Testing for Indeterminacy:
An Application to U.S. Monetary Policy

Technical Appendix

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Solving the New Keynesian Model

The simplified version of the DSGE model presented in Sections 5 of the paper can be expressed in terms of $\xi_t = [\xi_t^x, \xi_t^\pi]'$ as

$$\underbrace{\begin{bmatrix} 1 & \tau \\ 0 & \beta \end{bmatrix}}_{\Gamma_0} \xi_t = \underbrace{\begin{bmatrix} 1 & \tau\psi \\ -\kappa & 1 \end{bmatrix}}_{\Gamma_1} \xi_{t-1} + \underbrace{\begin{bmatrix} \tau & -1 & 0 \\ 0 & 0 & \kappa \end{bmatrix}}_{\Psi} \epsilon_t + \underbrace{\begin{bmatrix} 1 & \tau\psi \\ -\kappa & 1 \end{bmatrix}}_{\Pi} \eta_t. \tag{1}$$

Premultiply the system by

$$\Gamma_0^{-1} = \begin{bmatrix} 1 & -\tau/\beta \\ 0 & 1/\beta \end{bmatrix} \tag{2}$$

to obtain

$$\begin{aligned} \xi_t \quad = \quad & \underbrace{\begin{bmatrix} 1+\frac{\kappa\tau}{\beta} & \tau(\psi - \frac{1}{\beta}) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}}_{\Gamma_1^*} \xi_{t-1} \\ & + \underbrace{\begin{bmatrix} \tau & -1 & -\frac{\tau\kappa}{\beta} \\ 0 & 0 & \frac{\kappa}{\beta} \end{bmatrix}}_{\Psi^*} \epsilon_t + \underbrace{\begin{bmatrix} 1+\frac{\kappa\tau}{\beta} & \tau(\psi - \frac{1}{\beta}) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}}_{\Pi^*} \eta_t. \end{aligned} \tag{3}$$

The eigenvalues of Γ_1^* are a solution to the equation

$$0 = \det \begin{bmatrix} (1+\kappa\tau/\beta) - \lambda & \tau(\psi - 1/\beta) \\ -\kappa/\beta & 1/\beta - \lambda \end{bmatrix} \tag{4}$$

which can be rewritten as

$$0 = \lambda^2 - \lambda \left[1 + \frac{1}{\beta}(1+\kappa\tau) \right] + \frac{1}{\beta}(1+\kappa\tau\psi). \tag{5}$$

The solution of this quadratic equation is

$$\lambda_1, \lambda_2 = \underbrace{\frac{1}{2}\left(1 + \frac{\kappa\tau + 1}{\beta}\right)}_{l_1} \mp \underbrace{\frac{1}{2}\sqrt{\left(\frac{\kappa\tau+1}{\beta} - 1\right)^2 + \frac{4\kappa\tau}{\beta}(1-\psi)}\,.}_{l_2} \tag{6}$$

The eigenvectors have to satisfy the relationship

$$\begin{bmatrix} 1+\kappa\tau/\beta & \tau(\psi - 1/\beta) \\ -\kappa/\beta & 1/\beta \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = l_1 \begin{bmatrix} x \\ 1 \end{bmatrix} \mp l_2 \begin{bmatrix} x \\ 1 \end{bmatrix}. \tag{7}$$

Thus, x solves

$$-(\kappa/\beta)x + 1/\beta = l_1 \mp l_2, \tag{8}$$

which implies that

$$x = \frac{1}{\kappa}(1 - \beta l_1 \pm \beta l_2). \tag{9}$$

Thus, we obtain the following Jordan decomposition of Γ_1^*

$$\Gamma_1^* = J\Lambda J^{-1}, \tag{10}$$

where

$$J = \begin{bmatrix} \frac{1}{\kappa}(1 - \beta l_1 + \beta l_2) & \frac{1}{\kappa}(1 - \beta l_1 - \beta l_2) \\ 1 & 1 \end{bmatrix} \tag{11}$$

$$\Lambda = \begin{bmatrix} l_1 - l_2 & 0 \\ 0 & l_1 + l_2 \end{bmatrix} \tag{12}$$

$$J^{-1} = \frac{1}{2\beta l_2} \begin{bmatrix} \kappa & -1 + \beta l_1 + \beta l_2 \\ -\kappa & 1 - \beta l_1 + \beta l_2 \end{bmatrix}. \tag{13}$$

Let $w_t = J^{-1}\xi_t$. We can now write the LRE system in terms of the transformed variables

$$w_t = \Lambda w_{t-1} + J^{-1}\Psi^* \epsilon_t + J^{-1}\Pi^* \eta_t. \tag{14}$$

Determinacy

If both λ_1 and λ_2 are unstable then the (unique) solution is

$$\begin{aligned} \eta_t &= -\Pi_*^{-1}\Psi^* \epsilon_t \qquad (15) \\ &= -\begin{bmatrix} 1 + \frac{\kappa\tau}{\beta} & \tau(\psi - \frac{1}{\beta}) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}^{-1} \begin{bmatrix} \tau & -1 & -\frac{\tau\kappa}{\beta} \\ 0 & 0 & \frac{\kappa}{\beta} \end{bmatrix} \epsilon_t \\ &= -\frac{\beta}{1 + \kappa\tau/\beta - \kappa\tau/\beta + \kappa\tau\psi} \begin{bmatrix} 1/\beta & \tau(1/\beta - \psi) \\ \kappa/\beta & 1 + \kappa\tau/\beta \end{bmatrix} \begin{bmatrix} \tau & -1 & -\frac{\tau\kappa}{\beta} \\ 0 & 0 & \frac{\kappa}{\beta} \end{bmatrix} \epsilon_t \\ &= -\frac{1}{1 + \kappa\tau\psi} \begin{bmatrix} \tau & -1 & -\tau\kappa\psi \\ \kappa\tau & -\kappa & \kappa \end{bmatrix} \epsilon_t. \end{aligned}$$

The law of motion for output, inflation, and interest rates is given by

$$\begin{bmatrix} x_t \\ \pi_t \\ R_t \end{bmatrix} = \frac{1}{1+\kappa\tau\psi} \begin{bmatrix} -\tau & 1 & \tau\kappa\psi \\ -\kappa\tau & \kappa & -\kappa \\ 1 & \kappa\psi & -\kappa\psi \end{bmatrix} \begin{bmatrix} \epsilon_{R,t} \\ \epsilon_{g,t} \\ \epsilon_{z,t} \end{bmatrix}. \tag{16}$$

Indeterminacy

In order to solve the system under indeterminacy we have to calculate

$$J^{-1}\Psi^* = \frac{1}{2\beta l_2} \begin{bmatrix} \kappa & -1+\beta l_1+\beta l_2 \\ -\kappa & 1-\beta l_1+\beta l_2 \end{bmatrix} \begin{bmatrix} \tau & -1 & -\frac{\tau\kappa}{\beta} \\ 0 & 0 & \frac{\kappa}{\beta} \end{bmatrix}. \tag{17}$$

The (second) row of this vector corresponding to the unstable eigenvalue is

$$\Psi_x^J = \frac{1}{2\beta l_2} \begin{bmatrix} -\kappa\tau & \kappa & \kappa(\lambda_2 - 1) \end{bmatrix}, \tag{18}$$

since $1 + \tau\kappa/\beta + 1/\beta = 2l_1$ and $\lambda_2 = l_1 + l_2$. Moreover,

$$J^{-1}\Pi^* = \frac{1}{2\beta l_2} \begin{bmatrix} \kappa & -1+\beta l_1+\beta l_2 \\ -\kappa & 1-\beta l_1+\beta l_2 \end{bmatrix} \begin{bmatrix} 1+\frac{\kappa\tau}{\beta} & \tau(\psi-\frac{1}{\beta}) \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix} \tag{19}$$

and the (second) row of this matrix corresponding to the unstable eigenvalue is

$$\begin{aligned} \Psi_x^J &= \frac{1}{2\beta l_2} \begin{bmatrix} -\kappa - \kappa^2\tau/\beta - \kappa/\beta + \kappa l_1 - \kappa l_2 \\ -\kappa\tau(\psi - 1/\beta) + 1/\beta - l_1 + l_2 \end{bmatrix}' \\ &= \frac{1}{2\beta l_2} \begin{bmatrix} -\kappa\lambda_2 \\ \lambda_2 - 1 - \kappa\tau\psi \end{bmatrix}'. \end{aligned} \tag{20}$$

Thus, the stability condition can be expressed as

$$-\kappa\tau\epsilon_{R,t} + \kappa\epsilon_{g,t} + \kappa(\lambda_2 - 1)\epsilon_{z,t} - \kappa\lambda_2\eta_t^y + [\lambda_2 - 1 - \kappa\tau\psi]\,\eta_t^\pi = 0. \tag{21}$$

Note that in the paper we are using

$$\Pi_x^J = [-\kappa\lambda_2 \quad (\lambda_2 - 1 - \kappa\tau\psi)]$$

and

$$\Psi_x^J = \begin{bmatrix} -\kappa\tau & \kappa & \kappa(\lambda_2 - 1) \end{bmatrix},$$

which corresponds to a slightly different normalization of the matrix of eigenvectors J, than the one used in above derivations.

The singular value decomposition of Π_x^J yields

$$
\begin{aligned}
U_{.1} &= 1 \\
D_{11} &= d = \sqrt{(\kappa\lambda_2)^2 + (\lambda_2 - 1 - \kappa\tau\psi)^2} \\
V'_{.1} &= [-\kappa\lambda_2 \quad (\lambda_2 - 1 - \kappa\tau\psi)]/d \\
V'_{.2} &= [(\lambda_2 - 1 - \kappa\tau\psi) \quad \kappa\lambda_2]/d.
\end{aligned}
$$

Finding the Boundary of the Determinacy Region

To analyze the standard New Keynesian model without restrictions on the autocorrelation parameters ρ_R, ρ_g, and ρ_z it is useful to write it in terms of ξ_t^x, ξ_t^π, and R_t:

$$
\begin{bmatrix} \xi_t^x \\ \xi_t^\pi \\ R_t \end{bmatrix} = \underbrace{\begin{bmatrix} 1 + \frac{\tau\kappa}{\beta} + \tau(1-\rho_R)\psi_2 & -\frac{\tau}{\beta} + \tau(1-\rho_R)\psi_1 & \tau\rho_R \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 \\ (1-\rho_R)\psi_2 & (1-\rho_R)\psi_1 & \rho_R \end{bmatrix}}_{\widetilde{\Gamma}_1(\theta)} \begin{bmatrix} \xi_{t-1}^x \\ \xi_{t-1}^\pi \\ R_t \end{bmatrix} \quad (22)
$$

$$
+\widetilde{\Psi} \begin{bmatrix} \epsilon_{R,t} \\ g_t \\ z_t \end{bmatrix} + \widetilde{\Pi}\eta_t
$$

The stability properties of the system depend on the eigenvalues of $\widetilde{\Gamma}_1(\theta)$. At the boundary of the determinacy region it has to be the case that

$$
det(\widetilde{\Gamma}_1(\theta) - I_{3\times 3}) = 0 \quad (23)
$$

since the matrix $\widetilde{\Gamma}_1(\theta)$ has at least one unit eigenvalue. Tedious but straightforward algebra shows that (23) implies that

$$
\psi_1 = 1 - \frac{\beta\psi_2}{\kappa}\left(\frac{1}{\beta} - 1\right) \quad (24)
$$

which is the formula that we used to define the function $g(\theta)$ in Section 4.

Note that these calculations do not formally prove that (24) is the boundary of the determinacy region as they do not reveal the size of the other eigenvalues of $\widetilde{\Gamma}_1(\theta)$. To obtain a formal proof techniques as in Bullard and Mitra (2002) or Lubik and Marzo (2001) have to be applied.

Habit Formation

Suppose that the period utility function is of the form

$$\nu_t = \frac{[C_t/C_{t-1}^{\gamma}]^{1-1/\tau}}{1-1/\tau} \tag{25}$$

where C_{t-1}^{γ} is the habit formation reference consumption level. The resulting first-order condition is of the form

$$\lambda_t = C_t^{-1}\left[\left(\frac{C_t}{C_{t-1}^{\gamma}}\right)^{1-1/\tau} - \beta\gamma \mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t^{\gamma}}\right)^{1-1/\tau}\right]\right]. \tag{26}$$

Now define

$$u_t = \left(\frac{C_t}{C_{t-1}^{\gamma}}\right)^{1-1/\tau} \quad \text{and} \quad u_t^* = u_t - \beta\gamma \mathbb{E}_t[u_{t+1}].$$

We then have

$$\lambda_t = \frac{u_t^*}{C_t}, \tag{27}$$

which leads to the Euler equation

$$\frac{u_t^*}{C_t} = \mathbb{E}_t\left[\beta\frac{u_t^*}{C_{t+1}}R_t/\pi_{t+1}\right]. \tag{28}$$

In the steady state

$$u = C^{(1-\gamma)(1-1/\tau)}, \quad u^* = (1-\beta\gamma)u, \quad \text{and} \quad \lambda = u^*/C$$

Log-linearization gives us

$$\widetilde{u}_t^* - \widetilde{C}_t = \mathbb{E}_t[\widetilde{C}_{t+1}] - \mathbb{E}_t[\widetilde{C}_{t+1}] + (\widetilde{R}_t - \mathbb{E}_t[\widetilde{\pi}_{t+1}]), \tag{29}$$

and the two definitional equations

$$(1-\beta\gamma)\widetilde{u}_t^* = \widetilde{u}_t - \beta\gamma \mathbb{E}_t[\widetilde{u}_{t+1}] \tag{30}$$

$$\widetilde{u}_t = \left(1-\frac{1}{\tau}\right)\widetilde{C}_t - \gamma\left(1-\frac{1}{\tau}\right)\widetilde{C}_{t-1}. \tag{31}$$

Thus,

$$\begin{aligned}(1-\beta\gamma)\widetilde{u}_t^* &= \left(1-\frac{1}{\tau}\right)\widetilde{C}_t - \gamma\left(1-\frac{1}{\tau}\right)\widetilde{C}_{t-1} \\ &\quad -\beta\gamma\left(1-\frac{1}{\tau}\right)\mathbb{E}_t[\widetilde{C}_{t+1}] + \beta\gamma^2\left(1-\frac{1}{\tau}\right)\widetilde{C}_t.\end{aligned} \tag{32}$$

Combining (29) and (32) these three equations leads to

$$\begin{aligned}&\left[\tau + \frac{1+\beta\gamma^2+\gamma}{1-\beta\gamma}(1-\tau)\right]\widetilde{x}_t \\ &= \frac{\gamma(1-\tau)}{1-\beta\gamma}\widetilde{x}_{t-1} + \left[\tau + \frac{1+\beta\gamma(1+\gamma)}{1-\beta\gamma}(1-\tau)\right]\mathbb{E}_t[\widetilde{x}_{t+1}] \\ &\quad -\frac{\beta\gamma}{1-\beta\gamma}(\tau-1)\mathbb{E}_{t+2}[\widetilde{x}_{t+2}] - \tau(\widetilde{R}_t - \mathbb{E}_t[\widetilde{\pi}_{t+1}]) + g_t,\end{aligned} \tag{33}$$

where $\widetilde{C}_t$ is replace by $\widetilde{x}_t$ and g_t captures the residual effect of the discrepancy between output and consumption on the Euler equation.