat{\theta})q(\theta|\hat{\theta};\zeta)d\theta - h(\hat{\theta})\int \alpha_n(\theta|\hat{\theta})q(\theta|\hat{\theta};\zeta)d\theta. \tag{A-13}$$

The proposal density takes the form

$$q(\theta|\hat{\theta};\zeta) = (2\pi)^{-k/2}|c_1^2\Sigma_*|^{-1/2}\exp\left\{-\frac{1}{2}(\theta-\hat{\theta})'(c_1^2\Sigma_*)^{-1}(\theta-\hat{\theta})\right\} = \exp\big\{l(\theta|\hat{\theta};\zeta)\big\}.$$

Its derivatives are given by

$$q_\zeta(\theta|\hat{\theta};\zeta) = l_\zeta(\theta|\hat{\theta};\zeta)q(\theta|\hat{\theta};\zeta) \quad \text{and} \quad q_{\zeta\zeta}(\theta|\hat{\theta};\zeta) = \big[l_\zeta(\theta|\hat{\theta};\zeta)l_\zeta'(\theta|\hat{\theta};\zeta) + l_{\zeta\zeta}(\theta|\hat{\theta};\zeta)\big]q(\theta|\hat{\theta};\zeta).$$

The derivatives of the log density with respect to ζ, $l_\zeta(\theta|\hat{\theta};\zeta)$, and $l_{\zeta\zeta}(\theta|\hat{\theta};\zeta)$ are polynomial functions of θ and therefore integrable with respect to θ under the Gaussian proposal density $q(\theta|\hat{\theta};\zeta)$. This insight can be used to verify the differentiability condition of Assumption 4.

We now show that $\sqrt{N}(\hat{\zeta}_n - \zeta_n)$ is stochastically bounded. Recall that $\Sigma_n^* = \mathbb{V}_{\pi_n}[\theta]$ and define $\tilde{\Sigma}_n$ as the Monte Carlo approximation of Σ_n^* based on the particles $\{\theta_{n-1}^i, \tilde{W}_n^i\}_{i=1}^N$. We proceed recursively. To be concise, note that the asymptotic variances in Theorems 1 to 3 are functions of ζ_n. Given the recursive structure, we write $\tilde{\Omega}_n(h,\zeta_{2:n-1})$, $\hat{\Omega}_n(h,\zeta_{1:n-1})$, and $\Omega_n(h,\zeta_{2:n})$. According to Theorem 4:

$$\tilde{\Omega}_n(h,\hat{\zeta}_{2:n-1}) = \tilde{\Omega}_n(h,\zeta_{2:n-1}), \quad \hat{\Omega}_n(h,\hat{\zeta}_{2:n-1}) = \hat{\Omega}_n(h,\zeta_{2:n-1}), \quad \Omega_n(h,\hat{\zeta}_{2:n}) = \Omega_n(h,\zeta_{2:n}).$$

For $n=1$ we simply sample from the prior distribution which does not involve any adaptive choice of tuning parameters. Thus, we start with $n=2$. Let $c_2 = \hat{c}_2 = c^*$ such that $\hat{c}_2 - c_2 = 0$. Theorem 1 implies that $\sqrt{N}\big(vech(\tilde{\Sigma}_2) - vech(\Sigma_2^*)\big) = O_p(1)$. We deduce that $\sqrt{N}(\hat{\zeta}_2 - \zeta_2) = O_p(1)$. In turn we deduce from Theorem 3 and Theorem 4 that Assumption 2 is satisfied for $n=3$ with $\Omega_2(h,\zeta_2)$.

Now consider $n>2$ and assume that $\sqrt{N}(\hat{\zeta}_{n-1} - \zeta_{n-1}) = O_p(1)$ and Assumption 2 holds with

$\Omega_{n-1}(h, \zeta_{2:n-1})$. Conditional on Assumption 2 the proof of Theorem 1 is not affected by the adaptive choice of the transition kernel tuning parameters in the mutation step and the asymptotic covariance matrix is given by $\tilde{\Omega}_n(h, \zeta_{2:n-1})$. Thus, we deduce that $\sqrt{N}\big(vech(\tilde{\Sigma}_n) - vech(\Sigma_n^*)\big) = O_p(1)$.

Recall the definitions

$$\hat{c}_n = \hat{c}_{n-1}\hat{R}_{n-1}(\hat{\zeta}_{n-1}) \quad \text{and} \quad c_n = c_{n-1}R_{n-1}(\zeta_{n-1}),$$

where $R_{n-1}(\zeta_{n-1}) = \mathbb{E}_{\pi_{n-1}}\big[r_{n-1}(\theta_{n-1}, \zeta_{n-1})\big]$. A Taylor series approximation of $\hat{c}_n$ yields that

$$\begin{aligned}
\sqrt{N}(\hat{c}_n - c_n) &= c_{n-1}\sqrt{N}(\hat{c}_{n-1} - c_{n-1}) + R_{n-1}\sqrt{N}\big(\hat{R}_{n-1}(\hat{\zeta}_{n-1}) - R_{n-1}(\zeta_{n-1})\big) \\
&\quad + o_p\big(\sqrt{N}(\hat{c}_{n-1} - c_{n-1})\big) + o_p\big(\sqrt{N}(\hat{R}_{n-1} - R_{n-1})\big).
\end{aligned}$$

By the inductive assumption, $\sqrt{N}(\hat{c}_{n-1} - c_{n-1}) = O_p(1)$. Thus, it remains to be verified that $\sqrt{N}\big(\hat{R}_{n-1}(\hat{\zeta}_{n-1}) - R_{n-1}(\zeta_{n-1})\big)$ is also $O_p(1)$.

We begin with a Taylor series approximation of

$$\begin{aligned}
r_n(\hat{\theta}; \hat{\zeta}) &= r_n(\hat{\theta}; \zeta) - \left[\int \alpha_n(\theta|\hat{\theta}) q_\zeta(\theta|\hat{\theta}; \zeta) d\theta\right](\hat{\zeta} - \zeta) - \frac{1}{2}(\hat{\zeta} - \zeta)\left[\int \alpha_n(\theta|\hat{\theta}) q_{\zeta\zeta}(\theta|\hat{\theta}; \zeta_*) d\theta\right](\hat{\zeta} - \zeta) \\
&= r_n(\hat{\theta}; \zeta) + A_1(\hat{\theta}, \zeta)(\hat{\zeta} - \zeta) + \frac{1}{2}(\hat{\zeta} - \zeta)' A_2(\hat{\theta}, \zeta_*)(\hat{\zeta} - \zeta).
\end{aligned}$$

Using this Taylor series approximation we obtain the decomposition

$$\begin{aligned}
\sqrt{N}\big(\hat{R}_{n-1}(\hat{\zeta}_{n-1}) - R_{n-1}(\zeta_{n-1})\big) &= \sqrt{N}\left(\frac{1}{N}\sum_{i=1}\{\theta_{n-1}^i = \hat{\theta}_{n-1}^i\} - \frac{1}{N}\sum_{i=1}^N r_{n-1}(\hat{\theta}_{n-1}^i; \hat{\zeta}_{n-1})\right) \\
&\quad + \sqrt{N}\left(\frac{1}{N}\sum_{i=1}^N r_{n-1}(\hat{\theta}_{n-1}^i; \zeta_{n-1}) - \mathbb{E}_{\pi_{n-1}}\big[r_{n-1}(\theta_{n-1}, \zeta_{n-1})\big]\right) \\
&\quad + \left(\frac{1}{N}\sum_{i=1}^N A_1(\hat{\theta}_{n-1}^i, \zeta)\right)\sqrt{N}(\hat{\zeta}_{n-1} - \zeta_{n-1}) + o_p(1) \\
&= I + II + III, \quad \text{say.}
\end{aligned}$$

Term I: As in the proof of Theorem 4 the term can be rescaled such that its variance is determined by $\mathbb{V}_{K_n(\cdot|\hat{\theta}_n^i);\zeta_n}$ rather than $\mathbb{V}_{K_n(\cdot|\hat{\theta}_n^i);\hat{\zeta}_n}$. The arguments used in the proof of Theorem 3 can be modified to verify that conditional on $\hat{\mathcal{F}}_{n-1,N}$

$$\frac{\sqrt{\frac{1}{N}\sum_{i=1}^{N}\mathbb{V}_{K_{n-1}(\cdot|\hat{\theta}^i_{n-1};\zeta_{n-1})}[\cdot](W^i_{n-1})^2}}{\sqrt{\frac{1}{N}\sum_{i=1}^{N}\mathbb{V}_{K_{n-1}(\cdot|\hat{\theta}^i_{n-1};\hat{\zeta}_{n-1})}[\cdot](W^i_{n-1})^2}}\sqrt{N}\left(\frac{1}{N}\sum_{i=1}\{\theta^i_{n-1}=\hat{\theta}^i_{n-1}\}-\frac{1}{N}\sum_{i=1}^{N}r_{n-1}(\hat{\theta}^i_{n-1};\hat{\zeta}_{n-1})\right)$$

converges in distribution to a Gaussian limit. Thus, we can deduce that Term I is $O_p(1)$. Term II is $O_p(1)$ because according to Theorem 2 the term satisfies a central limit theorem. Term III is of the form $O_p(1)O_p(1)+o_p(1)$ because by the inductive assumption $\sqrt{N}(\hat{\zeta}_{n-1}-\zeta_{n-1})=O_p(1)$. Thus, overall we can deduce that $\sqrt{N}(\hat{c}_n-c_n)=O_p(1)$.

For $M>1$ or $N_{blocks}>1$ the representation of the transition kernel's density in (A-12) involves additional point masses and the expression for $\Psi(\hat{\theta},\zeta;h)$ in (A-13) becomes more complicated. For $\alpha<1$ it is no longer true that the conditional acceptance probability $\alpha_n(\theta|\hat{\theta};\zeta)$ is invariant to ζ. Due to the min operator in the definition of $\alpha_n(\cdot)$ there are points at which the function is no longer differentiable with respect to ζ. Thus, rather than verifying the differentiability of the integrants in (A-13) directly one has to integrate with respect to θ separately over the regions $\Theta_l=\left\{\theta \mid \pi_n(\theta)q(\hat{\theta}|\theta,\zeta)\leq\pi_n(\hat{\theta})q(\theta|\hat{\theta},\zeta)\right\}$ and $\Theta_u=\Theta\backslash\Theta_l$ and show that the boundary $\partial\Theta_l$ is a smooth function of ζ.

B The Smets-Wouters Model

B.1 Model Specification

The equilibrium conditions of the Smets and Wouters (2007) model take the following form:

$$\hat{y}_t = c_y\hat{c}_t + i_y\hat{i}_t + z_y\hat{z}_t + \varepsilon_t^g \tag{A-14}$$

$$\begin{aligned}\hat{c}_t =& \frac{h/\gamma}{1+h/\gamma}\hat{c}_{t-1} + \frac{1}{1+h/\gamma}E_t\hat{c}_{t+1} + \frac{wl_c(\sigma_c-1)}{\sigma_c(1+h/\gamma)}(\hat{l}_t - E_t\hat{l}_{t+1}) \\ &-\frac{1-h/\gamma}{(1+h/\gamma)\sigma_c}(\hat{r}_t - E_t\hat{\pi}_{t+1}) - \frac{1-h/\gamma}{(1+h/\gamma)\sigma_c}\varepsilon_t^b\end{aligned} \tag{A-15}$$

$$\hat{i}_t = \frac{1}{1+\beta\gamma^{(1-\sigma_c)}}\hat{i}_{t-1} + \frac{\beta\gamma^{(1-\sigma_c)}}{1+\beta\gamma^{(1-\sigma_c)}}E_t\hat{i}_{t+1} + \frac{1}{\varphi\gamma^2(1+\beta\gamma^{(1-\sigma_c)})}\hat{q}_t + \varepsilon_t^i \tag{A-16}$$

$$\hat{q}_t = \beta(1-\delta)\gamma^{-\sigma_c}E_t\hat{q}_{t+1} - \hat{r}_t + E_t\hat{\pi}_{t+1} + (1-\beta(1-\delta)\gamma^{-\sigma_c})E_t\hat{r}_{t+1}^k - \varepsilon_t^b \tag{A-17}$$

$$\hat{y}_t = \Phi(\alpha\hat{k}_t^s + (1-\alpha)\hat{l}_t + \varepsilon_t^a) \tag{A-18}$$

$$\hat{k}_t^s = \hat{k}_{t-1} + \hat{z}_t \tag{A-19}$$

$$\hat{z}_t = \frac{1-\psi}{\psi}\hat{r}_t^k \tag{A-20}$$

$$\hat{k}_t = \frac{(1-\delta)}{\gamma}\hat{k}_{t-1} + (1-(1-\delta)/\gamma)\hat{i}_t + (1-(1-\delta)/\gamma)\varphi\gamma^2(1+\beta\gamma^{(1-\sigma_c)})\varepsilon_t^i \tag{A-21}$$

$$\hat{\mu}_t^p = \alpha(\hat{k}_t^s - \hat{l}_t) - \hat{w}_t + \varepsilon_t^a \tag{A-22}$$

$$\begin{aligned}\hat{\pi}_t =& \frac{\beta\gamma^{(1-\sigma_c)}}{1+\iota_p\beta\gamma^{(1-\sigma_c)}}E_t\hat{\pi}_{t+1} + \frac{\iota_p}{1+\beta\gamma^{(1-\sigma_c)}}\hat{\pi}_{t-1} \\ &-\frac{(1-\beta\gamma^{(1-\sigma_c)}\xi_p)(1-\xi_p)}{(1+\iota_p\beta\gamma^{(1-\sigma_c)})(1+(\Phi-1)\varepsilon_p)\xi_p}\hat{\mu}_t^p + \varepsilon_t^p\end{aligned} \tag{A-23}$$

$$\hat{r}_t^k = \hat{l}_t + \hat{w}_t - \hat{k}_t^s \tag{A-24}$$

$$\hat{\mu}_t^w = \hat{w}_t - \sigma_l\hat{l}_t - \frac{1}{1-h/\gamma}(\hat{c}_t - h/\gamma\hat{c}_{t-1}) \tag{A-25}$$

$$\begin{aligned}\hat{w}_t =& \frac{\beta\gamma^{(1-\sigma_c)}}{1+\beta\gamma^{(1-\sigma_c)}}(E_t\hat{w}_{t+1} + E_t\hat{\pi}_{t+1}) + \frac{1}{1+\beta\gamma^{(1-\sigma_c)}}(\hat{w}_{t-1} - \iota_w\hat{\pi}_{t-1}) \\ &-\frac{1+\beta\gamma^{(1-\sigma_c)}\iota_w}{1+\beta\gamma^{(1-\sigma_c)}}\hat{\pi}_t - \frac{(1-\beta\gamma^{(1-\sigma_c)}\xi_w)(1-\xi_w)}{(1+\beta\gamma^{(1-\sigma_c)})(1+(\lambda_w-1)\epsilon_w)\xi_w}\hat{\mu}_t^w + \varepsilon_t^w\end{aligned} \tag{A-26}$$

$$\begin{aligned}\hat{r}_t =& \rho\hat{r}_{t-1} + (1-\rho)(r_\pi\hat{\pi}_t + r_y(\hat{y}_t - \hat{y}_t^*)) \\ &+r_{\Delta y}((\hat{y}_t - \hat{y}_t^*) - (\hat{y}_{t-1} - \hat{y}_{t-1}^*)) + \varepsilon_t^r.\end{aligned} \tag{A-27}$$

The exogenous shocks evolve according to

$$\begin{aligned}
\varepsilon_t^a &= \rho_a \varepsilon_{t-1}^a + \eta_t^a && \text{(A-28)}\\
\varepsilon_t^b &= \rho_b \varepsilon_{t-1}^b + \eta_t^b && \text{(A-29)}\\
\varepsilon_t^g &= \rho_g \varepsilon_{t-1}^a + \rho_{ga} \eta_t^a + \eta_t^g && \text{(A-30)}\\
\varepsilon_t^i &= \rho_i \varepsilon_{t-1}^i + \eta_t^i && \text{(A-31)}\\
\varepsilon_t^r &= \rho_r \varepsilon_{t-1}^r + \eta_t^r && \text{(A-32)}\\
\varepsilon_t^p &= \rho_r \varepsilon_{t-1}^p + \eta_t^p - \mu_p \eta_{t-1}^p && \text{(A-33)}\\
\varepsilon_t^w &= \rho_w \varepsilon_{t-1}^w + \eta_t^w - \mu_w \eta_{t-1}^w. && \text{(A-34)}
\end{aligned}$$

The counterfactual no-rigidity prices and quantities evolve according to

$$\begin{aligned}
\hat{y}_t^* &= c_y \hat{c}_t^* + i_y \hat{i}_t^* + z_y \hat{z}_t^* + \varepsilon_t^g && \text{(A-35)}\\
\hat{c}_t^* &= \frac{h/\gamma}{1+h/\gamma} \hat{c}_{t-1}^* + \frac{1}{1+h/\gamma} E_t \hat{c}_{t+1}^* + \frac{wl_c(\sigma_c - 1)}{\sigma_c(1+h/\gamma)} (\hat{l}_t^* - E_t \hat{l}_{t+1}^*) && \text{(A-36)}\\
&\quad - \frac{1-h/\gamma}{(1+h/\gamma)\sigma_c} r_t^* - \frac{1-h/\gamma}{(1+h/\gamma)\sigma_c} \varepsilon_t^b \\
\hat{i}_t^* &= \frac{1}{1+\beta\gamma^{(1-\sigma_c)}} \hat{i}_{t-1}^* + \frac{\beta\gamma^{(1-\sigma_c)}}{1+\beta\gamma^{(1-\sigma_c)}} E_t \hat{i}_{t+1}^* + \frac{1}{\varphi\gamma^2(1+\beta\gamma^{(1-\sigma_c)})} \hat{q}_t^* + \varepsilon_t^i && \text{(A-37)}\\
\hat{q}_t^* &= \beta(1-\delta)\gamma^{-\sigma_c} E_t \hat{q}_{t+1}^* - r_t^* + (1 - \beta(1-\delta)\gamma^{-\sigma_c}) E_t r_{t+1}^{k*} - \varepsilon_t^b && \text{(A-38)}\\
\hat{y}_t^* &= \Phi(\alpha k_t^{s*} + (1-\alpha)\hat{l}_t^* + \varepsilon_t^a) && \text{(A-39)}\\
\hat{k}_t^{s*} &= k_{t-1}^* + z_t^* && \text{(A-40)}\\
\hat{z}_t^* &= \frac{1-\psi}{\psi} \hat{r}_t^{k*} && \text{(A-41)}\\
\hat{k}_t^* &= \frac{(1-\delta)}{\gamma} \hat{k}_{t-1}^* + (1-(1-\delta)/\gamma)\hat{i}_t + (1-(1-\delta)/\gamma)\varphi\gamma^2(1+\beta\gamma^{(1-\sigma_c)})\varepsilon_t^i && \text{(A-42)}\\
\hat{w}_t^* &= \alpha(\hat{k}_t^{s*} - \hat{l}_t^*) + \varepsilon_t^a && \text{(A-43)}\\
\hat{r}_t^{k*} &= \hat{l}_t^* + \hat{w}_t^* - \hat{k}_t^* && \text{(A-44)}\\
\hat{w}_t^* &= \sigma_l \hat{l}_t^* + \frac{1}{1-h/\gamma} (\hat{c}_t^* + h/\gamma \hat{c}_{t-1}^*). && \text{(A-45)}
\end{aligned}$$

The steady state (ratios) that appear in the measurement equation or the log-linearized equilibrium conditions are given by

$$\gamma = \bar{\gamma}/100 + 1 \tag{A-46}$$

$$\pi^* = \bar{\pi}/100 + 1 \tag{A-47}$$

$$\bar{r} = 100(\beta^{-1}\gamma^{\sigma_c}\pi^* - 1) \tag{A-48}$$

$$r^k_{ss} = \gamma^{\sigma_c}/\beta - (1-\delta) \tag{A-49}$$

$$w_{ss} = \left(\frac{\alpha^\alpha (1-\alpha)^{(1-\alpha)}}{\Phi {r^k_{ss}}^\alpha} \right)^{\frac{1}{1-\alpha}} \tag{A-50}$$

$$i_k = (1 - (1-\delta)/\gamma)\gamma \tag{A-51}$$

$$l_k = \frac{1-\alpha}{\alpha} \frac{r^k_{ss}}{w_{ss}} \tag{A-52}$$

$$k_y = \Phi l_k^{(\alpha-1)} \tag{A-53}$$

$$i_y = (\gamma - 1 + \delta) k_y \tag{A-54}$$

$$c_y = 1 - g_y - i_y \tag{A-55}$$

$$z_y = r^k_{ss} k_y \tag{A-56}$$

$$wl_c = \frac{1}{\lambda_w} \frac{1-\alpha}{\alpha} \frac{r^k_{ss} k_y}{c_y}. \tag{A-57}$$

B.2 Additional Tables and Figures for the Analysis of the SW Model

The standard prior distribution for the SW model is summarized in Table A-1.

Table A-2 shows the posterior means as well as 90% equal-tail-probability credible intervals for the SW model with standard prior. We also report the standard deviation of posterior mean across the five repetitions of the posterior simulation.

Table A-3 shows the diffuse prior for the SW model.

Figure A-1 compares the standard and diffuse prior.

Table A-4 shows the posterior means as well as 90% equal-tail-probability credible intervals for the SW model with standard prior. We also report the standard deviation of posterior mean across the five repetitions of the posterior simulation.

Table A-5 compares the output of the fully adaptive version of the SMC algorithm to the output of the non-adaptive reference algorithm.

Table A-1: SW Model: Standard Prior

Parameter	Type	Para (1)	Para (2)	Parameter	Type	Para (1)	Para (2)
φ	Normal	4.00	1.50	α	Normal	0.30	0.05
σ_c	Normal	1.50	0.37	ρ_a	Beta	0.50	0.20
h	Beta	0.70	0.10	ρ_b	Beta	0.50	0.20
ξ_w	Beta	0.50	0.10	ρ_g	Beta	0.50	0.20
σ_l	Normal	2.00	0.75	ρ_i	Beta	0.50	0.20
ξ_p	Beta	0.50	0.10	ρ_r	Beta	0.50	0.20
ι_w	Beta	0.50	0.15	ρ_p	Beta	0.50	0.20
ι_p	Beta	0.50	0.15	ρ_w	Beta	0.50	0.20
ψ	Beta	0.50	0.15	μ_p	Beta	0.50	0.20
Φ	Normal	1.25	0.12	μ_w	Beta	0.50	0.20
r_π	Normal	1.50	0.25	ρ_{ga}	Beta	0.50	0.20
ρ	Beta	0.75	0.10	σ_a	Inv. Gamma	0.10	2.00
r_y	Normal	0.12	0.05	σ_b	Inv. Gamma	0.10	2.00
$r_{\Delta y}$	Normal	0.12	0.05	σ_g	Inv. Gamma	0.10	2.00
π	Gamma	0.62	0.10	σ_i	Inv. Gamma	0.10	2.00
$100(\beta^{-1}-1)$	Gamma	0.25	0.10	σ_r	Inv. Gamma	0.10	2.00
l	Normal	0.00	2.00	σ_p	Inv. Gamma	0.10	2.00
γ	Normal	0.40	0.10	σ_w	Inv. Gamma	0.10	2.00

Notes: Para (1) and Para (2) correspond to the mean and standard deviation of the Beta, Gamma, and Normal distributions and to the upper and lower bounds of the support for the Uniform distribution. For the Inv. Gamma distribution, Para (1) and Para (2) refer to s and ν, where $p(\sigma|\nu,s) \propto \sigma^{-\nu-1}e^{-\nu s^2/2\sigma^2}$.

Table A-2: SW Model with Standard Prior: Posterior Comparison

	RWMH			SMC		
Parameter	Mean	$[0.05, 0.95]$	STD(Mean)	Mean	$[0.05, 0.95]$	STD(Mean)
φ	5.70	[4.11, 7.48]	0.10	5.70	[4.12, 7.45]	0.03
σ_c	1.33	[1.14, 1.55]	0.02	1.33	[1.13, 1.54]	0.00
h	0.72	[0.65, 0.79]	0.01	0.72	[0.65, 0.79]	0.00
ξ_w	0.70	[0.59, 0.80]	0.02	0.70	[0.59, 0.80]	0.00
σ_l	1.90	[1.05, 2.88]	0.03	1.87	[1.01, 2.84]	0.02
ξ_p	0.65	[0.56, 0.74]	0.04	0.64	[0.54, 0.73]	0.00
ι_w	0.57	[0.35, 0.77]	0.02	0.57	[0.36, 0.77]	0.00
ι_p	0.26	[0.13, 0.42]	0.03	0.25	[0.12, 0.41]	0.00
ψ	0.55	[0.37, 0.73]	0.02	0.55	[0.37, 0.74]	0.00
Φ	1.58	[1.46, 1.71]	0.00	1.58	[1.46, 1.71]	0.00
r_π	2.04	[1.76, 2.34]	0.02	2.05	[1.77, 2.34]	0.01
ρ	0.81	[0.76, 0.85]	0.01	0.80	[0.76, 0.84]	0.00
r_y	0.09	[0.05, 0.13]	0.01	0.09	[0.05, 0.12]	0.00
$r_{\Delta y}$	0.23	[0.18, 0.27]	0.00	0.22	[0.18, 0.27]	0.00
π	0.69	[0.52, 0.87]	0.00	0.69	[0.52, 0.87]	0.00
$100(\beta^{-1} - 1)$	0.17	[0.08, 0.27]	0.00	0.17	[0.08, 0.27]	0.00
l	0.70	[-1.23, 2.62]	0.10	0.72	[-1.21, 2.65]	0.02
γ	0.42	[0.39, 0.45]	0.00	0.42	[0.39, 0.45]	0.00
α	0.19	[0.16, 0.22]	0.00	0.19	[0.16, 0.22]	0.00
ρ_a	0.96	[0.94, 0.97]	0.00	0.96	[0.94, 0.98]	0.00
ρ_b	0.22	[0.08, 0.38]	0.02	0.21	[0.08, 0.37]	0.00
ρ_g	0.98	[0.96, 0.99]	0.00	0.98	[0.96, 0.99]	0.00
ρ_i	0.73	[0.63, 0.82]	0.00	0.73	[0.63, 0.82]	0.00
ρ_r	0.15	[0.05, 0.26]	0.01	0.15	[0.06, 0.27]	0.00
ρ_p	0.89	[0.80, 0.96]	0.01	0.90	[0.80, 0.97]	0.00
ρ_w	0.97	[0.95, 0.99]	0.00	0.97	[0.95, 0.99]	0.00
μ_p	0.72	[0.54, 0.85]	0.09	0.69	[0.50, 0.84]	0.00
μ_w	0.85	[0.74, 0.93]	0.01	0.85	[0.73, 0.93]	0.00
ρ_{ga}	0.50	[0.35, 0.65]	0.00	0.50	[0.35, 0.65]	0.00
σ_a	0.47	[0.42, 0.52]	0.00	0.47	[0.42, 0.52]	0.00
σ_b	0.24	[0.20, 0.28]	0.00	0.24	[0.20, 0.28]	0.00
σ_g	0.54	[0.49, 0.59]	0.00	0.54	[0.49, 0.59]	0.00
σ_i	0.45	[0.38, 0.54]	0.00	0.45	[0.38, 0.54]	0.00
σ_r	0.25	[0.22, 0.28]	0.00	0.25	[0.22, 0.28]	0.00
σ_p	0.15	[0.12, 0.18]	0.02	0.14	[0.11, 0.17]	0.00
σ_w	0.25	[0.21, 0.28]	0.00	0.25	[0.21, 0.29]	0.00

Notes: Means and standard deviations are over 5 runs for each algorithm. The RWMH algorithms use 10 million draws with the first 5 million discarded. The average acceptance rate was roughly 30%. The SMC algorithms use 12,000 particles, 500 stages, $\lambda = 2.1$, a mixture proposal and 3 blocks in each MH step.

Table A-3: SW Model: Diffuse Prior

Parameter	Type	Para (1)	Para (2)	Parameter	Type	Para (1)	Para (2)
φ	Normal	4.00	4.50	α	Normal	0.30	0.15
σ_c	Normal	1.50	1.11	ρ_a	Uniform	0.00	1.00
h	Uniform	0.00	1.00	ρ_b	Uniform	0.00	1.00
ξ_w	Uniform	0.00	1.00	ρ_g	Uniform	0.00	1.00
σ_l	Normal	2.00	2.25	ρ_i	Uniform	0.00	1.00
ξ_p	Uniform	0.00	1.00	ρ_r	Uniform	0.00	1.00
ι_w	Uniform	0.00	1.00	ρ_p	Uniform	0.00	1.00
ι_p	Uniform	0.00	1.00	ρ_w	Uniform	0.00	1.00
ψ	Uniform	0.00	1.00	μ_p	Uniform	0.00	1.00
Φ	Normal	1.25	0.36	μ_w	Uniform	0.00	1.00
r_π	Normal	1.50	0.75	ρ_{ga}	Uniform	0.00	1.00
ρ	Uniform	0.00	1.00	σ_a	Inv. Gamma	0.10	2.00
r_y	Normal	0.12	0.15	σ_b	Inv. Gamma	0.10	2.00
$r_{\Delta y}$	Normal	0.12	0.15	σ_g	Inv. Gamma	0.10	2.00
π	Gamma	0.62	0.30	σ_i	Inv. Gamma	0.10	2.00
$100(\beta^{-1}-1)$	Gamma	0.25	0.30	σ_r	Inv. Gamma	0.10	2.00
l	Normal	0.00	6.00	σ_p	Inv. Gamma	0.10	2.00
γ	Normal	0.40	0.30	σ_w	Inv. Gamma	0.10	2.00

Notes: Para (1) and Para (2) correspond to the mean and standard deviation of the Beta, Gamma, and Normal distributions and to the upper and lower bounds of the support for the Uniform distribution. For the Inv. Gamma distribution, Para (1) and Para (2) refer to s and ν, where $p(\sigma|\nu, s) \propto \sigma^{-\nu-1}e^{-\nu s^2/2\sigma^2}$.

Figure A-1: SW Model: SW Prior Comparison

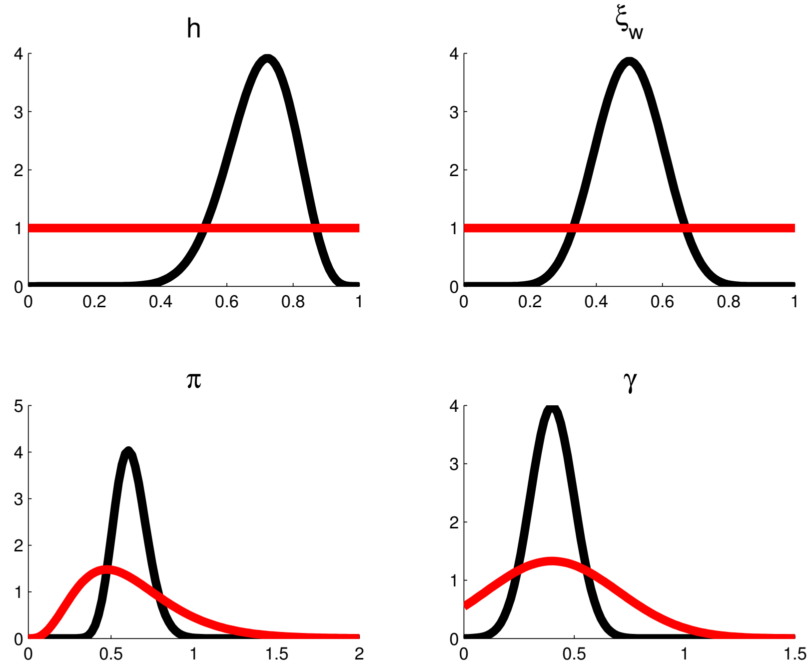


Table A-4: SW Model with Diffuse Prior: Posterior Comparison

	RWMH			SMC		
Parameter	Mean	[0.05, 0.95]	STD(Mean)	Mean	[0.05, 0.95]	STD(Mean)
φ	7.98	[4.16, 12.50]	1.03	8.12	[4.27, 12.59]	0.15
σ_c	1.65	[1.33, 2.02]	0.02	1.65	[1.34, 2.03]	0.01
h	0.69	[0.58, 0.78]	0.03	0.70	[0.59, 0.78]	0.00
ξ_w	0.93	[0.82, 0.99]	0.02	0.93	[0.80, 0.99]	0.01
σ_l	3.04	[1.41, 5.14]	0.15	3.06	[1.39, 5.26]	0.04
ξ_p	0.73	[0.62, 0.82]	0.03	0.72	[0.60, 0.82]	0.01
ι_w	0.72	[0.39, 0.96]	0.03	0.73	[0.37, 0.97]	0.02
ι_p	0.12	[0.01, 0.29]	0.02	0.11	[0.01, 0.29]	0.00
ψ	0.75	[0.50, 0.96]	0.01	0.75	[0.50, 0.96]	0.00
Φ	1.69	[1.48, 1.91]	0.04	1.71	[1.50, 1.94]	0.01
r_π	2.76	[2.11, 3.51]	0.03	2.78	[2.12, 3.52]	0.02
ρ	0.88	[0.84, 0.92]	0.01	0.88	[0.84, 0.92]	0.00
r_y	0.16	[0.09, 0.24]	0.01	0.15	[0.08, 0.24]	0.00
$r_{\Delta y}$	0.28	[0.22, 0.35]	0.01	0.28	[0.22, 0.35]	0.00
π	0.85	[0.43, 1.22]	0.02	0.85	[0.42, 1.23]	0.01
$100(\beta^{-1}-1)$	0.06	[0.00, 0.18]	0.00	0.06	[0.00, 0.19]	0.00
l	-0.01	[-2.92, 2.93]	0.16	-0.06	[-2.99, 2.92]	0.07
γ	0.41	[0.37, 0.44]	0.00	0.41	[0.37, 0.44]	0.00
α	0.17	[0.14, 0.20]	0.00	0.17	[0.14, 0.20]	0.00
ρ_a	0.97	[0.96, 0.98]	0.00	0.97	[0.96, 0.98]	0.00
ρ_b	0.21	[0.03, 0.48]	0.08	0.19	[0.03, 0.44]	0.01
ρ_g	0.99	[0.97, 1.00]	0.00	0.98	[0.97, 1.00]	0.00
ρ_i	0.72	[0.62, 0.83]	0.02	0.72	[0.61, 0.83]	0.00
ρ_r	0.05	[0.00, 0.14]	0.00	0.05	[0.00, 0.14]	0.00
ρ_p	0.91	[0.81, 0.99]	0.01	0.92	[0.81, 1.00]	0.01
ρ_w	0.69	[0.21, 0.99]	0.09	0.69	[0.21, 0.99]	0.04
μ_p	0.80	[0.54, 0.96]	0.10	0.77	[0.47, 0.98]	0.02
μ_w	0.63	[0.09, 0.98]	0.09	0.63	[0.09, 0.97]	0.04
ρ_{ga}	0.44	[0.26, 0.61]	0.02	0.43	[0.25, 0.61]	0.00
σ_a	0.46	[0.41, 0.51]	0.00	0.46	[0.41, 0.51]	0.00
σ_b	0.23	[0.18, 0.28]	0.01	0.24	[0.18, 0.29]	0.00
σ_g	0.55	[0.49, 0.60]	0.00	0.55	[0.50, 0.60]	0.00
σ_i	0.47	[0.39, 0.56]	0.01	0.46	[0.39, 0.55]	0.00
σ_r	0.24	[0.22, 0.27]	0.00	0.24	[0.22, 0.27]	0.00
σ_p	0.15	[0.11, 0.20]	0.04	0.14	[0.09, 0.23]	0.00
σ_w	0.25	[0.21, 0.29]	0.00	0.25	[0.22, 0.30]	0.00

Notes: Means and standard deviations are over 20 runs for each algorithm. The RWMH algorithms use 10 millions draws with the first 5 million discarded. The average acceptance rate was roughly 30%. The SMC algorithms use 12,000 particles, 500 stages, $\lambda = 2.1$, a mixture proposal and 3 blocks in each MH step.

Table A-5: SW Model: Diffuse Prior – Effects of Adaption

Parameter	Mean(Mean) Adaptive ($\hat{\zeta}_n$)	Non-Adaptive (ζ_n)	T-stat PIT	Std(Mean) Adaptive ($\hat{\zeta}_n$)	Non-Adaptive (ζ_n)	F-stat PIT
φ	8.1182	8.1737	0.15	0.1540	0.1786	0.26
σ_c	1.6510	1.6496	0.71	0.0090	0.0080	0.70
h	0.6976	0.6967	0.89	0.0026	0.0023	0.67
ξ_w	0.9302	0.9272	0.79	0.0091	0.0141	0.03
σ_l	3.0627	3.0652	0.42	0.0369	0.0420	0.28
ξ_p	0.7242	0.7233	0.63	0.0065	0.0095	0.05
ι_w	0.7277	0.7195	0.87	0.0196	0.0266	0.09
ι_p	0.1123	0.1108	0.90	0.0036	0.0041	0.26
ψ	0.7481	0.7510	0.03	0.0045	0.0051	0.29
Φ	1.7132	1.7091	1.00 (*)	0.0059	0.0039	0.96
r_π	2.7766	2.7911	0.01 (*)	0.0196	0.0201	0.45
ρ	0.8805	0.8808	0.22	0.0011	0.0014	0.15
r_y	0.1538	0.1565	0.00 (*)	0.0024	0.0032	0.09
$r_{\Delta y}$	0.2783	0.2790	0.05	0.0013	0.0011	0.78
π	0.8510	0.8514	0.46	0.0124	0.0135	0.35
$100(\beta^{-1}-1)$	0.0594	0.0596	0.27	0.0014	0.0015	0.45
l	-0.0614	-0.0598	0.47	0.0652	0.0794	0.19
γ	0.4050	0.4048	0.69	0.0012	0.0017	0.09
α	0.1689	0.1685	0.96	0.0008	0.0007	0.51
ρ_a	0.9695	0.9694	0.80	0.0003	0.0002	0.90
ρ_b	0.1856	0.1881	0.11	0.0065	0.0064	0.51
ρ_g	0.9849	0.9849	0.31	0.0003	0.0003	0.55
ρ_i	0.7210	0.7191	0.89	0.0039	0.0061	0.03
ρ_r	0.0546	0.0539	0.97	0.0010	0.0012	0.32
ρ_p	0.9164	0.9123	0.97	0.0067	0.0069	0.45
ρ_w	0.6898	0.7017	0.24	0.0398	0.0637	0.02 (*)
μ_p	0.7699	0.7617	0.86	0.0183	0.0280	0.03
μ_w	0.6324	0.6456	0.24	0.0445	0.0709	0.02 (*)
ρ_{ga}	0.4318	0.4318	0.49	0.0023	0.0026	0.29
σ_a	0.4585	0.4589	0.05	0.0008	0.0007	0.65
σ_b	0.2402	0.2398	0.83	0.0014	0.0014	0.46
σ_g	0.5464	0.5467	0.05	0.0008	0.0005	0.98 (*)
σ_i	0.4631	0.4644	0.03	0.0018	0.0025	0.08
σ_r	0.2392	0.2391	0.91	0.0004	0.0003	0.97
σ_p	0.1365	0.1374	0.22	0.0033	0.0040	0.18
σ_w	0.2545	0.2546	0.41	0.0014	0.0017	0.27

Notes: This table shows the Probability Integral Transforms (PIT) of the t and F statistics for tests for differences in means and variances of the posterior means from the SMC algorithm with adaptive and non-adaptive tuning parameters. Stars refer to PITs in the 5 percent extremes of the distribution. For the adaptive algorithm, we also generate a new sequence of random blocks $\{B_n\}$ for every repetition of the algorithm and we determine the resampling indicators $\{\rho_n\}$ adaptively. For the non-adaptive algorithm, we determine sequences $\{\rho_n\}$, $\{B_n\}$, and $\{\zeta_n\}$ in a trial run and hold these sequences fixed across repetitions of the algorithm.

C Schmitt-Grohé and Uribe (2010) Model

C.1 Steady State

$$\mu_{y_{ss}} = \mu_{a,ss}{}^{\frac{\alpha_k}{\alpha_k - 1}}\, \mu_{x,ss} \tag{A-58}$$

$$\mu_{k_{ss}} = \mu_{x,ss}\, \mu_{a,ss}{}^{\frac{1}{\alpha_k - 1}} \tag{A-59}$$

$$x_{g_{ss}} = \left(\frac{1}{\mu_{y_{ss}}}\right)^{\frac{1}{1-\rho_{xg}}} \tag{A-60}$$

$$\left(\frac{g}{y}\right)_{ss} = \frac{0.20}{x_{g_{ss}}} \tag{A-61}$$

$$\left(\frac{y}{k}\right)_{ss} = \frac{\frac{1}{\mu_{a,ss}\, \beta\, \mu_{y_{ss}}^{(-\sigma)}} - (1-\delta_0)}{\alpha_k\, \mu_{k_{ss}}} \tag{A-62}$$

$$\left(\frac{i}{k}\right)_{ss} = 1 - \frac{1-\delta_0}{\mu_{k_{ss}}} \tag{A-63}$$

$$\left(\frac{i}{y}\right)_{ss} = \left(\frac{i}{k}\right)_{ss} / \left(\frac{y}{k}\right)_{ss} \tag{A-64}$$

$$\left(\frac{c}{y}\right)_{ss} = 1 - x_{g_{ss}} \left(\frac{g}{y}\right)_{ss} - \left(\frac{i}{y}\right)_{ss} \tag{A-65}$$

$$\psi = \frac{\frac{\left(1 - \mu_{y_{ss}}^{(-\sigma)}\, \beta\, b\right) \alpha_h}{1+\mu_{ss}} \left(\frac{y}{c}\right)_{ss}}{h_{ss}^{\theta} \left(\frac{1}{\mu_{y_{ss}}}\right)^{\frac{1-\gamma}{\gamma}} \left(\theta \left(1 - \frac{b}{\mu_{y_{ss}}}\right) + \left(1 - \mu_{y_{ss}}^{(-\sigma)}\, \beta\, b\right) \left(\frac{y}{c}\right)_{ss} \frac{\alpha_h\, \frac{\gamma}{1-\beta\,(1-\gamma)\,\mu_{y_{ss}}^{1-\sigma}}}{1+\mu_{ss}}\right)} \tag{A-66}$$

$$kss = \left(\frac{\frac{y\_k_t}{l^{1-\alpha_k-\alpha_h}\, h_{ss}^{\alpha_h}}}{\mu_{k_{ss}}^{(-\alpha_k)}}\right)^{\frac{1}{\alpha_k - 1}} \tag{A-67}$$

$$\delta_1 = \mu_{k_{ss}}\, \alpha_k\, y\_k_t \tag{A-68}$$

C.2 Detrended Equilibrium

C.2.1 Optimality and Market Clearing Conditions

Investment Equation:

$$k_t = \left(1 - \left(\delta_0 + \delta_1\,(u_t - 1) + \frac{\delta_2}{2}\,(u_t-1)^2\right)\right) \frac{k_{t-1}}{\mu_{k_t}} + z_t^i\, i_t \left(1 - \frac{\kappa}{2}\left(\frac{i_t \mu_{k_t}}{i_{t-1}} - \mu_{k_{ss}}\right)^2\right) \tag{A-69}$$

Resource Constraint:

$$y_t = g_t \, x_{g_t} + i_t + c_t \tag{A-70}$$

Production Function:

$$y_t = z_t \left(u_t k_{t-1}/\mu_{kt}\right)^{\alpha_k} h_t^{\alpha_h} l^{1-\alpha_h-\alpha_k} \tag{A-71}$$

Value of Consumption Bundle:

$$v_t = c_t - b\frac{c_{t-1}}{\mu_{y_t}} - \psi h_t^{\theta} \, s_t \tag{A-72}$$

Geometric Average of past habit-adjusted consumption:

$$s_t = \left(c_t - b\frac{c_{t-1}}{\mu_{y_t}}\right)^{\gamma} \left(\frac{s_{t-1}}{\mu_{y_t}}\right)^{1-\gamma} \tag{A-73}$$

Consumption Decision:

$$\lambda_t = \zeta_t \, v_t^{-\sigma} - \frac{\gamma s_t p_t}{c_t - b\,\frac{c_{t-1}}{\mu_{y_t}}} - \beta \, b \mu_{y_{t+1}}^{-\sigma} \left(\zeta_{t+1} \, v_{t+1}^{-\sigma} - \frac{\gamma s_{t+1} p_{t+1}}{c_{t+1} - b\,\frac{c_t}{\mu_{y_{t+1}}}}\right) \tag{A-74}$$

Hours Decision:

$$\theta \, \psi s_t \, v_t^{-\sigma} \, \zeta_t \, h_t^{\theta-1} = \lambda_t \frac{\alpha_h y_t/h_t}{1+\mu_t} \tag{A-75}$$

Dynamics for the shadow price of past consumption:

$$p_t = \psi \zeta_t v_t^{-\sigma} h_t^{\theta} + \beta \, (1-\gamma) \, \mu_{y_{t+1}}^{1-\sigma} \, p_{t+1} \frac{s_{t+1}}{s_t} \tag{A-76}$$

Euler Equation:

$$\lambda_t \, q_t = \beta \, \lambda_{t+1} \, \mu_{at+1} \mu_{y_{t+1}}^{-\sigma} \left(\alpha_k \, u_{t+1} \frac{y_{t+1}}{k_t \, u_{t+1}/\mu_{kt+1}} + \left(1 - \left(\delta_0 + \delta_1 \, (u_{t+1}-1) + \frac{\delta_2}{2} \, (u_{t+1}-1)^2\right)\right) q_{t+1}\right) \tag{A-77}$$

Capacity Utilization:

$$q_t \, \left(\delta_1 + \delta_2 \left(u_t - 1\right)\right) = \alpha_k \frac{y_t}{u_t \, k_{t-1}/\mu_{kt}} \tag{A-78}$$

Dynamics of q_t:

$$\begin{aligned} \lambda_t \quad &= \quad q_t \lambda_t \, z_t^i \left(1 - \frac{\kappa}{2} \left(\frac{i_t \mu_{kt}}{i_{t-1}} - \mu_{kss}\right)^2 - \kappa \, \frac{i_t \mu_{kt}}{i_{t-1}} \left(\frac{i_t \mu_{kt}}{i_{t-1}} - \mu_{kss}\right)\right) \\ &+ \quad \beta \mu_{at+1} \mu_{y_{t+1}}^{-\sigma} \, q_{t+1} \lambda_{t+1} z_{t+1}^i \, \left(\frac{i_{t+1} \, \mu_{kt+1}}{i_t}\right)^2 \kappa \left(\frac{i_{t+1} \, \mu_{kt+1}}{i_t} - \mu_{kss}\right) \end{aligned} \tag{A-79}$$

C.2.2 Exogenous Processes and Trends

Stochastic trend in output:

$$\mu_{y_t} = \mu_{x_t}\, \mu_{a_t}^{\frac{\alpha_k}{\alpha_k - 1}} \tag{A-80}$$

Stochastic trend in capital and investment:

$$\mu_{k_t} = \mu_{x_t}\, \mu_{a_t}^{\frac{1}{\alpha_k - 1}} \tag{A-81}$$

Government Spending Trend:

$$x_{g_t} = \frac{x_{g_{t-1}}^{\rho_{xg}}}{\mu_{y_t}} \tag{A-82}$$

Capital-specific technology trend shock:

$$\log\left(\frac{\mu_{a_t}}{\mu_{a,ss}}\right) = \rho_a \log\left(\frac{\mu_{a_{t-1}}}{\mu_{a,ss}}\right) + \epsilon^0_{a,t} + \epsilon^4_{a,t-4} + \epsilon^8_{a,t-8} \tag{A-83}$$

Neutral technology trend shock:

$$\log\left(\frac{\mu_{x_t}}{\mu_{x,ss}}\right) = (\rho_x - 0.5) \log\left(\frac{\mu_{x_{t-1}}}{\mu_{x,ss}}\right) + \epsilon^0_{x,t} + \epsilon^4_{x,t-4} + \epsilon^8_{x,t-8} \tag{A-84}$$

Neutral technology shock:

$$\log(z_t) = \rho_z \log(z_{t-1}) + \epsilon^0_{z,t} + \epsilon^4_{z,t-4} + \epsilon^8_{z,t-8} \tag{A-85}$$

Investment-specific technology shock:

$$\log\left(z^i_t\right) = \rho_{z^i} \log\left(z^i_{t-1}\right) + \epsilon^0_{z^i,t} + \epsilon^4_{z^i,t-4} + \epsilon^8_{z^i,t-8} \tag{A-86}$$

Government Spending Shock

$$\log\left(\frac{g_t}{gss_t}\right) = \rho_g \log\left(\frac{g_{t-1}}{gss_t}\right) + \epsilon^0_{g,t} + \epsilon^4_{g,t-4} + \epsilon^4_{g,t-8} \tag{A-87}$$

Preference shock:

$$\log(\zeta_t) = \rho_\zeta \log(\zeta_{t-1}) + \epsilon^0_{\zeta,t} + \epsilon^4_{\zeta,t-4} + \epsilon^8_{\zeta,t-8} \tag{A-88}$$

Wage markup:

$$\log\left(\frac{\mu_t}{\mu_{ss}}\right) = \rho_\mu \log\left(\frac{\mu_{t-1}}{\mu_{ss}}\right) + \epsilon^0_{\mu,t} + \epsilon^4_{\mu,t-4} + \epsilon^8_{\mu,t-8} \tag{A-89}$$

C.2.3 Observation Equations

$$ygr_t = 100 \log\left(\frac{y_t \mu_{y_t}}{y_{t-1}}\right) + \epsilon^{me}_{y,t} \tag{A-90}$$

$$cgr_t = 100 \log\left(\frac{c_t \mu_{y_t}}{c_{t-1}}\right) \tag{A-91}$$

$$igr_t = 100 \log\left(\frac{i_t \mu_{a_t} \mu_{k_t}}{i_{t-1}}\right) \tag{A-92}$$

$$hgr_t = 100 \log\left(\frac{h_t}{h_{t-1}}\right) \tag{A-93}$$

$$ggr_t = 100 \log\left(\frac{g_t\, x_{g_t} \mu_{y_t}}{g_{t-1}\, x_{g_{t-1}}}\right) \tag{A-94}$$

$$zgr_t = 100 \log\left(\frac{z_t \mu_{x_t}^{1-\alpha_k}}{z_{t-1}}\right) \tag{A-95}$$

$$agr_t = 100 \log\left(\mu_{a_t}\right) \tag{A-96}$$

C.3 Replicating the Results in Schmitt-Grohé and Uribe (2012)

The description of the model in the previous subsections corresponds to the model presented in the text of Schmitt-Grohé and Uribe (2012). Their implementation is slightly different.

Wage Markup: The process for the wage markup shock actually operates on the *gross* markup,

$$1 + \mu.$$

Prior Normalizations: As noted in the text and Table 2 of SGU, the prior for θ is actually on $\theta - 1$. That is,

$$\theta \sim 1 + Gamma(4.00, 1.00).$$

Similarly, support for ρ_x is $[-0.5, 0.5]$, or

$$\rho_x \sim Beta(0.7, 0.2) - 0.5.$$

Finally, as noted in early drafts of SGU, the parameters $[b, \rho_{xg}, \rho_z, \rho_a, \rho_g, \rho_\mu, \rho_\zeta, \rho_{z^i}]$ are rescaled by 0.99, presumably to help identify the news innovations.

C.4 Additional Tables and Figures for the SGU Model

Table A-6 summarizes the SGU prior distribution for the news model.

Table A-7 compares the output of the posterior simulators for the news model under the SGU prior.

Figure A-2 shows some bimodal features of the posterior distribution.

Table A-6: News Model: SGU Prior Distribution

Parameter	Type	Para (1)	Para (2)	Parameter	Type	Para (1)	Para (2)
$\theta - 1$	Gamma	4.00	1.00	$\sigma_{z^i}^0$	Gamma	17.15	17.15
γ	Beta	0.50	0.29	$\sigma_{z^i}^4$	Gamma	7.00	7.00
κ	Gamma	4.00	1.00	$\sigma_{z^i}^8$	Gamma	7.00	7.00
δ_2/δ_1	Inv. Gamma	0.68	2.59	σ_z^0	Gamma	1.50	1.50
b	Beta	0.50	0.20	σ_z^4	Gamma	0.61	0.61
ρ_{x_g}	Beta	0.70	0.20	σ_z^8	Gamma	0.61	0.61
$\rho_z/0.99$	Beta	0.70	0.20	σ_μ^0	Gamma	1.19	1.19
$\rho_a/0.99$	Beta	0.50	0.20	σ_μ^4	Gamma	0.49	0.49
$\rho_g/0.99$	Beta	0.70	0.20	σ_μ^8	Gamma	0.49	0.49
$\rho_x + 0.5$	Beta	0.70	0.20	σ_g^0	Gamma	1.05	1.05
$\rho_\mu/0.99$	Beta	0.70	0.20	σ_g^4	Gamma	0.43	0.43
$\rho_\zeta/0.99$	Beta	0.50	0.20	σ_g^8	Gamma	0.43	0.43
$\rho_{z^i}/0.99$	Beta	0.50	0.20	σ_ζ^0	Gamma	6.30	6.30
$\sigma_{\mu_a}^0$	Gamma	0.31	0.31	σ_ζ^4	Gamma	2.57	2.57
$\sigma_{\mu_a}^4$	Gamma	0.13	0.13	σ_ζ^8	Gamma	2.57	2.57
$\sigma_{\mu_a}^8$	Gamma	0.13	0.13	σ_{ygr}^{me}	Uniform	0.00	0.30
$\sigma_{\mu_x}^0$	Gamma	0.45	0.45				
$\sigma_{\mu_x}^4$	Gamma	0.19	0.19				
$\sigma_{\mu_x}^8$	Gamma	0.19	0.19				

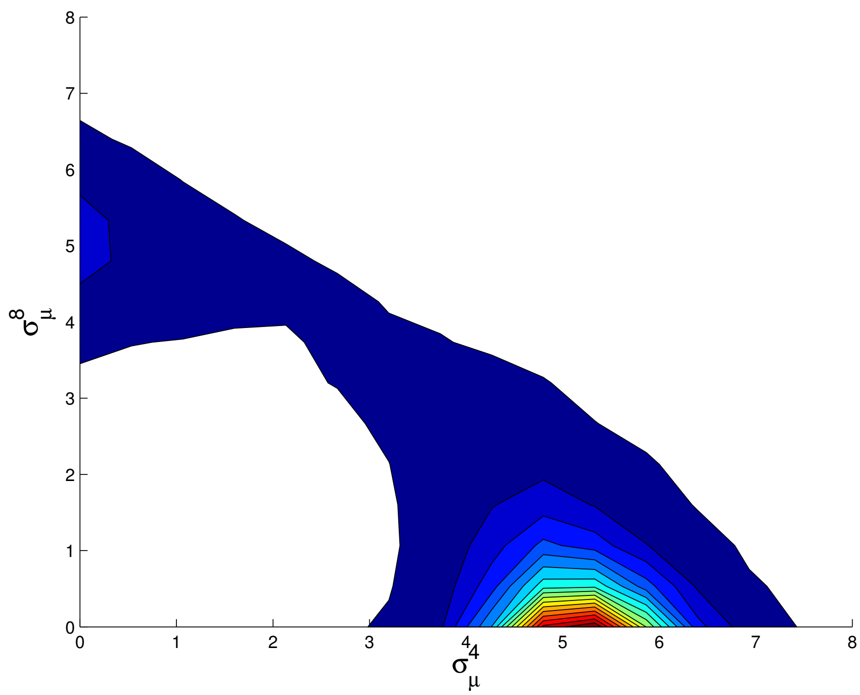
Notes: Para (1) and Para (2) correspond to the mean and standard deviation of the Beta, Gamma, and Normal distributions and to the upper and lower bounds of the support for the Uniform distribution. For the Inv. Gamma distribution, Para (1) and Para (2) refer to s and ν, where $p(\sigma|\nu, s) \propto \sigma^{-\nu-1} e^{-\nu s^2/2\sigma^2}$.

Table A-7: Posterior Comparison for News Model (SGU Prior)

	RWMH			SMC		
Parameter	Mean	$[0.05, 0.95]$	STD(Mean)	Mean	$[0.05, 0.95]$	STD(Mean)
σ_{μ}^{4}	4.33	[0.84, 5.92]	0.49	4.26	[0.28, 5.91]	0.24
σ_{μ}^{8}	1.34	[0.04, 4.83]	0.49	1.36	[0.03, 5.14]	0.24
$\sigma_{z^i}^{8}$	5.68	[0.87, 10.54]	0.30	5.59	[0.75, 10.59]	0.09
$\sigma_{z^i}^{4}$	3.08	[0.21, 7.80]	0.24	3.14	[0.21, 7.98]	0.04
σ_{ζ}^{0}	3.80	[0.51, 6.78]	0.22	3.82	[0.50, 6.77]	0.10
σ_{ζ}^{8}	2.62	[0.17, 6.07]	0.18	2.65	[0.17, 6.22]	0.11
$\sigma_{z^i}^{0}$	12.36	[9.05, 16.12]	0.09	12.27	[9.07, 15.84]	0.09
κ	9.33	[7.49, 11.40]	0.09	9.32	[7.48, 11.33]	0.05
θ	4.14	[3.22, 5.19]	0.05	4.13	[3.19, 5.18]	0.02
σ_{ζ}^{4}	2.44	[0.15, 6.04]	0.04	2.43	[0.15, 5.95]	0.09
σ_{μ}^{0}	0.92	[0.06, 2.39]	0.04	1.04	[0.06, 2.79]	0.04
σ_{g}^{0}	0.60	[0.06, 1.07]	0.03	0.62	[0.06, 1.08]	0.01
σ_{g}^{8}	0.41	[0.03, 0.98]	0.03	0.41	[0.03, 0.99]	0.01
$\sigma_{\mu_a}^{4}$	0.16	[0.01, 0.34]	0.01	0.16	[0.01, 0.34]	0.00
ρ_{z^i}	0.43	[0.21, 0.63]	0.01	0.43	[0.21, 0.63]	0.00
σ_{g}^{4}	0.57	[0.04, 1.06]	0.01	0.55	[0.04, 1.06]	0.02
$\sigma_{\mu_a}^{0}$	0.21	[0.02, 0.35]	0.01	0.21	[0.02, 0.35]	0.01
$\sigma_{\mu_x}^{8}$	0.12	[0.01, 0.29]	0.01	0.12	[0.01, 0.29]	0.00
$\sigma_{\mu_a}^{8}$	0.15	[0.01, 0.33]	0.01	0.16	[0.01, 0.34]	0.00
ρ_{x_g}	0.64	[0.37, 0.83]	0.01	0.67	[0.40, 0.86]	0.03
$\sigma_{\mu_x}^{0}$	0.36	[0.18, 0.52]	0.01	0.37	[0.19, 0.54]	0.01
σ_{z}^{0}	0.66	[0.56, 0.74]	0.01	0.65	[0.54, 0.74]	0.01
$\sigma_{\mu_x}^{4}$	0.10	[0.01, 0.26]	0.01	0.10	[0.01, 0.27]	0.00
σ_{z}^{4}	0.13	[0.01, 0.31]	0.01	0.13	[0.01, 0.32]	0.00
ρ_{ζ}	0.19	[0.08, 0.31]	0.00	0.19	[0.08, 0.32]	0.00
σ_{z}^{8}	0.12	[0.01, 0.28]	0.00	0.12	[0.01, 0.29]	0.00
ρ_{x}	0.88	[0.68, 0.99]	0.00	0.86	[0.65, 0.99]	0.01
δ_2/δ_1	0.42	[0.31, 0.55]	0.00	0.42	[0.31, 0.56]	0.00
ρ_{a}	0.48	[0.39, 0.57]	0.00	0.48	[0.38, 0.57]	0.00
ρ_{z}	0.91	[0.85, 0.96]	0.00	0.91	[0.84, 0.96]	0.00
ρ_{g}	0.96	[0.92, 0.99]	0.00	0.96	[0.92, 0.99]	0.00
ρ_{μ}	0.98	[0.95, 1.00]	0.00	0.98	[0.95, 1.00]	0.00
b	0.92	[0.89, 0.94]	0.00	0.92	[0.89, 0.94]	0.00
σ_{ygr}^{me}	0.30	[0.30, 0.30]	0.00	0.30	[0.30, 0.30]	0.00
γ	0.00	[0.00, 0.00]	0.00	0.02	[0.00, 0.01]	0.01

Notes: Means and standard deviations are over 20 runs for each algorithm. The RWMH algorithms use 10 million draws with the first 5 million discarded. The SMC algorithms use 30,048 particles and 500 stages.

Figure A-2: News Model: Bivariate Contour Plot of σ_μ^4 and σ_μ^8



Notes: The figure shows a This figure shows contour plots for bivariate kernel density estimates of the posteriors for $[\sigma_\mu^4, \sigma_\mu^8]$ from the SMC simulator, conditional on $\gamma < 0.01$.