SYMPLECTIC INTEGRATORS FOR SOLAR SYSTEM DYNAMICS

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ABSTRACT

Many problems in solar system dynamics are described by Hamiltonians of the form $H = H_{Kep} + \epsilon H_{pert}$, $\epsilon \ll 1$, where $H_{Kep}$ is the usual Hamiltonian for the Kepler two-body problem and $\epsilon H_{pert}$ represents (for example) much weaker perturbations from the planets. We review symplectic integrators for Hamiltonians of this kind, focusing on methods that exploit the integrability of $H_{Kep}$. We show that the long-term errors in these integrators can be reduced by a factor of order $\epsilon$ by suitable starting procedures, for example, by starting with a very small stepsize and gradually increasing the stepsize to its final value. The resulting integrators are easily the best available for a wide range of solar system problems.

1. INTRODUCTION

The speed of modern digital computers allows us to examine the evolution of solar system orbits over very long times (up to $10^5$ orbits) by direct numerical integration. Long integrations imply the need for accurate and robust integration algorithms.

For the past several decades long integrations have mostly used high-order multistep integration methods in Cartesian coordinates. Multistep methods fit polynomials to the positions and accelerations at several previous time steps to compute the latest position. The most popular of these is the 13 step Störmer integrator, which has been used for many long integrations, including the 845 Myr integration of the outer planets by Sussman & Wisdom (1988). A disadvantage of Störmer methods (in common with most other integration schemes) is that the errors in position grow quadratically in time. This problem is remedied by the symmetric multistep integrators (Lambert & Watson 1976) in which position errors grow linearly; recently Quinlan & Tremaine (1990) have derived high-order1 forms of these suitable for solar system work. Quinn et al. (1991) use the 12 step symmetric multistep method to integrate all of the planets for 3 Myr.

Another approach is to use the so-called symplectic integrators; these are algorithms specific to Hamiltonian systems, that enforce exactly certain conservation laws characteristic of Hamiltonian dynamics—see Channell & Scovel (1990) for an elementary review, or Sanz-Serna (1992) for a more formal one. For example, symplectic integrators conserve phase-space volume and hence have no spurious dissipation. A disadvantage is that high-order symplectic algorithms are much more complicated than multistep algorithms of the same order.

The construction of symplectic integrators is simplified if the Hamiltonian can be written in the form

$$H = T(p) + V(q),$$

where $p$ and $q$ are canonical momenta and coordinates. The Hamiltonian for the nonrelativistic gravitational $N$-body problem has this form if $q$ and $p$ are Cartesian coordinates and momenta ($T$ and $V$ are simply the kinetic and potential energy). Gladman & Duncan (1990) used a fourth-order symplectic integrator in Cartesian coordinates to follow the evolution of $10^5$ test particles in the outer solar system for $2 \times 10^4$ yr.

A special feature of solar system dynamics is that the perturbations to integrable Kepler motion are usually small, that is, the motion is described by a Hamiltonian,

$$H = H_{Kep} + \epsilon H_{pert},$$

$\epsilon \ll 1$ being the ratio of planetary to solar mass or (for satellite motion) the ratio of tidal to planetary force. This suggests that the integration should be carried out in action-angle variables of the Kepler problem, in which the unperturbed motions are simple (actions are constant and angles increase linearly with time). A difficulty with this approach is that the perturbation Hamiltonian $\epsilon H_{pert}$ is cumbersome to evaluate in action-angle variables.

In this paper we shall be concerned with a particular class of integrators introduced by Wisdom & Holman (1991; hereafter WH). One of these was used by WH to follow the outer solar system for 1 Gyr, and by Sussman & Wisdom (1992) to follow all the planets for 100 Myr. Such integrators were also suggested independently by Kinoshita et al. (1991; hereafter KY). These integrators characteristically make use of two sets of variables: action-angle variables to integrate the Kepler Hamiltonian $H_{Kep}$ and Cartesian coordinates to integrate the perturbation Hamiltonian $\epsilon H_{pert}$. In KYN the integrators are called "modified
symplectic integrators” and in WH simply “N-body maps,” but we adopt the more descriptive name “mixed variable symplectic” (or MVS) integrators; thus MVS2 denotes the second-order integrator and so on.

An important property of the MVS integrators is that they follow exactly (apart from computer roundoff errors) the dynamics of a “surrogate” Hamiltonian

$$\bar{H} = H + H_{\text{err}}.$$  \hfill (3)

$H_{\text{err}}$ depends on the stepsize $\tau$, and can be expressed as a (formal) power series in $\tau$. For an integrator of order $n$, $H_{\text{err}} \sim O(\tau^n)$. The second- and fourth-order integrators appear so far to be the most useful, though higher order ones are known. The desirable properties of symplectic integrators, such as absence of secular energy errors and linear (rather than quadratic) error growth in the longitudes, follow immediately from the existence of $\bar{H}$. It is also convenient for analyzing the integrators that the properties of the integrators are completely described by $H_{\text{err}}$.

The contribution of this paper is to refine the MVS integrators. We show that the dominant long-term error arises from a constant error in the mean motion of $O(\tau^3)$; then we describe special starting procedures which can eliminate this error, leaving a mean error in the mean motion that is only $O(\tau^5)$. The resulting improvement makes the MVS integrators easily the best available for a wide range of solar system problems.

Section 2 of this paper briefly reviews the relevant aspects of the theory of symplectic integrators. Section 3 describes our new starting procedures, while Sec. 4 gives an alternative interpretation that leads to the same conclusion while avoiding most of the Hamiltonian formalism. Section 5 contains numerical tests that verify our analysis and illustrate the improvements offered by the new starting procedures. Section 6 is a summary. We do not consider general relativistic corrections in the main part of this paper, but discuss them briefly in the Appendix.

2. SOME THEORY FOR SYMPLECTIC INTEGRATORS

“Symplectic” is a generalization of “area-preserving.” In a $2N$-dimensional phase space $(p, q)$—within which Hamiltonian systems might live—a symplectic transformation is one that preserves the areas projected by an arbitrary closed loop on any surface $p\,dq$; in other words $dp \wedge dq$ is preserved. (Other names are contact or canonical transformation.) In particular, time evolution under a Hamiltonian is symplectic.

We may think of a numerical integrator applied to a Hamiltonian system as a mapping

$$(p, q)_{t=0} \rightarrow (p, q)_{t=\tau},$$  \hfill (4)

where $\tau$ is the stepsize, and this mapping is iterated. A symplectic integrator is one that guarantees that the mapping (4) is symplectic (apart from roundoff errors). This ensures that the integrator will preserve much of the geometric structure of the original Hamiltonian flow, but in itself says nothing about the accuracy.

We now specialize to Hamiltonians of the form

$$H = H_A + H_B,$$  \hfill (5)

where $H_A$ and $H_B$, considered separately, are both integrable. The solar system Hamiltonian in Eq. (2) is of this form, for $H_{\text{ker}}$ by itself is integrable, and so is $H_{\text{err}}$ since it depends only on position, not velocity. Another example is any Hamiltonian such as (1) that is a sum of kinetic energy and coordinate-dependent potential.

For a system of the form (5), Hamilton’s equations are

$$\dot{z} = [z, H_A + H_B],$$  \hfill (6)

where $z = (p, q)$ and $[,]$ are Poisson brackets. We introduce the operators

$$A \equiv [\cdot, H_A] \quad \text{and} \quad B \equiv [\cdot, H_B].$$  \hfill (7)

The formal solution of (6) is

$$z(\tau) = \exp[\tau(A + B)]z(0).$$  \hfill (8)

Since we have assumed that $H_A$ and $H_B$ are integrable, we know how to calculate $\exp(\tau A)$ and $\exp(\tau B)$. To develop a symplectic integrator we approximate the true evolution operator $\exp[\tau(A + B)]$ by a product (composition) of terms like $\exp(\tau A)$ and $\exp(\tau B)$. This can be done via the Baker–Campbell–Hausdorff (BCH) identity:

$$\exp A \exp B = \exp(A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A - B, [A, B]]$$
$$+ \langle \text{more terms} \rangle),$$  \hfill (9)

where $[,]$ is a commutator, and $\langle \text{more terms} \rangle$ consist entirely of nested commutators of $A$ and $B$.

As a simple example, consider

$$\exp(\tau A)\exp(\tau B) = \exp(\tau(A + B) + \langle \text{error} \rangle).$$  \hfill (10)

It is a straightforward to show that $\langle \text{error} \rangle$ can be written as $[\cdot, H_{\text{err}}]$. In other words, the sequence of operations on the left side of (10) is the same as evolving for time $\tau$ under the Hamiltonian

$$H_A + H_B + H_{\text{err}}.$$  \hfill (11)

From (10) and the BCH identity, it is straightforward to derive

$$H_{\text{err}} = (\tau/2)\{H_A, H_B\} + O(\tau^2).$$  \hfill (12)

A single timestep $\tau$ of the integrator described by the left

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3The existence of a surrogate Hamiltonian is not necessary for linear error growth in longitude. More general symplectic integrators than the ones considered here have no (known) surrogate Hamiltonian, but do show linear error growth; also, the symmetric multi-step integrators show linear error growth even though symplecticity is not easily defined for multi-step methods.

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side of (10) consists of first advancing the dynamical system for a time $\tau$ under the influence of $H_B$ alone, then advancing for $\tau$ under the influence of $H_A$ alone. Since the error Hamiltonian $H_{\text{err}}$ is $O(\tau)$, the error in a single timestep is $O(\tau^2)$; thus this integrator is first order.

We denote the first-order symplectic integrator described by the left side of (10) as S1. If $H_A$ and $H_B$ are separable in Cartesian coordinates and momenta [as in Eq. (1)] we may call the integrator CS1, while if the actions for $H_A$ are not canonically conjugate to the actions of $H_B$ ("mixed variables") we call the integrator MVS1.

To derive a higher order integrator, consider

$$\exp(i\tau A)\exp(\tau B)\exp(i\tau A) = \exp(\tau(A + B) + \{\text{error}\}), \quad (13)$$

Using the BCH identity it is straightforward to derive

$$H_{\text{err}} = \frac{\tau^2}{12} \{[H_A, H_B], H_B + \frac{1}{2}H_A\} + O(\tau^4), \quad (14)$$

so the method described by the left side of (13) is second order. Note the close relation of the second- and first-order integrators: $N$ steps of the second-order integrator are equivalent to advancing the system for a time $\frac{\tau}{N}$ under $H_A$, then taking $N-1$ steps of the first-order integrator, then advancing the system again for a time $\frac{\tau}{N}$ under $H_A$. The extra half-steps provide a simple way to improve the accuracy of the integrator from first to second order. For the case where $H_A = \frac{1}{2}p^2$ and $H_B = V(q)$ the integrator is simply leapfrog.4

A fourth-order integrator is obtained if we concatenate three second-order steps with sizes in the ratio $1:-2^{1/3}:1$. Yoshida (1990) gives formulas for higher order integrators.

From the symmetry of the formulas, it is clear that the integrators described in this section are time reversible, apart from roundoff.

The BCH identity was first used to systematize the derivation of integration formulas by Forest & Ruth (1990). That the errors are described by a Hamiltonian $H_{\text{err}}$ was demonstrated by Yoshida (1991). For further details on the material covered in this section, the reader is referred to these papers in addition to WH and KYN. Also of interest is the analysis by Wisdom & Holman (1992) of MVS2 at large stepsizes (when $H_{\text{err}}$ would not be dominated by the leading term of a power series in $\tau$, as we have assumed here).

For application to solar system integrations using a Hamiltonian of the form (2), we note that advancing under the influence of $H_{\text{Kepler}}$ alone is trivial if we use action-angle variables or orbital elements. Advancing under $eH_{\text{pert}}$ alone is trivial in Cartesian components (the velocities change while the positions stay fixed). As WH point out, the whole operation of transforming from Cartesian coordinates to orbital elements, advancing the mean anomaly, and transforming back is efficiently encapsulated in Gauss's $f$ and $g$ functions.

### 3. Starting Procedures

The surrogate Hamiltonian $\tilde{H} = H + H_{\text{err}}$ can be used to estimate the errors in an integration, as illustrated by KYN. We now show that in the case of MVS integrators applied to Hamiltonians of the solar system form (2), the leading source of error can be more than identified, it can be removed.

The most important measure of the accuracy of an integrator in most solar system problems is the mean error in the mean motion, since this determines both (a) the rate at which the error in position grows (because most of the position error is in the longitude), and (b) whether mean-motion resonances and near-resonances in the original dynamical system are also present in the numerical integration.

In the following discussion we analyze the surrogate Hamiltonian $\tilde{H}$ using perturbation theory (treating $H_{\text{err}}$ as a perturbation on $H$) to compare the frequencies $\tilde{\omega}$ of the surrogate system with the frequencies $\omega$ of the actual system, and then discuss how to reduce the differences. We consider only regular orbits, and in particular assume that $H$ has at least approximate actions and angles. We also assume that $H_{\text{err}}$ is small enough (i.e., $\tau$ is small enough) that perturbation theory gives useful results at first order.

We consider MVS2, but similar arguments apply at higher orders. Substituting the Hamiltonian (2) into the expression (14) for $H_{\text{err}}$, we have

$$H_{\text{err}} = \frac{\tau^2}{24} \{[H_{\text{Kepler}} H_{\text{pert}}, H_{\text{Kepler}}\} + O(\tau^2). \quad (15)$$

We will evaluate the Poisson brackets in terms of the action-angle variables of the actual system $H$. Suppose $J_i$, $\theta$ are the actions and angles of $H$, and that $\omega = (\partial H/\partial \theta)$ are the canonical frequencies. Now $J_i$, $\theta$ differ from the action-angles of $H_{\text{Kepler}}$ (the Delaunay elements) at $O(\epsilon)$:

$$J_1 = L + O(\epsilon); \quad J_2 = G + O(\epsilon); \quad J_3 = H + O(\epsilon);$$

$$\theta_1 = I + O(\epsilon); \quad \theta_2 = g + O(\epsilon); \quad \theta_3 = h + O(\epsilon). \quad (16)$$

Since $H_{\text{Kepler}}$ depends only on $L$, $H_{\text{Kepler}}$ will depend on $J_2$, $J_3$, $\theta_2$, $\theta_3$ only at $O(\epsilon)$, and hence $\omega_2$, $\omega_3$ are $O(\epsilon)$. Thus

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4 Leapfrog is usually written as

$$q_{i+1/2} = q_{i} + \frac{1}{2}p_{i}, \quad p_{i+1/2} = p_{i} - \tau \frac{\partial V}{\partial q_{i}} \quad q_{i+1} = q_{i+1/2} + \frac{1}{2}p_{i+1/2}.$$ 

But it is easy to see that this form is equivalent to the one in the text above (see also Ruth 1983).

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5 A minor notational degeneracy: in the following equation "$\tilde{H}$" refers to a Delaunay element, but everywhere else in this paper it denotes the Hamiltonian. The Delaunay elements themselves are derived in, for example, Sec. 143 of Plummer (1960) or Sec. 9.9 of Brouwer & Clemence (1961).
\[ H_{\text{err}} = \frac{\epsilon r^2}{24} \left( \frac{\partial H_{\text{Kep}}}{\partial \mathbf{r}} \right)^2 \left( \frac{\partial^2 H_{\text{pert}}}{\partial \mathbf{r}^2} \right) + O(\epsilon^3 r^2). \] (17)

We can write \( H_{\text{err}} \) in terms of Fourier components as
\[ H_{\text{err}} = \epsilon r^2 \sum_m X_m(\mathbf{J}) \exp(i m \cdot \theta) + O(\epsilon^3 r^2), \] (18)
where \( m \) is a vector of integers \( m_1, m_2, m_3 \) and \( X_m = 0 \) if \( m_1 = 0 \) because of the \( (\partial^2 H_{\text{pert}}/\partial \theta^2) \) term in (17). Thus \( H_{\text{err}} \) differs from \( H_{\text{pert}} \) in having no secular terms (terms independent of the mean anomaly \( I \)) at \( O(\epsilon) \).

We now focus on the errors caused by the terms in (18) that are \( O(\epsilon^3 r^2) \). From perturbation theory, we have for the actions and frequencies of the surrogate system, to \( O(\epsilon^3) \):
\[ \mathbf{J} = \mathbf{J} - \epsilon r^2 \sum_m \frac{m X_m}{\omega \cdot m} \exp(i m \cdot \theta) \] (19a)
\[ \tilde{\omega}(\mathbf{J}) = \omega(\mathbf{J}) + \epsilon r^2 \frac{\partial X_0}{\partial \mathbf{J}} = \omega(\mathbf{J}). \] (19b)

The problem of small denominators does not arise because \( \omega \cdot m \) is near-zero only if \( m_1 = 0 \), in which case \( X_m = 0 \).

Since \( \mathbf{J} \) is fixed in the surrogate system (\( \mathbf{H} \)), and \( \mathbf{J} \) is fixed in the real system (\( H \)), the frequency error will be zero (i.e. \( \tilde{\omega} = \omega \)) to the order \( O(\epsilon^3) \) of Eq. (19) if and only if, at the start of the integration, \( \mathbf{J} = \mathbf{J}_0 \) to the same order. If this condition is satisfied, the frequency error be of \( O(\epsilon^3 r^2) \), a significant improvement since \( \epsilon \approx 10^{-3} \) for most solar system problems.

Problem: How can we start an integrator (in general of order \( n \)) so that the action for the surrogate system equals the action for the actual system to \( O(\epsilon^n) \)? There are at least two possibilities.

(i) The warm start: A slow change in a Hamiltonian system preserves the actions (adiabatic invariants), so one way is to run the integrator while gradually increasing the stepsize from 0 to \( \tau \) (in effect, slowly changing from \( H \) to \( \mathbf{H} \)). If this is done over time \( T \), the change in actions is
\[ \Delta J \approx \epsilon^3 \sum_m \frac{m X_m}{\omega \cdot m} \left( \frac{\epsilon^3}{\omega^2 T} \right) \] (20)

We call this "warming up" the integrator. There are many possible ways to warm up the integrator; what we do is to integrate forward in time for \( S \) steps, backward for \( 2S \) steps, and then forward again for \( S \) steps, increasing the stepsize by \( \tau/(4S) \) at each step; this process returns to the initial point in time. Note the curious fact that artificially introducing a small error in the initial conditions in this way results in a significant reduction in error at later times.

(ii) The iterated start: Equation (19) shows that \( \mathbf{J} \) oscillates on the orbital time scale. This suggests that we try starting an MVS integrator from different points on an orbit, testing the accuracies over some reasonable time span using an integration with shorter stepsize for reference, and then picking the best starting point (which occurs at the phase of the oscillation when \( \mathbf{J} \) is closest to \( \mathbf{J}_0 \)).

With these starting procedures, the dominant errors consist of short-period errors in all three actions and long-period errors in \( J_2 \) and \( J_3 \) that are \( O(\epsilon^3) \), and mean errors in the actions that are \( O(\epsilon^3 r^2) \). By reducing the mean error by \( O(\epsilon) \) the starting procedures substantially improve the accuracy of the mean motions in the numerical integration.

In Sec. 5 we test both of these starting procedures. We refer to integrators that are run without special starting procedures as having "cold" starts.

4. ANOTHER INTERPRETATION

In this section we present an alternative derivation of some of our results, which follows more closely WH's approach to the MVS integrators.

To derive a symplectic integrator for solar system problems we replace the Hamiltonian (2) by
\[ \mathbf{H} = H_{\text{Kep}} + \epsilon H_{\text{pert}} = H_{\text{Kep}} + \epsilon H_{\text{pert}} \sum_{k=-\infty}^{\infty} \delta(t-t_0-k\tau), \] (21)

that is, we multiply \( H_{\text{pert}} \) by a packet fence of delta functions. Setting \( \mathbf{H} = \mathbf{H} + \epsilon H_{\text{pert}} \) as usual, we have
\[ H_{\text{err}} = \epsilon H_{\text{pert}} \sum_{k=-\infty}^{\infty} \delta(t-t_0-k\tau) - 1. \] (22)

If \( \tau \) is small (compared to the typical orbital period), then the error Hamiltonian contains only high-frequency terms, which should have little effect on the motion (see WH). Between the times when the delta functions act, the system described by \( \mathbf{H} \) evolves under the influence of \( H_{\text{Kep}} \) alone; this evolution is straightforward to compute since \( H_{\text{Kep}} \) is integrable. The evolution across the delta functions is described solely by \( H_{\text{pert}} \), and is easy to calculate since \( H_{\text{pert}} \) depends only on the coordinates. The resulting integrator is equivalent to the integrator MVS1 of Sec. 2 if we start and end the integration just before one of the delta functions; the integrator is equivalent to MVS2 if we start and end midway between the delta functions. Higher order integrators can be derived using several delta functions in a single timestep \( \tau \); however, this method is somewhat less general than the method used in Sec. 2 to derive integrators, because it does not allow evolution in the negative time direction, such as occurs in the integrator MVS4 mentioned in Sec. 2.

We now ask what effect the error Hamiltonian \( H_{\text{err}} \) has on the actions \( \mathbf{J} \) of the exact Hamiltonian \( H \) when the integrator is used to follow the system over the time interval \([0, \tau] \). We have \( \mathbf{J}(t) = -\partial H/\partial \theta = -\partial H_{\text{err}}/\partial \theta \). Since \( H_{\text{err}} \) is \( O(\epsilon) \) we have
\[ \mathbf{J}(t) = \mathbf{J}_0 - \int_0^t \left( \frac{\partial H_{\text{err}}}{\partial \theta} \right) dt' + O(\epsilon^3), \] (23)
where the integrand is evaluated at the unperturbed phase-space coordinates \( \mathbf{J} = \mathbf{J}_0 = \mathbf{J}(t=0), \mathbf{\theta} = \mathbf{\theta}_0 + \omega t, \omega = (\partial H/\partial \mathbf{J})_{\mathbf{J}_0} \).
A typical term in the perturbation Hamiltonian is the real part of

$$H_{\text{pert}} = Y_n(\mathbf{J}) \exp(i\mathbf{n} \cdot \mathbf{\omega}),$$

where \( \mathbf{n} \) is a vector of integers. (For simplicity, we neglect secular terms with \( n_1 = 0 \), so that \( n \cdot \mathbf{\omega} = 0 \).) Using Eq. (22), we may then evaluate (23) to find

$$J(t) = J_0 + \sum_{k=k_1}^{k_2} \exp(i\lambda(t_0 + k\tau)) \left[ \frac{\exp(i\lambda t) - 1}{i\lambda} - \tau \frac{\exp(i\lambda t_0) - 1}{i\lambda} \right] + O(e^2),$$

where \( \lambda = n \cdot \mathbf{\omega} \), and \( k_1 \) and \( k_2 \) are the smallest and largest integers such that \( t_0 + k\tau \in [0, \tau] \). Evaluating the sum,

$$J(t) = J_0 + \sum_{k=k_1}^{k_2} \exp(i\lambda(t_0 + k\tau)) \left[ \frac{\exp(i\lambda t) - 1}{i\lambda} - \tau \frac{\exp(i\lambda t_0) - 1}{i\lambda} \right] + O(e^2).$$

We now ask for the time-averaged value of \( J(t) \), which we denote \( \langle J \rangle \). All terms involving exponentials with \( t \) or \( k\tau \) in the argument average to zero, which leaves

$$\langle J \rangle = J_0 + \sum_{k=k_1}^{k_2} \frac{\exp[i\lambda(t_0 + k\tau)]}{i\lambda} = J_0 + \frac{1}{2\sin \frac{i\lambda \tau}{2}} + O(e^2).$$

In the MVS2 integrator the first delta function is encountered \( \frac{1}{\tau} \) after the start of the integration so \( t_0 + k\tau = \frac{1}{\tau} \), and we have

$$\langle J \rangle = J_0 + \sum_{k=k_1}^{k_2} \frac{\exp[i\lambda(t_0 + k\tau)]}{i\lambda} = J_0 + \frac{1}{2\sin \frac{i\lambda \tau}{2}} + O(e^2).$$

The error Hamiltonian causes a shift in the mean action \( \Delta \mathbf{J} = \langle \mathbf{J} \rangle - J_0 \), which leads to a frequency shift \( \Delta \mathbf{w} = (\partial^2 H / \partial \mathbf{J}^2) \Delta \mathbf{J} \). For MVS2 the shift is \( O(\tau^2) \) since the quantity in braces in (28) is \( O(\tau^2) \). This shift is the dominant source of position error in long integrations, since the resulting longitude error grows linearly with time.

A clue to how to eliminate this error comes from examining an artificial problem in which \( H_{\text{pert}} \) is multiplied by \( \exp[y(1 - T)] \) where \( T \) and \( y \) are positive constants. It is straightforward to show that the only effect of this modification on Eq. (26) for \( J(t) \) is to replace \( i\lambda t \) by \( y + i\lambda t \), and \( i\mathbf{n} \cdot \mathbf{\omega}_0 \) by \( i\mathbf{n} \cdot \mathbf{\omega}_0 - y T \). We now average \( J(t) \) over the interval \([0, T]\), letting the interval \( T \) infinity while the growth rate \( y \) is fixed. We find precisely the result (27) except that the mean error \( \langle J \rangle - J_0 \) is multiplied by \( \exp(-y T) \). Thus if \( y T > 1 \), the mean error is reduced from \( O(e) \) to \( O(e^2) \).

This result suggests that the shift in the mean action can be removed to leading order if the error Hamiltonian is near zero at the start of the integration and grows slowly to the steady-state value associated with the stepsize \( \tau \). The simplest way to accomplish this is to start with a small stepsize and slowly increase the stepsize to \( \tau \) over many orbits; this is the "warm start" of Sec. 3. Other possibilities exist, for example, switching gradually from a highly accurate integrator to the integrator that will be used for the main part of the integration. A third possibility arises because Eq. (27) shows that the shift in the mean action oscillates on the orbital timescale as a function of the starting phase \( \theta_0 \). Thus the shift can be drastically reduced if the integration is started at the appropriate time; this is the basis for the "iterated start" described in Sec. 3.

5. NUMERICAL RESULTS

For our tests we integrated a Sun-Jupiter-asteroid system. The asteroid is started off with \( a = 2.6 \) AU (between the 3:1 and 8:3 resonances), \( e = 0.25 \) and \( i = 0.2 \) rad. The computations use heliocentric coordinates. When displaying results, we quote times in "orbits"—this refers to the period of the initial oscillating orbit (about 4 yr). Because different integrators use different numbers of force evaluations per step, we quote not the stepsize but the number of force evaluations per orbit. To compute the errors in an orbit integration, we ran the orbit again using a smaller stepsize (small enough to reduce the errors at least fivefold further).

In the MVS integrators, most of the computer time is consumed in advancing the orbit under \( H_{\text{Kep}} \), and evolving under \( H_{\text{pert}} \) takes only about 10% of the total machine cycles. However, for a system with \( N \) interacting planets, the cycles consumed by \( H_{\text{Kep}} \) will scale as \( N \), whereas the cycles consumed by \( H_{\text{pert}} \) will scale as \( N^2 \). A step with one of the multistep integrators takes about 60% of the time taken for a step governed by \( H_{\text{Kep}} \) in the MVS integrators. (These estimates are implementation dependent of course, but we believe our programs are reasonably efficient.)

Figure 1 compares the MVS integrators with others in the literature. Note especially the improvement gotten by warming up the stepsize (compare the curves marked MVS2 and WH2). If much higher accuracies are required than shown in this figure the symmetric multistep integrators described by Quinlan & Tremaine (1990) perform best, simply because of their higher order. On the other hand, for larger stepsizes than those shown here, the low order of the MVS integrators improves their performance relative to the multistep integrators.

The integrators in Fig. 1 all have error in the mean longitude or mean anomaly that grows linearly with time, except for the Störmer integrator which has quadratic error growth. The latter was, until recently, the usual choice for long term solar system integrations, but it is easily outperformed by the newer integrators.

Figures 2(a) and 2(b) illustrate that MVS2 is a second-order integrator and show the improvements from warming up in more detail. The integrations shown by circles were warmed up to the final timestep over about 4000 orbits (the exact duration was varied slightly and at ran-
dom to verify that the results do not depend on the exact length of the warmup, provided the warmup is long enough. The integrations shown by crosses were given a brief warmup over a few orbits (the exact duration also varying at random); this warmup is far too short to reduce the error but mimics the effect of starting the integrations at random phases of the orbit. With fewer than about 30 force evaluations per orbit the warmup works poorly, and with much fewer it fails altogether.

Figure 3 is analogous to Fig. 2(b) but uses MVS4 with iterated starts. In this case, roundoff error began to limit the accuracy in the mean anomaly. (This is also the reason why the MVS4 integration in Fig. 1 was run with fewer evaluations than the others.)

Finally, Fig. 4 illustrates that MVS2 with a warm start really does have longitude error $\propto \epsilon^2$.

6. SUMMARY

We have studied a class of numerical integrators for solar system problems, that has recently been introduced by Wisdom & Holman (1991; WH) and by Kinoshita et al. (1991 KYN). These are symplectic integrators that apply to systems in which the dynamics can be described by a Hamiltonian of the form $H_{\text{keep}} + \epsilon H_{\text{pert}}, \epsilon \ll 1$, where $H_{\text{keep}}$ describes Keplerian motion and $\epsilon H_{\text{pert}}$ describes planetary perturbations (or tidal perturbations for satellites). This exact form is not essential; what is required is that the Hamiltonian should be the sum of two parts, each of which is integrable when considered in isolation.

Operationally, the integrators are easy to describe. In the second-order integrator a step (of size $\tau$, say) involves three operations, being: (i) evolve the system under $H_{\text{keep}}$ alone for time $\frac{1}{2}\tau$; (ii) evolve the system under $\epsilon H_{\text{pert}}$ alone for time $\tau$; (iii) evolve again under $H_{\text{keep}}$ alone for $\frac{1}{2}\tau$. (See KH for algorithmic details.) At the end of (iii)—but not after (i) or (ii)—the time evolution is exactly that of a “surrogate” system with Hamiltonian $\tilde{H}$, differing from $H$ by $O(\epsilon^2)$. Higher order integrators are also known.

The contribution of this paper has been to identify the leading error in these integrators and show how it can be
removed. This error comes basically from the actions of the surrogate system \( \hat{H} \) being slightly different from the actions of the actual system \( H \). The error in the actions (which depends on the phase of the orbit) leads to an error in the frequencies which in turn causes a linearly growing error in the phase. We show from theory and then verify numerically that a suitable starting procedure, for example, starting with a very small step size and then gradually increasing the step size to its final value, eliminates this error. This simple refinement can reduce long term errors by a factor of order \( \epsilon \) (two orders of magnitude in our example of an asteroid orbit). We show comparisons with other algorithms in the literature and argue that these new integrators, with suitable starting procedures, are the best available for a wide range of solar system problems.

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### APPENDIX

Accurate integrations of planetary orbits may require corrections for relativistic effects, which have fractional amplitude of \( O(GM/c^2r) \), where \( r \) is the radius (\( \sim 10^{-8} \) for \( r = 1 \) AU). The Hamiltonian for the Sun and planets may be written \( H \) where (Landau & Lifshitz 1975)

\[
H_{\text{classical}} = \frac{p_0^2}{2M_0} + \sum_i \frac{p_i^2}{2m_i} - G\mu \sum_i \frac{m_i}{r_i} - G \sum_{i<j} \frac{m_i m_j}{r_{ij}},
\]

\[
H_{\text{grav}} = \frac{G^2 \mu^2}{2c^2} \sum_i m_i \rho_i \sum_j \frac{p_j^4}{8m_j c^2} - \frac{3G \mu}{2c^2} \sum_i \rho_i.
\]

Here \( \mu \) and \( m_i \) are the masses of the Sun and planets, \( (\mathbf{p_0}, \mathbf{q_0}) \) and \( (\mathbf{p}_i, \mathbf{q}_i) \) are their momenta and coordinates, \( r_i = |\mathbf{q}_i - \mathbf{q}_j|, r_j = |\mathbf{q}_j - \mathbf{q}_0| \), and the sums are over all planets. We have neglected smaller relativistic corrections of fractional amplitude \( O(GM/c^2r_i) \) or \( O(GM/c^2r_j)^3 \). The coordinates correspond to the harmonic or isotropic form of the Schwarzschild metric (Weinberg 1972).

Unfortunately the Hamiltonian \( H_{\text{classical}} + H_{\text{grav}} \) cannot be written in the separable form (5). Since the relativistic corrections are small, a useful approximate correction is to replace \( H_{\text{grav}} \) with a potential \( U_{\text{ap}}(r) \) that mimics the principal effects of \( H_{\text{grav}} \), especially the apse precession. The relativistic apse precession is correctly reproduced if (Nobili & Roxburgh 1986)

\[
U_{\text{ap}}(r) = \frac{3(GM_0)^2}{c^2 r^2}.
\]

However, this potential does not yield the same mean motion as \( H_{\text{grav}} \). The mean motion can be corrected by adding a potential \( \sim r^{-3} \), which leaves the apse precession unaffected:

\[
U_{\text{ap}}(r) = \frac{3(GM_0)^2}{c^2 a r} \left[ \frac{1}{(1-e^2)^{1/2}} - 1 \right] - \frac{3(GM_0)^2}{c^2 r^3},
\]

where \( a \) and \( e \) are the semi-major axis and eccentricity of the orbit under consideration. Obviously, this form is only useful if \( a \) and \( e \) are approximately constant over many orbits.

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