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A Technical Primer on Auction Theory I: Independent Private Values*

by

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PREFACE

This primer rigorously introduces the auction model of “risk neutral bidders with independent private values”. The model is central to auction theory, and its structure is the same as a many models used in information economics. Results are derived regarding the nature of equilibria, the effects of entry fees and reserve prices, revenue equivalence, and the design of optimal auctions. Wdely applicable concepts are emphasized, such as revealed preference logic, the single-crossing property, and the Revelation Principle. Intended readers are economics graduate and advanced undergraduate students, and all economists who want to examine auction theory in detail.

CONTENTS

1. ENVIRONMENT

The environment has one seller and several potential buyers.[1] The seller initially owns a single, indivisible object. The seller does not know how much any buyer would be willing to pay for it. If the seller were to know each buyer's value for the object, she could just approach the buyer who values it most and bargain with him over a price. This strategy is infeasible when she does not know their values. The *reason* the seller holds an auction is because her information about the possible buyers is imperfect; the auction is intended to produce the best sale price in part by identifying the best buyer.

The number of potential buyers is $n > 1$, which is commonly known. Even so, a buyer need not know when he bids how many other buyers also submit bids — some buyers might choose to not bid. We adopt the usual misleading practice of referring to a potential buyer as a "bidder," even though he need not bid.

The remaining assumptions describe a simple environment in which we can easily study auctions. These assumptions are plausible for some, but not all, situations.

(A1) *Private values:* The private information of a bidder is his own value for the object, and it does not depend on what the other bidders know.

The *value* of bidder i, denoted v_i, is the maximum amount of money he would be willing to pay for the object.[2] Assumption (A1) states that a bidder's value is known only to himself, and that it is his only private information. Everything else in the model is commonly known to everybody (including assumption (A1) and those that follow). In game theory jargon, v_i is the *type* of bidder i. When the identity of the bidder is unimportant, we may refer to *any* bidder whose value is some v as a *type v bidder*.

Assumption (A1) applies, e.g., to art auctions if each bidder knows his own personal evaluation of a painting, regardless of what the others might think. The key aspect of (A1) is that a bidder's value should not change if he were to learn another bidder's private information. Thus, (A1) applies even if a bidder is uncertain about his value, as long as its expectation (estimate) does not depend on the other bidders' information. (If a bidder is uncertain about his value, v_i is to be interpreted as his estimated value.)

[An example of an auction for which (A1) is less sensible is an oil tract auction. Each bidder for an oil tract is typically allowed to privately conduct seismic testing of the tract before bidding. A bidder's (estimated) value depends on the test results. Plausibly, the bidder would form a different estimate of the amount of oil contained in the tract if he were to also learn the test results of other bidders. This would violate (A1).[3]]

(A2) *Independent types:* $v_1, ..., v_n$ are independently distributed.

Recall from (A1) that each bidder is ignorant of the others' types. We make the "Bayesian" assumption that bidder i believes $v_1,..., v_{i-1}, v_{i+1},..., v_n$ are random variables to which he can attribute a joint probability distribution. Assumption (A2) requires each bidder to believe the others' types are distributed independently of his own. Knowing his own type does not tell a bidder anything about the others' types; thus, the beliefs of one bidder about the actions of the others will not be a function of his own type.

Assumption (A2) is, e.g., sensible for art auctions. [Like (A1), it is not so sensible for oil auctions. If the test results of one bidder in an oil auction suggest a lot of oil, chances are high that the other bidders also received positive test results. In this case the bidders have dependent types.[4]]

(A3) *Symmetry:* Each random variable v_i has the same distribution $F(\cdot)$.[5]

The symmetry assumption says two things. First, each pair of bidders have the same beliefs about how the value of a third bidder is distributed. Second, each single bidder believes that the values of any pair of other bidders are identically distributed.

(A4) *Risk neutrality:* The bidders are risk neutral.

Assumption (A4) implies that each bidder maximizes expected profit. In most of the auctions we shall consider, a bidder pays some amount p_w if he wins the auction and another amount p_ℓ if he loses. Given that the bidder values the object at v_i dollars, his profit is $v_i - p_w$ if he wins and $-p_\ell$ if he loses. His expected profit is therefore

$$\Pr[\text{win}]\,(v_i - p_w) \;+\; \Pr[\text{lose}]\,(-p_\ell). \tag{1.1}$$

The probabilities Pr[i wins] and Pr[i loses], and payments p_w and p_ℓ, depend on the rules of the auction and the behavior of the bidders. Bidder i acts to maximize (1.1).[6]

The seller is assumed to be risk neutral. Her value, v_S, is the minimum amount of money she is willing to accept for the object, her "opportunity cost" of relinquishing it (or porducing it to order). Though not often pertinent, v_S is assumed to be commonly known.

2. FOUR STANDARD AUCTIONS

1. *Second Price Auction* (*SPA*). The bidders simultaneously submit sealed bids. The high bidder wins and pays a price equal to the second highest bid. This auction was invented in 1961 by William Vickrey. Though rarely used,[7] second price auctions are of central theoretical importance.

2. *English Auction* (*EA*). Bids are oral. The auctioneer starts the bidding at some price. The bidders proclaim successively higher bids until no bidder is willing to bid higher. The bidder who submitted the final bid wins and pays a price equal to his bid. This auction is commonly used for art, used cars, etc.

3. *First Price Auction* (FPA). The bidders simultaneously submit sealed bids. The high bidder wins and pays a price equal to his bid. This auction is commonly used for selling mineral rights, e.g., oil tract leases.

4. *Dutch Auction* (*DA*). The price continuously declines on a "wheel" in front of the bidders until one yells "stop." That bidder wins and pays the price at which the wheel stopped. This auction is used to sell flowers in Holland.

Each auction has a rule for breaking ties. For modeling purposes, where we abstract from such things as personalities and bargaining skills, the appropriate assumption in our symmetric environment is that ties are broken without bias. A k-way tie is decided by flipping a "k-sided coin"; each of the high bidders wins with probability $1/k$.

A *reserve price*, denoted by r, is one parameter of each auction. This is a number which determines the lower bound on acceptable bids. In the first and second price auctions, all submitted bids must be greater than or equal to r; no sale occurs if no such bids are submitted. In the second price auction the reserve price will be the price the winning bidder pays if he is the only one to submit a bid (r is then, in a sense, the second highest "bid"). In the English auction the auctioneer starts the bidding at r, and no sale occurs if no bidder is willing to bid at least that much. In the Dutch auction, no sale occurs if no bidder stops the wheel before the price drops to r. In each auction we assume the reserve price is announced before the bidding starts, so that all bidders know it before they bid.

An *entry fee,* denoted by c, is another parameter of each auction. This is an amount a bidder must pay in order to submit a bid. Each bidder i knows his value v_i before he must decide whether to participate in the auction by paying the entry fee c.

If we want to be explicit about the reserve price r and entry fee c, we write SPA(r, c), EA(r, c), FPA(r, c), or DA(r, c) for the different auctions.

Other rules for auctions can be concocted. Each bidder might be *paid* an amount that depends on his bid; or the k^{th} highest bid might win; or the winner might pay the sum of all bids; or all bidders might pay the amount of their own bids; or …. Such auctions might seem crazy, but they must be — and will be — considered in order to determine an "optimal auction," e.g., one that maximizes the seller's expected profit.

3. PROBABILITY PRELIMINARIES

As a normalization, assume each v_i is in the interval [0, 1].

We use a "~" to emphasize randomness. Thus, $\tilde{v}_i$ is, to the seller and the bidders $j \neq i$, the random variable whose realization is the value of bidder i. The distribution $F(\cdot)$ of $\tilde{v}_i$ is a nondecreasing function giving the probability that $\tilde{v}_i$ is in intervals of the form $(-\infty, v]$:

$$F(v) = \Pr[\tilde{v}_i \leq v].$$

Since $\tilde{v}_i$ is surely in [0, 1], we know $F(0) = 0$ and $F(1) = 1$.

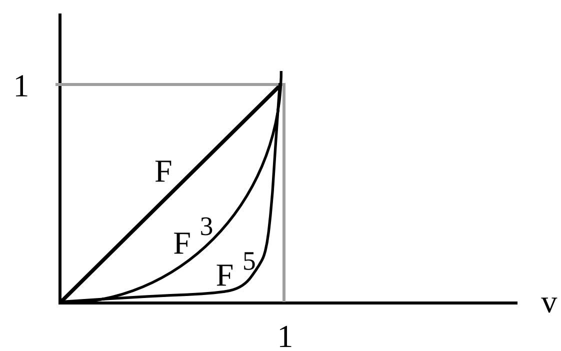
Assume $F(\cdot)$ has a continuous and positive derivative on [0, 1], the density function $f(\cdot) = F'(\cdot)$. Thus, $\tilde{v}_i$ is "continuously distributed." Given any number v, the probability that $\tilde{v}_i$ is equal to v is zero. Consequently,

$$\Pr[\tilde{v}_i < v] = \Pr[\tilde{v}_i \le v] = F(v).$$

Two derived random variables that will be important are $\tilde{v}_{(1)}$ and $\tilde{v}_{(2)}$, the highest and the second highest elements of $\{\tilde{v}_1, \ldots, \tilde{v}_n\}$.[8] By independence, the distribution of $\tilde{v}_{(1)}$ is

$$F_{(1)}(v) = \Pr[\tilde{v}_{(1)} \le v] = \prod_{i=1}^{n} \Pr[\tilde{v}_i \le v] = F(v)^n. \tag{3.1}$$

For $v < 1$, raising n lowers $F_{(1)}(v) = F(v)^n$:



Thus, increasing the sample size n increases the probability of finding $\tilde{v}_{(1)}$ near its upper bound, 1. For any positive ε, the probability that the highest of the values exceeds 1–ε, which is $1 - F_{(1)}(v) = 1 - F(1-\varepsilon)^n$, increases to 1 as $n \to \infty$.

Differentiate $F_{(1)}(v)$ to obtain its density function:

$$f_{(1)}(v) = nF(v)^{n-1}f(v). \tag{3.2}$$

Turning to $\tilde{v}_{(2)}$, the second highest of a sample of size n, its distribution function is

$$F_{(2)}(v) = \Pr[\tilde{v}_{(2)} \le v] = F(v)^n + n[F(v)^{n-1} - F(v)^n]. \tag{3.3}$$

To derive (3.3), note that it is the sum of the probabilities of n+1 distinct ways in which the second highest value can be less than v. One way is if all values are less than v; the term $F(v)^n$ is the probability of this event. Another way is for exactly one $\tilde{v}_i$ to be greater than v; the probability of this is $(1 - F(v))F(v)^{n-1} = F(v)^{n-1} - F(v)^n$. This term appears n times, one for each i, to yield the second term in (3.3).

Differentiating (3.3) yields the density function of $\tilde{v}_{(2)}$,

$$f_{(2)}(v) = n(n-1)F(v)^{n-2}[1 - F(v)]f(v). \tag{3.4}$$

Another useful random variable is $\tilde{y} = \max(\tilde{v}_2, \ldots, \tilde{v}_n)$, the maximum of n–1 independent drawings from the distribution $F(\cdot)$. To bidder 1, this random $\tilde{y}$ is the maximum of the values of the other bidders, with the following distribution and density functions:

$$G(y) = F(y)^{n-1} \quad \text{and} \quad g(y) = (n-1)F(y)^{n-2}f(y). \tag{3.5}$$

A simple distribution is the uniform, for which

$$F(v) = v \text{ and } f(v) = 1 \text{ for every } v \in [0, 1]. \tag{3.6}$$

A uniform random variable $\tilde{v}_i$ is equally likely to be anywhere in [0, 1]. Its mean is the midpoint, $E[\tilde{v}_i] = 1/2$. The maximum $\tilde{v}_{(1)}$ of a sample of size n has the functions:

$$F_{(1)}(v) = v^n \quad \text{and} \quad f_{(1)}(v) = nv^{n-1}. \tag{3.7}$$

$$E[\tilde{v}_{(1)}] = \int_0^1 v[nv^{n-1}]dv = \frac{n}{n+1}. \tag{3.8}$$

Note that $E[\tilde{v}_{(1)}]$ increases to 1 as $n \to \infty$, so $\tilde{v}_{(1)}$ is more likely to be near 1 if n is large.

Similar statements apply to $\tilde{v}_{(2)}$, the second highest of a sample of n uniform random variables. Its density and expectation are, respectively,

$$f_{(2)}(v) = n(n-1)v^{n-2}(1-v), \tag{3.9}$$

$$E[\tilde{v}_{(2)}] = \int_0^1 v f_{(2)}(v)dv = \frac{n-1}{n+1}. \tag{3.10}$$

From (3.8) and (3.10), we obtain the intuitive inequality $E[\tilde{v}_{(2)}] < E[\tilde{v}_{(1)}]$. The distribution and density functions of $\tilde{y} = \max(\tilde{v}_2, \ldots, \tilde{v}_n)$ are

$$G(y) = y^{n-1} \quad \text{and} \quad g(y) = (n-1)y^{n-2}. \tag{3.11}$$

4. SECOND PRICE AUCTIONS

We begin with the concept of a strategy in any sealed bid auction. As in any game, a strategy maps each of a player's information sets into a feasible action. In a sealed bid auction, a bidder's information sets are indexed by his type. So his strategy is a function of his type, *i*.e., a function $b_i(\cdot)$ with domain [0, 1].

The set of actions available to a bidder in a sealed bid auction consists of not submitting a bid, and all the possible bids that can be submitted. We represent this set of actions as $\{\text{No}\} \cup [r, \infty)$. Action "No" denotes not bidding, and the numbers in the interval $[r, \infty)$ are the possible acceptable bids (recall that an acceptable bid cannot be less than the reserve price r). Thus, a *strategy* for bidder i is a function $b_i(\cdot)$ mapping [0, 1] to $\{\text{No}\} \cup [r, \infty)$. If $b_i(v_i) = \text{No}$, the bidder does not bid when his value is v_i; otherwise $b_i(v_i)$ is the number he submits as his bid.

Consider now SPA(r, c), the second price auction with reserve price r and entry fee c. Suppose a bidder bids $b \geq r$ and has value v. Let z be the maximum of the other bids if there are any other bids, and otherwise let $z = r$. The rules of this auction imply that this bidder's profit is given by the following function:

$$A(b, z, v) = \begin{cases} 0 & \text{if } b < z \\ (v - z)p(b) - c & \text{if } b = z \\ v - z & \text{if } b > z, \end{cases}$$

where $p(b)$ is the probability this bidder wins a tie at b. We know that $0 \leq p(b) \leq 1$; as we will see, the nature of $p(b)$ will not matter. The bidder chooses a bid that maximizes his expected profit. Generally he will not know the maximum of the other bids, but will instead view it as a random variable $\tilde{z}$ distributed according to some distribution. He accordingly chooses his bid b to maximize the expected profit, $E[A(b, \tilde{z}, v)]$.

Our first important result is that if the entry fee is $c = 0$, then each bidder in the second price auction has a *dominant strategy*. That is, for each i and v_i, there is a bid $b(v_i)$ which is optimal for the bidder regardless of what he believes the other bidders are doing; it maximizes $E[A(b, \tilde{z}, v)]$ regardless of what distribution of $\tilde{z}$ is used to take the expectation. The bidder's dominant is to bid if and only if his value exceeds the reserve price, and in that case to bid "truthfully," *i*.e., to submit a bid equal to his value.[9]

THEOREM 4.1: *In the* SPA(r, 0)*, the following is each bidder's dominant strategy*:[10]

$$b(v) = \begin{cases} v & \text{if } v > r \\ \text{No} & \text{if } v < r. \end{cases}$$

PROOF: Consider first the case $v < r$. Then "No" (weakly) dominates any bid. To see this, note that "No" surely yields zero profit. But submitting a bid, since $v - r < 0$ and the price a winner pays will be at least r, results in negative profit if it wins (which it does if the other bidders do not bid), and zero profit (as $c = 0$) if it loses.

Consider now the case $v \geq r$.

Contemplate a hypothetical situation in which you are a bidder, and you *know* that the highest of your competitors' bids is a number z. The number z is the price you will pay if you win the auction. Your bid — call it b — determines only whether you win; you win if $b > z$ and you lose if $b < z$. If $v - z > 0$ you want to win, and so any bid $b > z$ is optimal for you. *The bid $b = v$ is such a bid.* Similarly, if $v - z < 0$ you want to lose, and so any bid $b < z$ is optimal for you. *The bid $b = v$ is such a the bid*. We now know that if you know the highest of your competitors' bids, then $b = v$ is a best bid for you when your value is v. (Bid $b = v$ is not your only best bid, but that is irrelevant.)

Now we consider the situation in which you do not know the highest of your competitors' bids. This step is actually trivial. The argument in the previous paragraph does not depend on the precise value of the number z. This means that its conclusion must be true regardless of what z is: $b = v$ is always optimal. Therefore, *even if you do not know the maximum of your competitors' bids*, $b = v$ is an optimal bid.

This simple argument is a bit slippery. We clarify it by taking a more formal tack. Referring back to the expression for a bidder's profit $A(b, z, v)$, consider Figure 4.1. It shows $A(b, z, v)$ as b varies in three different cases: $z < v$, $z = v$, and $z > v$.

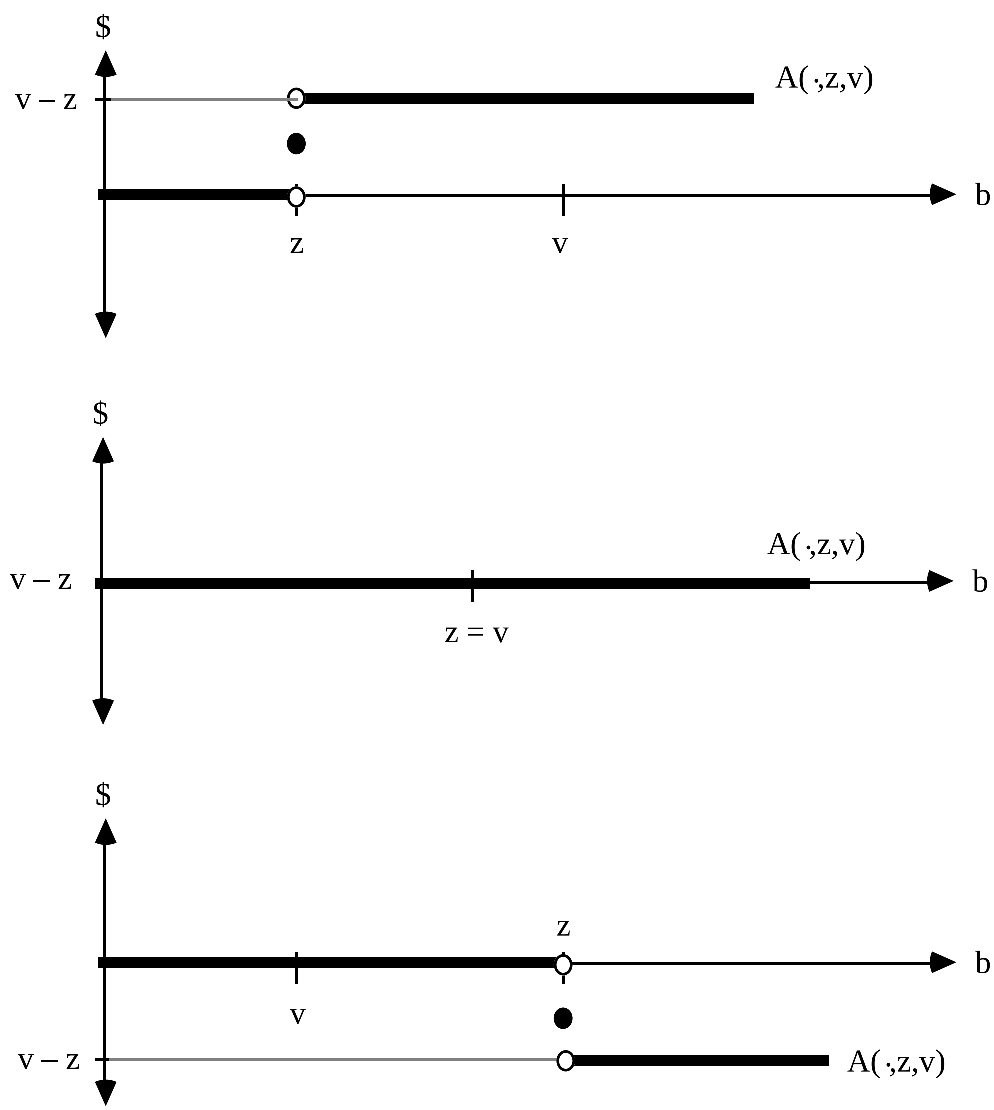


Figure 4.1

The figure shows that in each case, $b = v$ maximizes $A(b, z, v)$, which is the first step of the argument presented above. The second step, that in which you don't know z, follows from standard Bayesian decision theory. Not knowing z means that you believe it is a random variable $\tilde{z}$ with some density function $h(\cdot)$.[11] Denote by $B(b, v)$ your expected profit when you bid b and your value is v. Then $B(b, v)$ is the expectation of $A(b, \tilde{z}, v)$:

$$B(b, v) \;=\; \int_{\infty}^{-\infty} A(b, z, v)h(\mathrm{z})dz. \tag{4.1}$$

Since density functions are nonnegative, and since we already established in the first step of the argument that $A(b, z, v) \leq A(v, z, v)$ for every (b, z, v), we know that

$$\int_{\infty}^{-\infty} A(b, z, v)h(z)dz \;\leq\; \int_{\infty}^{-\infty} A(b, z, v)h(z)dz. \tag{4.2}$$

Thus, $B(b, v) \leq B(v, v)$, so that bid $b = v$ gives you greater expected profit than any other bid. Because this argument holds regardless of your beliefs about the behavior of your competitors, *i*.e., regardless of the nature of the density function $h(\cdot)$, it shows that the identity function $b(v) = v$ is a dominant strategy. [12]

We can now say something about the sale price, *i*.e., the price paid to the seller by the winning bidder, in a second price auction. Consider the auction with a zero reserve price and a zero entry fee: let $\tilde{w}^{2}$ be the sale price in the SPA(0, 0) when the bidders play their dominant strategies. These strategies entail truthful bidding, and so the bidder with the highest value wins and pays a price equal to the second highest value. Thus,

$$\tilde{w}^{2} = \tilde{v}_{(2)}. \tag{4.3}$$

The expected sale price, $E[\tilde{w}^{2}]$, is the seller's expected revenue from holding the auction. (If her own value for the object is $v_S = 0$, $E[\tilde{w}^{2}]$ is also her expected profit.) From (4.3),

$$E[\tilde{w}^{2}] = E[\tilde{v}_{(2)}]. \tag{4.4}$$

In the uniform case, from (3.10) and (4.4), we see that $E[\tilde{w}^{2}] = \dfrac{n-1}{n+1.}$.

Remark: Second price auctions have other equilibria, though they are not very plausible. Consider SPA(0, 0). Suppose that regardless of their values, bidders 2,..., *n* all choose "No", and bidder 1 bids $b = 1$. This is an equilibrium. Given that bidder 1 bids $b = 1$, another bidder will pay a price of 1, which is greater than his value, if he submits a bid that wins. Thus, "No" is indeed a best reply of each of the bidders 2,..., *n*. Bidder 1, on the other hand, wins by bidding, and pays the reserve price $r = 0$ for the object. As his value is greater than 0, bidding $b = 1$ is a best reply for him.

The strategies in this equilibrium are (weakly) dominated. For example, note that bidder 1 would make just as much profit by bidding $b = v_1$ as by bidding $b = 1$, *if* the other bidders chose "No" as they are supposed to do, but that $b = v_1$ is strictly better if one of the other bidders happens to submit a bid greater than v_1. One of these bidders just might "tremble" and submit any bid less than 1, since such bids are also best replies to the bid $b = 1$ of bidder 1. If one of the bidders 2,..., n might "tremble" in this way, then bidder 1 takes an unnecessary risk by bidding $b = 1$. Similarly, submitting the bid $b = v_i$ dominates "No" for each of the bidders $i = 2,..., n$, given that the entry fee is zero.

How does a positive entry fee affect these results? Observe first that truthful bidding is no longer a dominant strategy for any bidder! With an entry fee, the decision whether to submit a bid depends on how one thinks the others are bidding. For example, if bidder 1 believes some other bidder always submits a very high bid, say $b > 1$, then bidder 1 cannot win by submitting a bid less than or equal to his value $v_1 \leq 1$. Hence, submitting a bid guarantees a loss that is no less than the entry fee. Bidder 1's best reply is "No".

However, it remains true that for any given profile of strategies for the other bidders, if a bidder's best reply is to submit a bid, then he can do no better than to bid his value. The argument the same as before: $b_i = v_i$ is always no worse, and sometimes better, than any other bid (even though not bidding might be better).

A positive entry fee also causes bidders with values very close to the reserve price to not bid. To see this, note that if a bidder bids, he pays the fee $c > 0$. His profit gross of this fee is at most $v_i - r$, since r is the lowest price he can pay if he wins. So, if v_i is only slightly above r, then $v_i - r < c$ and the bidder should refrain from bidding.

The *marginal value*, the value v_0 which a bidder's value must exceed in order to find it worthwhile bidding, must be formally derived as part of the equilibrium. But we can use our intuition to guess it. First, we should expect a bidder with value v_0 to be indifferent between bidding and not bidding, *i*.e., his expected profit from bidding should be zero. As a bidder's best bid is his value, this bidder's best bid is v_0. Since any other bidder who bids will bid his value, and that value will exceed v_0, the bidder bidding v_0 wins exactly if the others don't bid. The probability this happens is the probability that the others' values are less than v_0, which is $G(v_0)$. And since this bidder who bids v_0 wins only if no other bid is submitted, he will pay a price of r when he wins. His expected profit is therefore $(v_0 - r)G(v_0) - c$. Setting this expression equal to zero determines the marginal value v_0 as a function of r and c. That is, the marginal value is the $v_0(r, c)$ defined as the solution to the following equation:

$$(v_0 - r)G(v_0) = c. \tag{4.5}$$

Remark: For $r \geq 0$ and $0 \leq c \leq 1-r$, the following facts are easy to verify. [Draw the graph of $\pi = (v - r)G(v)$ as a function of v, and note where it crosses the horizontal line $\pi = c$.] First, (4.5) is satisfied by a unique $v_0 \in [0, 1]$. Second, this v_0 satisfies $v_0 \geq r$, strictly if and only if $c > 0$. Third, this v_0 satisfies $v_0 \leq 1$, strictly if and only if $c < 1-r$. Unless $c \leq 1-r$, no bidder bids: a winning bidder pays a price of at least r, and so his expected profit from bidding is bounded above by $(v - r)(1) - c \leq 1 - r - c$. A bidders' expected profit from bidding is therefore negative if $c > 1- r$.

Theorem 4.2 describes the equilibrium of a second price auction with an entry fee and a reserve price. The equilibrium it describes is the same as that of Theorem 4.1 if $c = 0$. [The proof is relatively concise and may be skipped at a first reading.]

THEOREM 4.2: *Assume* $0 \leq c \leq 1-r \leq 1$. *Then an equilibrium of* SPA(r, c) *consists of each bidder using the following strategy* (*where* $v_0(r, c)$ *is the solution of* (4.5)):

$$b_i(v) = \begin{cases} v & \text{if } v \geq v_0(r, c) \\ \text{No} & \text{if } v < v_0(r, c). \end{cases} \tag{4.6}$$

PROOF: If $v < r$, bidding $b_i(v) =$ No is clearly a best reply. So we can restrict attention to a bidder with value $v \geq r$. We show that if the other bidders bid according to (4.6), then so should this bidder. Let $v_0 = v_0(r, c)$, and recall that $v_0 \geq r$.

First, suppose the bidder bids $b \in [r, v_0]$. Because the other bidders use (4.6), this bid wins exactly if the others' values are less than v_0, which occurs with probability $G(v_0)$. If he wins, the price is the reserve price r. The bidder's expected profit gross of the entry fee is $\pi(v, b) = (v - r)G(v_0)$. Observe that $\pi_b(v, b) = 0$ for $b \in [r, v_0]$.

Now suppose the bidder bids $b > v_0$. He wins if $\tilde{y} \le b$, where $\tilde{y}$ is the maximum of the other bidders' values. If he wins he pays price

$$p(\tilde{y}) = \begin{cases} r & \text{if } \tilde{y} < v_0 \\ \tilde{y} & \text{if } \tilde{y} \ge v_0. \end{cases}$$

The bidder's expected gross profit is therefore

$$\pi(v, b) = \int_0^b [v - p(y)]g(y)dy. \qquad (4.7)$$

Hence $\pi_b(v, b) = [v - p(b)]g(b) = (v - b)g(b)$, since $b > v_0$. We conclude that $(v - b)\pi_b(v, b) \ge 0$. This inequality holds for all $b \ge r$, given the previous paragraph. Therefore $\pi(v, b) \le \pi(v, v)$ for any $b \ge r$: among all bids, $b = v$ is optimal. (This argument is made in more detail in Section 6; see Lemma 6.2 and Figure 6.2.)

Consequently, a best reply for the bidder is $b = v$ if $\pi(v, v) \ge c$, and "No" if $\pi(v, v) < c$. It remains only to show that $\pi(v, v) \ge c$ if and only if $v \ge v_0$. The expressions above imply $\pi(v, v)$ is continuous on $[r, \infty)$, and $\pi(v_0, v_0) = c$. For $v \in [r, v_0)$,

$$\pi(v, v) = (v - r)G(v_0) < (v_0 - r)G(v_0) = c,$$

by the definition of v_0 in (4.5). For $v > v_0$,

$$\pi(v, v) = \int_0^v [v - p(y)]g(y)dy.$$

Since $p(y) < v$ for all $y < v$ (using $v > v_0 \ge r$), we see that $\pi(v, v)$ strictly increases on $[v_0, \infty)$. Hence, $\pi(v, v) > \pi(v_0, v_0) = c$ for all $v > v_0$. This finishes the proof.

5. ENGLISH AUCTIONS

Recall that an English auction is the usual oral auction in which bidders yell bids. It is a complicated object to study. For example, consider what a strategy for a bidder might be. A bidder can remain silent for a while, then yell out a couple of bids in quick succession, or never bid until it looks like the bidding is slowing down, or, ... The possibilities are endless and complicated. A realistic extensive form model of an English auction would be very complicated indeed.

However, a simple model of an English auction captures the relevant phenomena. This tractable model is called a "button auction." In the button auction, each bidder presses a button in front of him as the "standing bid" continuously increases. A bidder can release his button at any time; he has irreversibly "dropped out" of the bidding at the moment he releases his button. The auction is over once there is only one bidder pressing a button. That bidder wins and pays a price equal to the value of the standing bid at the moment the last bidder dropped out. We also assume a bidder holding down his button does not see how many other bidders are still holding down their buttons.[13]

The set of possible actions for a bidder in this button auction is the same as in a sealed bid auction, $\{\text{No}\} \cup [r, \infty)$. The "No" action corresponds to never pressing the button. A number $b \geq r$ represents releasing the button when the standing bid reaches b, provided no other bidder released his button first. A strategy for bidder i is a function $b_i(\cdot)$ from the set of possible values, $[0, 1]$, to the set of possible actions, $\{\text{No}\} \cup [r, \infty)$. This is the standard definition of a strategy as a map from information sets into actions.

Suppose for simplicity that $r = c = 0$. Then a bidder cannot lose money by adopting the strategy of pressing his button as long as the standing bid is below his own value. If a bidder releases his button before the standing bid reaches his value, he will lose, even though there is still a chance of winning and paying a price below his value. The bidder should therefore not release the button before the standing bid reaches his value. On the other hand, if he holds the button down after the standing bid exceeds his value, he takes a chance on "winning" and taking a loss, since the price he would have to pay in this case would be greater than his value. This argument shows that releasing his button when the standing bid reaches his value is a bidder's optimal strategy, regardless of what the other bidders do. Just as in SPA(0, 0), the strategy defined by $b(v_i) = v_i$ is a dominant strategy. [More generally, it is straightforward to show that the equilibrium of auction SPA(r, c), as described in Theorem 4.2, is also an equilibrium of auction EA(r, c).]

Thus, in SPA(0, 0) the item will be sold to the bidder with the highest value, $\tilde{v}_{(1)}$. This bidder pays a price equal to the level reached by the standing bid at the moment the last of the other bidders drops out. This last bidder to drop out will be the bidder with the second highest value, $\tilde{v}_{(2)}$. Letting $\tilde{w}^{E}$ denote the sale price in EA(0,0), we see that

$$\tilde{w}^{E} = \tilde{v}_{(2)}. \tag{5.1}$$

Comparing (5.1) to (5.3), we see that EA(0, 0) and SPA(0, 0) have the same sales prices:

$$\tilde{w}^{E} = \tilde{w}^{2}. \tag{5.2}$$

The two sale prices are equal for any realization of the variables $\tilde{v}_1, ..., \tilde{v}_n$. This implies that the expected sale prices are equal: $E[\tilde{w}^{E}] = E[\tilde{w}^{2}]$.

6. FIRST PRICE AUCTIONS

We turn now to the first price auction, FPA(r, c). Recall that in this auction, each bidder either does not bid, or submits a sealed bid no less than the reserve price r; the high bidder wins; the price the winner pays is equal to his own bid; and every bidder who submits a bid pays the entry fee c.

For a moment, consider intuitively the differences between the first and second price auctions. Put yourself in the shoes of a bidder in a first price auction. Suppose your value is $v = .5$. Can it be optimal to bid .5? Certainly not, at least if you think there is a chance your competitors will bid less than .5. By bidding .5 you would make zero profit even if you win, because you would pay a price equal to your value if you win. If you bid slightly less than .5, you might have a slightly lower probability of winning, but at least you would have a positive gain if you did win.

How much less than .5 should you bid? Well, it depends on what you think your competitors are bidding. For example, if you thought they were all bidding .25, you should bid only slightly more than .25, as that would be the lowest price at which you could get the item. But if you thought they were probably bidding close to .5, you should bid close to .5 yourself in order to have a chance at winning. If you thought they were all bidding greater than .5, you would have no incentive to bid at all. This argument suggests that a bidder in the first price auction generally does not have a dominant strategy; his optimal strategy depends on what he thinks the other bidders are doing.

Let's be more formal. A strategy for bidder i in a first price auction with zero reserve price is again a mapping $b_i(\cdot)$ from his set of possible values, [0, 1], to his set of possible actions, $\{\text{No}\} \cup [r, \infty)$. As before, "No" represents not bidding and the numerical actions represent possible bids. Given a profile of strategies $\langle b_1(\cdot), \ldots, b_n(\cdot)\rangle$, one for each player, we can define a "probability-of-winning" function for bidder i:

$$\hat{Q}_i(b) \;=\; \text{Prob}\big[i \text{ wins} \mid i \text{ bids } b \text{ and each } j \neq i \text{ bids according to } b_j(\cdot)\big].$$

Luckily, we will not need to compute this function in general. From the point of view of bidder i, if he expects the others to play according to the given strategy profile, his probability of winning is $\hat{Q}_i(b)$ if he bids b. His expected profit, excluding the entry fee c, when his value is v_i is

$$\pi_i(v_i, b) = (v_i - b)\hat{Q}_i(b). \tag{6.1}$$

The bidder's net expected profit is thus $\pi_i(v_i, b) - c$, which must not be negative if the bidder is acting rationally (as he can guarantee himself zero profit by not bidding). Profile $\langle b_1(\cdot), \ldots, b_n(\cdot)\rangle$ is a (*Bayesian-Nash*) *equilibrium* (in pure strategies) if for each i and v_i, bid $b_i(v_i)$ is a *best reply* to strategies $\langle b_j(\cdot)\rangle_{j\neq i}$:

$$b_i(v_i) = \text{No} \;\Rightarrow\; \pi_i(v_i, b) \leq c \text{ for all } b \geq r,$$
$$b_i(v_i) \neq \text{No} \;\Rightarrow\; \pi_i(v_i, b_i(v_i)) \geq c \text{ and } \pi_i(v_i, b_i(v_i)) \geq \pi_i(v_i, b) \text{ for all } b \geq r.$$

So, a bidder does not bid only if he cannot bid profitably, and he bids only if he makes nonnegative profit, in which case his bid maximizes his expected profit. (A bidder who is indifferent between not bidding and submitting his best bid can take either action.)

In a *symmetric equilibrium*, each strategy $b_i(\cdot)$ is equal to the same $b(\cdot)$. To simplify matters, consider for the remainder of this section only the first price auction with zero reserve price and zero entry fee, FPA(0, 0). Its equilibrium consists of any bidder with any value $v > 0$ submitting the following bid:

$$b^*(v) = \int_0^v y\left[\frac{g(y)}{G(v)}\right]dy. \tag{6.2}$$

We shall prove that $b^*(\cdot)$ is an equilibrium, and that it is the only symmetric equilibrium. (Although it is true that asymmetric equilibria fail to exist, we will not prove this.) First, however, we make three observations about the equilibrium.

(1) Interpretation

A bidder's equilibrium bid is equal to the expectation of the maximum of his competitors' values conditional on that value being less than his own:

$$b^*(v) = E[\tilde{y} \mid \tilde{y} \le v].$$

This is easy to see. Recall that $g(y)$ is the density function of $\tilde{y}$, the maximum of n–1 of the bidders' values. The probability that event $\{\tilde{y} \le v\}$ occurs is $G(v) = F(v)^{n-1}$. Hence, $g(y)/G(v)$ is the conditional density of $\tilde{y}$, given $\{\tilde{y} \le v\}$. The right side of (6.2) is therefore the expected value of $\tilde{y}$ conditional on $\{\tilde{y} \le v\}$.

(2) Underbidding and Competition

From (6.2) we see that bidders with positive values bid less than their values, $b^*(v) < v$. We see this clearly by integrating (6.2) by parts and substituting $F(x)^{n-1} = G(x)$ to obtain

$$b^*(v) = v - \int_0^v \left(\frac{F(y)}{F(v)}\right)^{n-1} dy. \tag{6.2′}$$

The amount of underbidding is measured by the integral in (6.2′). Since the integration variable y is less than v, the ratio $F(y)/F(v)$ is less than one. It therefore diminishes to zero as $n \to \infty$; as competitive pressure increases in that the number of bidders grows, the amount of underbidding goes to zero. (Obtaining this intuitive result is a useful check of a model — we want our models to yield results we "just know, intuitively" are right.)

(3) Revenue Equivalence

Denote the equilibrium sale price in FPA(0, 0) as $\tilde{w}^1$. Its expectation can be shown to equal that of $\tilde{w}^2$ and, hence, of $\tilde{w}^{\mathrm{E}}$. The seller's expected revenue is the same in FPA(0, 0), SPA(0, 0), and EA(0, 0)! The next section explores this surprising result.

For now, observe merely that revenue equivalence is not *too* surprising.[14] In FPA(0, 0) the bidders bid less than their true values, but in SPA(0, 0) they bid their true values. So bids are higher in the SPA. But the sale price in the FPA is the highest instead of the second highest bid. Casual thought might suggest that which of these opposing forces should dominatewould depend on the distribution of values. The revenue equivalence result is that in fact, these two forces always actually offset each other.

Remark: Though they have the same expectation, $\tilde{w}^1$ and $\tilde{w}^2$ are not the same random variable. [A risk averse seller can be shown to prefer $\tilde{w}^1$ (Matthews, 1980).] This is in contrast to the English and second price auctions, where $\tilde{w}^2$ and $\tilde{w}^{\mathrm{E}}$ are both equal to $\tilde{v}_{(2)}$.

In the uniform example, calculation from either (6.2) or (6.2′) yields

$$b^*(v) = \frac{(n-1)v}{n}.$$

The expected sale price, calculated from this and (3.8), is

$$E[\tilde{w}^1] = E[b^*(\tilde{v}_{(1)})] = E\left[\frac{(n-1)\tilde{v}_{(1)}}{n}\right] = \left[\frac{n-1}{n}\right]\left[\frac{n}{n+1}\right] = \frac{n-1}{n+1}.$$

Recall that $E[\tilde{w}^2] = \frac{n-1}{n+1}$. Hence, revenue equivalence indeed holds.

Theorem 6.1 below is the central result of this section. The arguments used to prove it are economically interesting and of general use in information economics.

THEOREM 6.1: *In the* FPA(0, 0),

(*i*) *the strategy* $b^*(\cdot)$ *defined by* (6.2) *is a symmetric equilibrium, and*

(ii) *if* $b(\cdot)$ *is any symmetric equilibrium, then* $b(v) = b^*(v)$ *for all* $v > 0$.

We start the proof by making some "inspired guesses," and then go back to rigorously prove the guesses. This technique is how the first economist to study auctions might have originally derived the equilibrium. That economist might have made the following good guesses about the equilibrium $b(\cdot)$ of FPA(0, 0) being sought:

Guess A: All types of bidder submit bids: $b(v) \neq$ No for all $v \geq 0$.
Guess B: Bidders with greater values bid higher: $b(v) > b(z)$ for all $v > z$.
Guess C: The function $b(\cdot)$ is differentiable.

Guess A is natural because, as $r = c = 0$, even bidders with low values should find it worthwhile to bid (except type $v = 0$). Guess B is also intuitive; a bidder with a higher willingness-to-pay has a greater should be expected to bid higher. Guess C, on the other hand, is based more on optimism than intuition, since differentiable functions are easy to work with. Obviously, many functions other than $b^*(\cdot)$ satisfy Guesses A – C. We now show that the only possible equilibrium satisfying them is $b^*(\cdot)$.

LEMMA 6.1: *If* $b(\cdot)$ *is a symmetric equilibrium of* FPA(0, 0) *satisfying Guesses A – C, then* $b(\cdot) = b^*(\cdot)$.

PROOF: The plan is to use Guesses A – C to show that $\pi(v, b)$ is differentiable in b. Then, since $b(v)$ is optimal for a type v bidder, $b = b(v)$ satisfies $\pi_b(v, b(v)) = 0$. As this holds for all types v, a necessary differential equation is obtained, and its solution is $b^*(\cdot)$.

Let's do it. From Guesses B and C, $b(\cdot)$ has an inverse function, which we denote as $\varphi(\cdot)$, that satisfies $b(\varphi(b)) = b$ for all numbers b in the range of $b(\cdot)$. Value $v = \varphi(b)$ is the value a bidder must have in order to bid b when using strategy $b(\cdot)$.

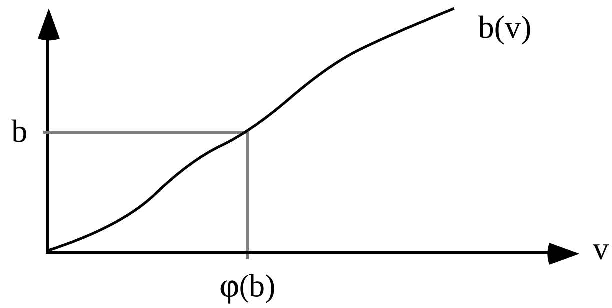


Figure 6.1

Because $b(\cdot)$ strictly increases, the probability that bidder i wins if he bids b is

$$\hat{Q}(b) = F(\varphi(b))^{n-1} = G(\varphi(b)). \tag{6.3}$$

Why? Well, observe that since $b(\cdot)$ is strictly increasing and the other bidders' values are continuously distributed, we can ignore ties: another bidder j bids the same b only if his value is precisely $v_j = \varphi(b)$, which occurs with probability zero. So the probability of bidder i winning with bid b is equal to the probability of the other bidders bidding no more than b. Because $b(\cdot)$ strictly increases, this is the probability that each other bidder's value is no more than $\varphi(b)$. This probability is $G(\varphi(b)) = F(\varphi(b))^{n-1}$, the probability that the maximum of the other bidders' values is no more than $\varphi(b)$.

Thus, the expected profit of a type v bidder who bids b when the others use $b(\cdot)$ is

$$\pi(v, b) = (v - b)G(\varphi(b)).$$

Because of Guess C, the inverse function $\varphi(\cdot)$ is differentiable. Letting $\varphi'(b)$ be its derivative at b, the partial derivative of $\pi(v, b)$ with respect to b is

$$\pi_b(v, b) = -G(\varphi(b)) + (v - b)g(\varphi(b))\varphi'(b).$$

Since $b(\cdot)$ is an equilibrium, $b(v)$ is an optimal bid for a type v bidder, i.e., $b = b(v)$ maximizes $\pi(v, b)$. The first order condition $\pi_b(v, b(v)) = 0$ holds:

$$-G(\varphi(b(v))) + (v - b(v))g(\varphi(b(v))\varphi'(b(v)) = 0.$$

Use $\varphi(b(v)) = v$ and $\varphi'(b(v)) = 1/b'(v)$ to write this as

$$-G(v)) + \frac{(v - b(v))g(v)}{b'(v)} = 0.$$

This is a differential equation that almost completely characterizes the exact nature of $b(\cdot)$. It is easy to solve. Rewrite it as

$$G(v)b'(v) + g(v)b(v) = vg(v). \tag{6.4}$$

Since $g(v) = G'(v)$, the left side is the derivative of $G(v)b(v)$. So we can integrate both sides from any v_0 to any v (using "y" to denote the dummy integration variable):

$$G(v)b(v) - G(v_0)b(v_0) = \int_{v_0}^{v} yg(y)dy. \tag{6.5}$$

From Guess A, all types bid, so we can take $v_0 \to 0$. We know $b(0) \geq 0$, as $r = 0$. Hence, $G(v_0)b(v_0) \to 0$ as $v_0 \to 0$. Take this limit in (6.5) and divide by $G(v)$ to obtain

$$b(v) = \int_0^v y\left[\frac{g(y)}{G(v)}\right]dy = b^*(v).$$

This is the desired result. ♕

Technical Aside: Write (6.4) as $b' = (v - b)g(v)/G(v)$. Notice that its right side blows up as $v \to 0$, since $G(0) = 0$. Thus, as a differential equation for $b(\cdot)$, it does not satisfy the Lipschitz condition required for the theorem which concludes that a differential equation has a unique solution for any inital value of $b(0)$. In fact, (6.4) has a solution $b(\cdot)$ only for three initial values of $b(0)$: $-\infty$, 0, and ∞. It is easy to verify that a function $b(\cdot)$ on $(0, 1]$ is a solution if and only if for some K, $b(v) = b(v; K) \equiv b^*(v) + K/G(v)$ for all $v \in (0, 1]$. If $K > 0$, $b(v; K)$ decreases in v when v is small, and so $b(\cdot; K)$ is not an equilibrium. If $K < 0$, $b(v; K) \to -\infty$ as $v \to 0+$, and so $b(\cdot; K)$ is not an equilibrium if the reserve price is finite. But for any entry fee $c \geq 0$, if $K \leq -c$ then $b(\cdot; K)$ is an equilibrium of FPA($-\infty$, c). This is of no practical interest, as no real auction has a reserve price $r = -\infty$.

We now prove that $b^*(\cdot)$ is actually an equilibrium. Its derivation in Lemma 6.1 shows that it satisfies a first-order condition for equilibrium; it is shown in the proof of Lemma 6.2 that it satisfies a "pseudoconcavity" property that is a kind of second-order condition.

LEMMA 6.2: $b^*(\cdot)$ *is a symmetric equilibrium of* FPA(0, 0).

PROOF: Because $b^*(\cdot)$ satisfies Guesses A – C, the argument used in Lemma 6.1 shows that the expected profit of a type v bidder who bids b when the others use $b^*(\cdot)$ is

$$\pi(v, b) = (v - b)G(\varphi^*(b)), \tag{6.6}$$

where $\varphi^*(\cdot)$ is the inverse of $b^*(\cdot)$. We must prove $b = b^*(v)$ maximizes $\pi(v, b)$.

We first observe that "No" is not preferred to $b^*(v)$. This follows from the observation that $\pi(v, b^*(v)) \geq 0$, which is a consequence of (6.6) and $b^*(v) \leq v$ for all v (see (6.2′)).

To prove $b^*(v)$ maximizes $\pi(v, b)$, we prove $\pi(v, \cdot)$ is pseudoconcave,[15] i.e., that the derivative $\pi_b(v, b)$ is nonnegative if $b < b^*(v)$, and nonpositive if $b > b^*(v)$. This then implies $\pi(v, b)$ is maximized at $b = b^*(v)$ (since $\pi(v, b)$ is continuous in b).

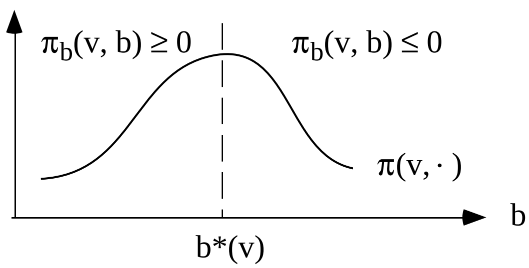


Figure 6.2

The key is to show that $\pi_{vb} > 0$. Differentiate (6.6) to obtain $\pi_v(v, b) = G(\varphi^*(b))$, and then differentiate again to obtain

$$\pi_{vb}(v, b) = G(\varphi^*(b))g(\varphi^*(b))\varphi^{*\prime}(b).$$

Hence, $\pi_{vb}(v, b) > 0$ for all $v \in (0, 1)$ and $b \in (0, b^*(1))$.

Now we can show $\pi_b(v, b) \geq 0$ for all $\hat{b} \in [0, b^*(v))$. (A similar argument proves $\pi_b(v, b) \leq 0$ for all $b > b(v)$.) Choose $\hat{b} \in [0, b^*(v))$. Let $\hat{v}$ be the type who is supposed to bid $\hat{b}$, so that $b^*(\hat{v}) = \hat{b}$. Since $\hat{b} < b^*(v)$, $\hat{v} < v$. Therefore, since $\pi_{vb} > 0$,

$$\pi_b(v, \hat{b}) \geq \pi_b(\hat{v}, \hat{b}).$$

Because $\hat{b} = b^*(\hat{v})$, the proof of Lemma 6.1 implies $\pi_b(\hat{v}, \hat{b}) = 0$. Thus, $\pi_b(v, \hat{b}) \geq 0$. ☐

To finish the proof of Theorem 6.1, we now need only to prove that $b^*(\cdot)$ is the only symmetric equilibrium. In light of Lemma 6.1, we need only show that any symmetric equilibrium of FPA(0, 0) satisfies Guesses A-C. This is done in a sequence of lemmas. You will see that much of the proof of Lemma 6.1 is actually redundant. (The proof of differentiability, Guess C, directly derives the differential equation (6.4).)

Lemma 6.3 validates Guess A, and proves some other useful facts. Its proof is more technical than those that follow; it can be skipped at first reading.

LEMMA 6.3 (GUESS A): *If* $b(\cdot)$ *is a symmetric equilibrium of* FPA(0, 0), *then for all* $v > 0$: $b(v) \neq$ No, $\hat{Q}(b(v)) > 0$, *and* $b(v) < v$. *That is, every bidder type* $v > 0$ *bids, wins with positive probability, and makes positive profit when he wins*.

PROOF: We first show that almost every type bids. That is, letting $Q^0 = \Pr[b(\tilde{v}_i) = \text{No}]$ be the equilibrium probability that a bidder does not bid, we prove $Q^0 = 0$.

Assume $Q^0 > 0$. Let $v > 0$. If a type v bidder unilaterally deviates from the equilibrium by bidding 0, his probability of winning still exceeds the probability that no other bidder bids: $\hat{Q}(0) \geq (Q^0)^{n-1} > 0$. Thus, $(v - 0)\hat{Q}(0) > 0$, and type v makes positive profit. His optimal action is therefore to bid. We conclude that $b(v) \neq$ No for all $v > 0$, which implies $Q^0 = 0$. This contradicts the assumed $Q^0 > 0$. Hence, $Q^0 = 0$.

The equilibrium probability of {No Sale} is $(Q^0)^n = 0$. Hence, for any $v \geq 0$, $\Pr[\text{No Sale and } \tilde{v}_i \leq v\ \forall i] = 0$. Consequently,

$$\begin{aligned} F(v)^n &= \Pr[\tilde{v}_i \leq v\ \forall i] \\ &= \Pr[\text{Sale and } \tilde{v}_i \leq v\ \forall i] + \Pr[\text{No Sale and } \tilde{v}_i \leq v\ \forall i] \\ &= \Pr[\text{Sale and } \tilde{v}_i \leq v\ \forall i]. \end{aligned}$$

Let E_i be the event $\{\text{Sale to bidder } i \text{ and } \tilde{v}_i \leq v\}$. Then $\Pr(E_i) = \int_0^v \hat{Q}(b(x)) f(x)dx$. As the union of the disjoint events E_i contains $\{\text{Sale and } \tilde{v}_i \leq v\ \forall i\}$,

$$\begin{aligned} \Pr[\text{Sale and } \tilde{v}_i \leq v\ \forall i] &\leq \Pr(\cup E_i) \\ &= \Sigma \Pr(E_i) = n \int_0^v \hat{Q}(b(x)) f(x)dx. \end{aligned}$$

Putting these expressions together gives us

$$F(v)^n \leq n \int_0^v \hat{Q}(b(x)) f(x) dx.$$

This shows that for all $v > 0$, the interval $[0, v]$ contains an x for which $\hat{Q}(b(x)) > 0$. The profit of type x is $(x - b(x))\hat{Q}(b(x)) \geq 0$, as not bidding is an option. Hence, $b(x) \leq x$. The profit of type $v > x$, as bidding $b = b(x)$ is an option, satisfies

$$(v - b(v))\hat{Q}(b(v)) \geq (v - b(x))\hat{Q}(b(x)) > 0.$$

We conclude that $\hat{Q}(b(v)) > 0$ and $b(v) < v$ for every $v > 0$.

Turning to Guess B, Lemma 6.4 shows that it holds weakly, *i*.e., any equilibrium $b(\cdot)$ is weakly increasing. The *revealed preference* proof of the result is of general importance.

LEMMA 6.4: *Any symmetric equilibrium* $b(\cdot)$ *of the* FPA(0, 0) *weakly increases on* (0, 1].

PROOF: Let $v > z > 0$. We must show $b(z) \leq b(v)$. Type v prefers bid $b(v)$ to bid $b(z)$; *i*.e., the chosen action of type v "reveals" that he prefers it to the action chosen by type z:

$$(v - b(v))\hat{Q}(b(v)) \geq (v - b(z))\hat{Q}(b(z)). \tag{6.7}$$

Similarly, the type z bidder prefers $b(z)$ to $b(v)$:

$$(z - b(z))\hat{Q}(b(z)) \geq (z - b(v))\hat{Q}(b(v)). \tag{6.8}$$

Add these inequalities and cancel terms to obtain

$$(v - z)[\hat{Q}(b(v)) - \hat{Q}(b(z))] \geq 0. \tag{6.9}$$

As $v > z$, we see that $\hat{Q}(b(v)) \geq \hat{Q}(b(z))$: type v has a (weakly) greater win probability.[16] By Lemma 6.3, $\hat{Q}(b(v)) > 0$ and $z - b(z) > 0$. We divide (6.8) by these terms to get,[17]

$$\frac{\hat{Q}(b(z))}{\hat{Q}(b(v))} \geq \frac{z - b(v)}{z - b(z)}.$$

Because the left side is no larger than one, this shows that

$$1 \geq \frac{z - b(v)}{z - b(z)}.$$

Multiplying both sides by $z - b(z)$ yields $z - b(z) \geq z - b(v)$. This proves $b(z) \leq b(v)$.

Revealed Preference Picture

Because revealed preference arguments like that used to prove Lemma 6.3 are so important, let us digress for a moment to make a picture.

The problem faced by a type z bidder, when the behavior of the other bidders results in his probability-of-winning function being $\hat{Q}(\cdot)$, can be written in the following way:

$$\underset{b,\,Q}{\text{maximize}}\ \ (z-b)Q \quad \text{such that}\ \ Q=\hat{Q}(b).$$

Writing it this way means that we are viewing the type z bidder's problem as one of choosing an optimal point (b, Q) on the curve determined by the equation $Q = \hat{Q}(b)$.

View $(z-b)Q$ as the "utility function" of a type z bidder over points (b, Q). His utility decreases in the bid b, and increases (if $b < z$) in the probability Q of winning. Thus, the indifference curves of this utility function in $b-Q$ space are upward sloping (in the region where $b < z$), and utility increases to the northwest.

The maximization problem written above is conceptually the same as the standard consumer problem of maximizing utility subject to a budget constraint, where the budget constraint is $Q = \hat{Q}(b)$. We cannot draw the curve $Q = \hat{Q}(b)$ because we do not know much about it yet; in fact, the point of this exercise is to discover its properties.

We do know, however, that point $(b(z), \hat{Q}(b(z))$ is the point on the curve $Q = \hat{Q}(\cdot)$ that lies on the highest indifference curve of the utility function $(z-b)Q$. This indifference curve is the thicker of the two curves shown in Figure 6.3 below.

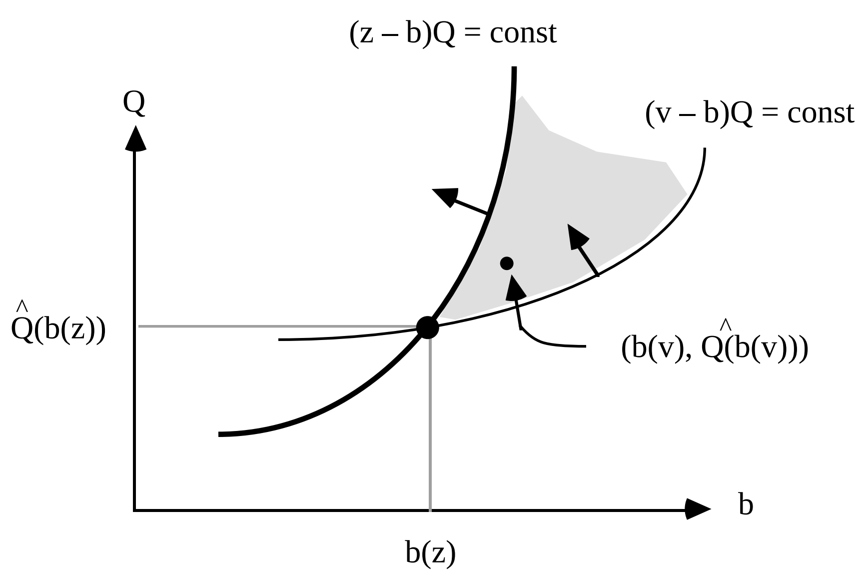


Figure 6.3

The thin indifference curve is that of a type $v > z$ bidder which passes through point $(b(z), \hat{Q}(b(z))$. The thin curve is flatter than the thick one; this property is very important and is called the *single-crossing property*.[18] The single-crossing property here means that when his value is higher, the bidder is willing to submit a higher bid in order to get the same increase ΔQ in his probability of winning.

The type v bidder faces the same sort of problem as does the type z bidder. He too must choose an optimal point on the curve $Q = \hat{Q}(b)$. His optimal point, $(b(v), \hat{Q}(b(v)))$, must lie in the shaded region. This is because of revealed preference: the type z bidder prefers $(b(z), \hat{Q}(b(z)))$ over $(b(v), \hat{Q}(b(v)))$, but the type v bidder has the opposite preference. This means that the point $(b(v), \hat{Q}(b(v)))$ must lie below the thick indifference curve and above the thin one, which is the shaded region.

This observation shows that the curve $Q = \hat{Q}(b)$ must go into the shaded region from the point $(b(z), \hat{Q}(b(z)))$. Thus, $\hat{Q}(b(v)) \geq \hat{Q}(b(z))$ and $b(z) \leq b(v)$.

LEMMA 6.5 (GUESS B): *Any symmetric equilibrium* $b(\cdot)$ *of the* FPA(0, 0) *strictly increases on* (0, 1].

PROOF: Assume $b(\cdot)$ only weakly increases. Then $v > z \geq 0$ and bid $\bar{b}$ exist such that $b(\hat{v}) = \bar{b}$ if $\hat{v} \in (z, v)$, and $b(\hat{v}) < \bar{b}$ if $\hat{v} < z$, and $b(\hat{v}) > \bar{b}$ if $\hat{v} > v$. This is shown in Figure 6.4 below (for the case of a continuous b(·)).

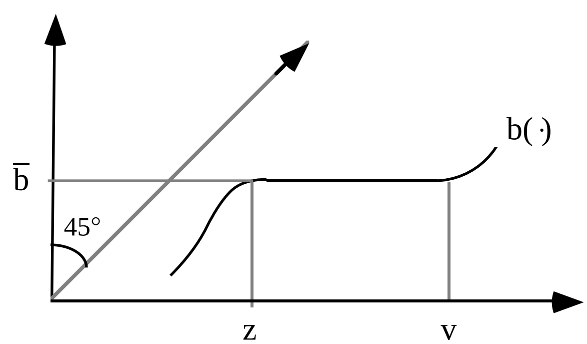


Figure 6.4

Consider $\hat{v} \in (z, v)$. The flat results in a positive probability that this type will tie, with other bidders whose types are in (z, v), when he bids $\bar{b}$. Bidding slightly more would reduce the tie probability to zero. This causes the win probability to jump up, a discrete benefit bounded above zero. Because it is obtained at only the infinitesimal cost of bidding slightly more than $\bar{b}$, the benefit of this action is greater than its cost, and thus $\bar{b}$ cannot be this bidder's optimal bid — contradiction.

To make the argument more formal, note that the flat implies

$$\lim_{\varepsilon \to 0+} \hat{Q}(\bar{b} + \varepsilon) > \hat{Q}(\bar{b}).$$

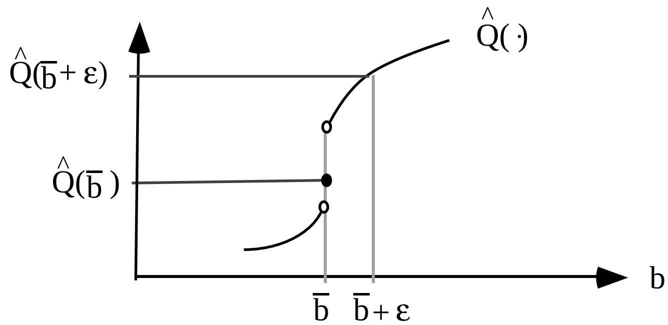


Figure 6.5

By Lemma 6.3, $v > \bar{b}$. Hence,

$$\lim_{\varepsilon \to 0+} (\hat{v} - \bar{b} - \varepsilon)\hat{Q}(\bar{b} + \varepsilon) > (\hat{v} - \bar{b})\hat{Q}(\bar{b}).$$

So for small $\varepsilon > 0$, bid $\bar{b} + \varepsilon$ gives type $\hat{v}$ more than his equilibrium profit, a contradiction. This proves $b(\cdot)$ has no flat, and so Lemma 6.4 implies $b(\cdot)$ strictly increases.

Because $b(\cdot)$ strictly increases, the probability that type $v > 0$ wins is just the probability $G(v)$ that his type is higher than the other bidders' types. The following verifies this result formally:

$\hat{Q}(b(v))$	=	Pr[i wins when bids $b(v)$]	
	=	Pr[$b(\tilde{v}_j) \leq b(v)$ for all $j \neq i$]	as the probability of a tie is zero because $b(\cdot)$ has no flats
	=	Pr[$\tilde{v}_j \leq v$ for all $j \neq i$]	as $b(\cdot)$ strictly increases
	=	$G(v)$.	

Using this, we now finish the proof of Theorem 6.1 by proving the validity of Guess C.

LEMMA 6.6 (GUESS C): *Any symmetric equilibrium* $b(\cdot)$ *of the* FPA(0, 0) *is differentiable on* (0,1].

PROOF: Let $v > z > 0$. By the revealed preference inequalities, (6.7) and (6.8),

$$z[G(v) - G(z)] \;\le\; b(v)G(v) - b(z)G(z) \;\le\; v[G(v) - G(z)]. \tag{6.10}$$

Define $P(x) = b(x)G(x)$. Use this in the middle term of (6.10), and divide by $v - z$:

$$z\left(\frac{G(v) - G(z)}{v - z}\right) \le \left(\frac{P(v) - P(z)}{v - z}\right) \le v\left(\frac{G(v) - G(z)}{v - z}\right). \tag{6.11}$$

Now, $G(\cdot)$ is differentiable, with derivative $g(\cdot)$. Thus,

$$\operatorname*{limit}_{z \to v} \left(\frac{G(v) - G(z)}{v - z}\right) = g(v).$$

Holding v fixed and letting $z \to v$ in all three terms of (6.11), we see that the left and right terms both converge to $vg(v)$. The middle term is sandwiched, and so it too converges to $vg(v)$. By the definition of a derivative, this proves that $P(\cdot)$ is differentiable at v, with $P'(v) = \mathrm{v}g(v)$. Since $b(v) = P(v)/G(v)$, the derivative $b'(v)$ also exists.[19]

Remark: Note that the middle term in (6.10) is the expected cost of changing one's bid from $b(z)$ to $b(v)$. The left term is the expected benefit to type z of the change in bid, and the right term is the expected benefit to type v. So, (6.10) states that the expected cost of changing a bid from $b(z)$ to $b(v)$ is greater than its expected benefit to a type z bidder, but less than its expected benefit to a type v bidder.

7. DUTCH AUCTIONS

Recall that in a Dutch auction, a "wheel" in front of the bidders turns at a regular pace so that the price it indicates steadily falls. The first bidder to yell "stop" wins the object and pays the price the wheel indicates at the time he stops it. Each bidder pays an entry fee c if he chooses to enter the room with the wheel (participate in the auction), and the wheel stops with the object unsold if the price falls to the reserve price r.

Any bidder's action is either "Not bidding," which means never stopping the wheel, or a number (a "bid") at which to stop the wheel if it falls that far. Once again, the set of actions for a bidder can be denoted as $\{\text{No}\} \cup [r, \infty)$. A strategy for bidder i is a function $b_i(\cdot)$ from, [0, 1], his possible values, to $\{\text{No}\} \cup [r, \infty)$.

Let us compare a Dutch auction and a first price auction with the same reserve price r and entry fee c. Consider an action profile, $(b_1,..., b_n)$, where each b_i is a number or "No". If all these actions are "No", then in the FPA no bidder submits a bid, and in the DA no bidder stops the wheel. Each bidder's payoff is zero in either auction. The other case occurs if at least one bidder bids. Suppose b_i is the only highest bid. Then in the FPA, bidder i wins and has payoff $v_i - b_i - c$, and the others obtain zero or $-c$ payoffs, depending on whether they bid. In the DA, no bidder stops the wheel before it reaches b_i, and so bidder i stops it at b_i; again bidder i wins and has payoff $v_i - b_i - c$, and the others have zero or $-c$ payoffs, depending on whether they participated. These payoffs have to be multiplied by $1/k$ if there are k highest bids in the action profile, but still the conclusion holds: any action profile, if played in both auctions, results in the same payoffs for all the bidders.

The conclusion of this argument is that the (reduced) normal form games corresponding to the Dutch and first price auctions (with the same reserve price and entry fees) are identical. They both have the same strategy sets, and the same action profile gives each bidder the same payoff in the two auctions. Thus, any strategy profile $(b_1(\cdot),..., b_n(\cdot))$ will give each bidder the same payoff. This shows that the two auctions are *strategically equivalent.* This is the strongest kind of equivalence we have seen, and it also holds in more general information environments. Two games that are strategically equivalent are to all intents and purposes the same game, and *a fortiori they must* have the same equilibria. Theorem 7.1 therefore applies to the Dutch auction; the $b^*(\cdot)$ defined in (7.2) is the unique equilibrium of the DA(0, 0), and its sale price, $\tilde{w}^{\mathrm{D}}$, is equal to $\tilde{w}^{1}$, the sale price of the FPA(0, 0).

A More Accurate Treatment

The observation that the Dutch and first price auctions are strategically equivalent is more subtle than we have made it seem. The obvious way of writing down extensive forms for the two auctions does result in a difference. The extensive form of the first price auction is very simple, with Nature moving first to choose the type profile and then each bidder simultaneously choosing a bid or a "No"; the strategies for this extensive form are functions $b_i(\cdot)$: $[0, 1] \rightarrow \{\text{No}\} \cup [r, \infty)$.

In the Dutch auction, a bidder has more information upon which to act, namely, the current price reached by the wheel. His information sets in the extensive form are indexed by a pair of numbers, (v, p), where v is his value and p is the price the wheel has reached. A strategy in the Dutch auction (for a participating bidder) should be a function

$$\beta_i(\cdot,\cdot)\text{: } [0, 1] \times (r, \infty) \rightarrow \{\text{stop wheel, don't stop wheel}\}.$$

The interpretation is that $\beta_i(v, p)$ = "stop wheel" if the type v bidder is to stop the wheel at price p, and $\beta_i(v, p)$ = "don't stop wheel" otherwise.

Now, such a strategy may have much redundancy in it. If the bidder plans to stop the wheel at a price p, then what he plans to do if the wheel gets to a lower price is irrelevant; the wheel cannot get to a lower price because he himself would stop it at p. Given a strategy $\beta_i(\cdot,\cdot)$, for each v let $b_i(v)$ be the maximum price at which $\beta_i(v, p)$ = "stop wheel"; if $\beta_i(v, p)$ = "don't stop wheel" for all $p > r$, let $b(v)$ = "No".[20] Then, in the language of game theory, two strategies $\beta_i(\cdot,\cdot)$ and $\hat{\beta}_i(\cdot,\cdot)$ are *equivalent strategies* if they give rise to the same function $b_i(\cdot)$, *i*.e., if the bidder stops the wheel at the same point according to each strategy. Given any strategy profile, no bidder's payoff will change if bidder i switches to an equivalent strategy.

The *reduced normal form* of a game is defined from its normal form by identifying as a single strategy each set of equivalent strategies. The functions $b_i(\cdot)$ we discussed above this box are actually strategies of the reduced normal form of the Dutch auction.

8. ABSTRACT AUCTIONS

We have seen that the four auctions SPA(0, 0), EA(0, 0), FPA(0, 0), and DA(0, 0) all yield the same expected sale price, given that the bidders play the equilibria we have derived. This surprising result is referred to as the "Revenue Equivalence Theorem." In this section we explore more deeply the ways in which auctions can be equivalent, considering as we do so many other kinds of auctions. The thrust of the argument is that the equilibrium behavior of bidders can nullify the differences in rules between auctions.

Before we start, observe that the equivalency we are discussing now is in terms of expected payoffs. Two equivalent auctions yield the same expected profit to the seller and to each type of bidder. The sale prices of two auctions that are equivalent in this sense can be random variables with different distributions, in which case only a risk neutral seller would necessarily regard them as equivalent.

Consider an arbitrary auction, one of the ones we have studied, with or without a reserve price or an entry fee, or even any crazy kind of auction we might dream up (e.g., all bidders pay their bid, even if they lose, or the winner is the one who submits the lowest bid, or ...). What is it that a bidder really cares about in this auction? Because he is risk neutral, he only cares about two variables, his *probability of winning*, Q, and his *expected payment*, P. If a type v bidder has chosen an action so that his probability of winning is Q and his expected payment is P, his expected profit is $Qv - P$.

Consider an equilibrium of the auction. Take the viewpoint of a particular bidder, say Paul, when the other bidders play according to the given equilibrium. Let $b(\cdot)$ be Paul's equilibrium strategy, so that $b(v)$ is his best action when his value is v. (At this level of generality, $b(v)$ may be more complicated than a bid. This does not matter.)

Paul's action, b, together with the rules of the auction, the strategies of the other bidders, and the probability distribution over the values of the other bidders, determine Paul's probability of winning and his expected payment. This means we can write Q and P as functions of b, say $\hat{Q}(b)$ and $\hat{P}(b)$.

Examples: In FPA(0, 0), we see that

$$\hat{Q}(b) = G(\varphi(b)) \quad \text{and} \quad \hat{P}(b) = G(\varphi(b))\,b. \tag{8.1}$$

In SPA(0, 0), we obtain $\hat{Q}(b) = G(b)$ and

$$\hat{P}(b) = \Pr[\tilde{y} \le b]E[\tilde{y} \mid \tilde{y} \le b] = \int_0^b yg(y)\,dy. \tag{8.2}$$

Paul's problem can be viewed as choosing b to maximize $\hat{Q}(b)v - \hat{P}(b)$. Because $b(v)$ is his optimal action when his value is v, action $b = b(v)$ solves this problem.

Instead of examining this problem directly, it is convenient to perform a "change of variables." Suppose that instead of submitting a bid to the seller himself, Paul programs a computer to do it for him. (He may want to do this, for example, because he plans to go to the beach on the day of the auction.) When he programs the computer he does not yet know his value. But he does know his optimal strategy, $b(\cdot)$, which he can program into the computer. On the day of the auction, when he knows his value, he can simply call up his computer (from the beach on his cellular telephone) and report his value. The computer then calculates a bid, using the function $b(\cdot)$, and submits it to the seller.

Paul must still make a choice: instead of reporting his true value to his computer, he could report some other value. Letting z denote the value he reports, Paul's problem is:

$$\underset{z}{\text{Maximize}}\; Q(z)v - P(z), \tag{8.3}$$

where the functions $Q(\cdot)$ and $P(\cdot)$ are defined to be the composition of the computer's rule of action, $b(\cdot)$, and the auction functions $\hat{Q}(\cdot)$ and $\hat{P}(\cdot)$:

$$Q(z) = \hat{Q}(b(z)) \quad \text{and} \quad P(z) = \hat{P}(b(z)).$$

Examples: Referring to the box above, we see that in FPA(0, 0),

$$Q(z) = G(\varphi(b(z))) = G(v) \quad \text{and} \quad P(z) = G(\varphi(b(z)))b(z) = G(z)b(z).$$

In SPA(0, 0), since $b(z) = z$, we obtain $Q(z) = G(z)$ and

$$P(z) = \hat{P}(b(z)) = \int_0^z yg(y)\,dy.$$

Luckily for Paul, problem (8.3) is trivial. After all, since the computer is programmed with his optimal strategy, it has his best interests at heart. If Paul reports his true value v, the computer will take action $b(v)$, which is Paul's best action when his value really is v. If he lies to the computer by reporting $z \neq v$, it will take action $b(z)$, which cannot be better and may be worse than $b(v)$. Thus, $z = v$ solves problem (8.3).

The moral of the story is, "do not lie to your computer." The reasoning we have followed is known as the *Revelation Principle*, a key device in information economics.[21]

The argument applies to any bidder. But their equilibrium strategies need not be the same, and so we resort to subscripts. Let $Q_i(z)$ and $P_i(z)$ denote bidder i's probability of winning and expected payment (from his point of view) when he reports to *his* computer that his value is z. Then, if his type is v, the equilibrium expected profit of bidder i is

$$\Pi_i(v) = Q_i(v)v - P_i(v). \tag{8.4}$$

Revenue Equivalence is a consequence of the following more fundamental theorem. What information do we need in order to know the equilibrium expected profit of bidder i when his value is v_i? In order to know the number $\Pi_i(v_i)$, expression (8.4) tells us that we need to know two numbers, $Q_i(v_i)$ and $P_i(v_i)$. In order to know the entire function $\Pi_i(\cdot)$, it seems as though we need to know both functions, $Q_i(\cdot)$ and $P_i(\cdot)$. In fact, however, we need less information. Because each type of bidder i optimizes, the function $\Pi_i(\cdot)$ actually depends only on one function, $Q_i(\cdot)$, and one number, $\Pi_i(0)$.

THEOREM 8.1: *The expected profit of bidder i in any equilibrium of any auction depends only on his equilibrium probability-of-winning function,* $Q_i(\cdot)$, *and the equilibrium profit of his lowest type,* $\Pi_i(0)$. *Specifically,*

$$\Pi_i(v) = \Pi_i(0) + \int_0^v Q_i(y)dy. \tag{8.5}$$

PROOF: Since $z = v$ solves (8.3), the first order condition holds: $Q_i'(v)v - P_i'(v) = 0$. Differentiate (8.4) to get $\Pi_i'(v) = Q_i'(v)v + Q_i(v) - P_i'(v)$. Put these two equations together to obtain $\Pi_i'(v) = Q_i(v)$. Integrate this to obtain (8.5).[22] ⚲

Aside on "Envelopes"

The reasoning in Theorem 8.1 is important and of general use. It relies on an *envelope property* that is exhibited by any set of maximization problems like (8.3).

Suppose, so we can make a picture, that Paul has only a finite number of possible types: $v_1,..., v^m$, numbered so that $v_1 < v^2 < \cdots < v^m$. Let $Q^k = Q(v^k)$ and $P^k = P(v^k)$.

Problem (8.3) can be viewed as choosing a pair (P, Q) from among those pairs for which some z exists such that $(P, Q) = (P(z), Q(z))$. If the only z's which Paul can report are $v_1,..., v^m$, then he can only choose one of the pairs $(P^1, Q^1),..., (P^m, Q^m)$. Since $z = v^k$ is the solution to (8.3) when his type is v^k, his optimal pair is then (P^k, Q^k).

Figure 8.2 below tries to illustrates the situation. The horizontal axis indexes the possible types. The vertical axis measures expected profit. Paul's expected profit when he is type v^k and chooses (P^j, Q^j) is measured by the vertical distance at $v = v^k$ from the horizontal axis to line j. Line j is the graph of the equation $\Pi = vQ^j - P^j$. The fact that (P^k, Q^k) is his optimal pair when his type is v^k follows from line k being the highest line at $v = v^k$.

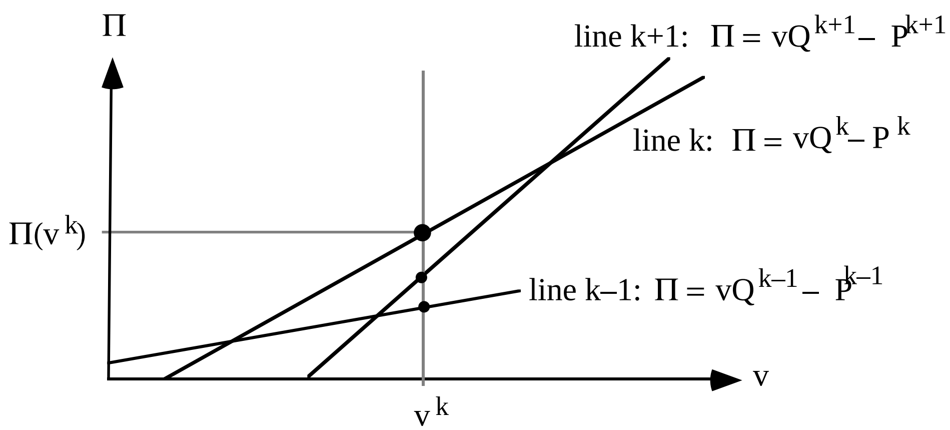


Figure 8.1

Because this is true for each v^k, the graph of $\Pi(\cdot)$ is the *upper envelope* of these lines. Going to a continuum of types, with v^{k-1}, v^k, and v^{k+1} being just three of them, we have Figure 8.2:

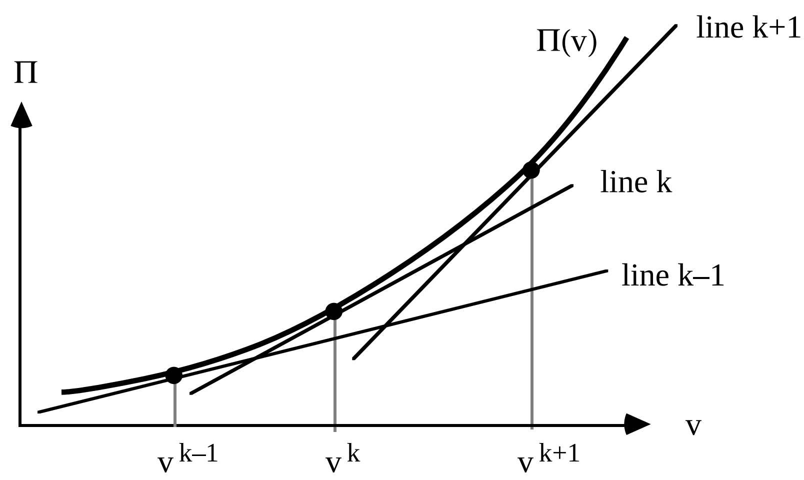


Figure 8.2

The key feature of the envelope property is the tangency at $v = v^k$ between line k and the graph of $\Pi(v)$: they have the same slopes at this point. The slope of line k is $Q^k = Q(v^k)$. This shows that $\Pi'(v^k) = Q(v^k)$, which is the derivative version of (8.5).

Economically, $\Pi'(v^k) = Q(v^k)$ means that the equilibrium profit of a type v slightly greater than v^k is nearly equal to what it would be if type v were to take the same action as type v^k. That is, increasing v from v^k, but holding the action fixed, increases the payoff along line k. Only by allowing the bidder to re-optimize by changing his action will his payoff go up from line k to $\Pi(v)$. The *envelope theorem* says that this second way in which the bidder's payoff increases when his type increases is of "second order" importance, insignificant for small changes from v^k.

We now discuss some applications of (8.4) and (8.5).

8.1 *Deriving Equilibria*

Expressions (8.4) and (8.5) can be used to derive auction equilibria easily. Consider the FPA(r, c). Let us "guess" that an equilibrium takes the form of all bidders with types less than a marginal type v_0 not bidding, and all other types bidding according to a strictly increasing function $b(\cdot)$. Then, in equilibrium, a bidder wins only if his type is greater than all the other bidders' types and greater than v_0. The equilibrium probability-of-winning function is,

$$Q_i(v) = \begin{cases} 0 & \text{if } v < v_0 \\ G(v) & \text{if } v \geq v_0. \end{cases} \tag{8.6}$$

The marginal and non-participating types have zero profit: $\Pi_i(v) = 0$ for $v \in \{0, v_0]$. Expressions (8.4) and (8.5), for $v \geq v_0$, become (dropping subscripts):

$$\Pi(v) = G(v)v - P(v). \tag{8.4′}$$

$$\Pi(v) = \int_{v_0}^{v} G(y)dy. \tag{8.5′}$$

The expected payment of type $v \geq v_0$ is

$$P(v) = c + G(v)b(v). \tag{8.7}$$

Solving these three equations for $b(v)$ shows that for $v \geq v_0$,

$$b(v) = v - \int_{v_0}^{v} \left(\frac{G(y)}{G(v)}\right) dy - \frac{c}{G(v).} \tag{8.8}$$

This is the equilibrium bidding function, completely specified except for the marginal type. To find v_0, note first that a bidder with type v_0 wins only if no other bidder bids. His probability of winning is $G(v_0)$, even if he bids less than $b(v_0)$. Since lowering his bid does not affect his probability of winning, and since $b(v_0)$ is his optimal bid, it must be as low as possible: $b(v_0) = r$. Therefore, from (8.8),

$$r = v_0 - \frac{c}{G(v_0).} \tag{8.9}$$

This equation uniquely determines the marginal type: $v_0 = v_0(r, c)$. Equation (8.9) is the same as (4.5), and so the marginal type is the same in both auctions, SPA(r, c) and FPA(r, c). (Refer to Theorem 4.2, and the Remark following (4.5).) Integrating (8.8) by parts and using (8.9) yields a generalization of (7.2): $b(v)$ = ”No” for all $v < v_0$, and for all $v \geq v_0$,

$$b(v) = \int_{v_0}^{v} y\left(\frac{g(y)}{G(v)}\right) dy + r\left(\frac{G(v_0)}{G(v)}\right). \tag{8.10}$$

This derivation was based on the "guess" that precisely the types greater than some marginal type bid, and they bid according to a strictly increasing bid function. The $b(\cdot)$ we have found satisfies these guesses, and proving it is actually an equilibrium can be done by checking the pseudoconcavity condition, as in Lemma 6.2.

8.2 *Equivalent Auctions*

Suppose that in two auctions A and B, bidder i has the same equilibrium expected profit when his type is the lowest possible, $\Pi_i^A(0) = \Pi_i^B(0)$, and his equilibrium probability of winning is the same in the two auctions for all his types, $Q_i^A(\cdot) = Q_i^B(\cdot)$. Then by (8.5), the expected profit of any type of the bidder is the same in both auctions, $\Pi_i^A(v) = \Pi_i^B(v)$. To the bidder, the auctions are (expected) payoff equivalent.

For example, consider the four auctions SPA(r, c), EA(r, c), FPA(r, c), and DA(r, c). In each of them, the marginal type v_0 is the same, determined by (8.9) (or (4.5)), and the probability-of-winning function is the same, shown in (8.6). In each auction, the expected profit of the lowest type of bidder is $\Pi_i(0) = 0$. Thus, every type of every bidder views the four auctions as equivalent.

It is now not surprising that the seller also views these auctions as equivalent, in terms of her expected profit. Her expected profit from an auction can, using (8.4) and (8.5), also be put in terms of just the numbers $\Pi_i(0)$, and the functions $Q_i(\cdot)$, for $i = 1,\dots, n$. She therefore obtains the same expected profit from all auctions that have the same $\Pi_i(0)$ numbers and $Q_i(\cdot)$ functions.

To show this rigorously, let $\Pi_S(v_S)$ be the seller's expected profit in a given auction when her value for the object is v_S. We show that it can be expressed as

$$\Pi_S(v_S) = v_S + \sum_{i=1}^{n} \left\{ \int_0^1 \left(v_i - \frac{1-F(v_i)}{f(v_i)} - v_S \right) Q_i(v_i) f(v_i) dv_i \ - \ \Pi_i(0) \right\}. \tag{8.11}$$

This expresses $\Pi_S(v_S)$ solely in terms of the $\Pi_i(0)$ numbers and $Q_i(\cdot)$ functions. Proving (8.11) is a "mere" calculation based on (8.4) and (8.5).

COROLLARY 8.1: *The seller's expected profit from any auction is given in terms of its equilibrium quantities* $\Pi_i(0)$ *and* $Q_i(\cdot)$ *by* (8.11).

PROOF: For now, drop subscripts. From (8.4) and (8.5),

$$P(v) \;=\; Q(v)v - \Pi(v) \;=\; Q(v)v - \left(\Pi(0) \;+ \int_0^v Q(y)dy\right).$$

A bidder's expected payment to the seller is therefore

$$\int_0^1 P(v)f(v)dv \;=\; \int_0^1 Q(v)vf(v)dv \;-\; \int_0^1\int_0^v Q(y)f(v)dydv \;-\; \Pi(0). \tag{8.12}$$

Now, $\displaystyle \int_0^1\int_0^v Q(y)f(v)dydv \;=\; \int_0^1\int_y^1 Q(y)f(v)dvdy \;=\; \int_0^1 Q(y)(1-F(y))dy.$

Substitute the last integral, after changing its integration variable to v, for the middle term on the right of (8.12). This yields

$$\int_0^1 P(v)f(v)dv \;=\; \int_0^1 \left(v - \frac{1-F(v)}{f(v)}\right)Q(v)f(v)dv \;-\; \Pi(0). \tag{8.13}$$

Restoring the subscripts i and summing over i yields the sum of the bidders' expected payments to the seller. Her expected profit is this expected revenue plus her own value times the probability of not making a sale. Her probability of making a sale to bidder i is

$$\int_0^1 Q_i(v_i)f(v_i)dv_i \;.$$

Summing this over i gives the probability of a sale, so that

$$1 \;-\; \sum_{i=1}^{n} \int_0^1 Q_i(v_i)f(v_i)dv_i$$

is the probability of no sale. Multiplying this by v_S and adding the result to the summation over i of (8.13) yields (8.11). ♀

To check (8.11), calculate it using $v_S = 0$, $Q(v) = F(v)^{n-1}$, $\Pi(0) = 0$, $F(v) = v$, and $f(v) = 1$. This should yield $(n–1)/(n+1)$, the seller's expected revenue in the four standard auctions with $r = c = 0$ in the uniform case.

8.3 *Optimal Reserve Prices and Entry Fees*

Let us find the seller's optimal entry fee and reserve price in the first or second price auction. As we have seen, a choice of (r, c) results in a marginal type $v_0(r, c)$ defined by (8.9). Each bidders' equilibrium probability-of-winning function is given by (8.6), and his equilibrium expected profit is zero when is type is zero. Hence, letting $v_0 = v_0(r, c)$, the seller's expected profit in either auction is, from (8.11),

$$\Pi_S(v_S) = v_S + n\left[\int_{v_0}^{1}\left(v - \frac{1 - F(v)}{f(v)} - v_S\right)F(v)^{n-1}f(v)dv\right]. \tag{8.14}$$

The seller's choice of (r, c) affects her expected profit only in so far as they affect the marginal type. If v_0^* is a marginal type v_0 that maximizes (8.14), any (r, c) satisfying $v_0(r, c) = v_0^*$ is optimal. For the distributions $F(\cdot)$ we usually work with, the function

$$v - \frac{1 - F(v)}{f(v)} - v_S \tag{8.15}$$

increases in v. In this case there is a number such that (8.15) is negative for v less than this number and positive for v greater than this number. Setting v_0 equal to this number maximizes (8.14), since it makes the interval of integration precisely equal to the interval on which the integrand is positive. The following equation then determines v_0^*:

$$v_0^* - \frac{1 - F(v_0^*)}{f(v_0^*)} - v_S = 0. \tag{8.16}$$

Even if (8.15) does not increase in v, the optimal v_0 satisfies (8.16); the problem then is that (8.16) may have multiple solutions, and only some maximize the seller's profit.

Notice something remarkable: the seller's optimal marginal value v_0^* does not depend on the number of bidders. Thus, the pairs (r, c) of optimal reserve price and entry fee combinations also do not depend on the number of bidders.

8.4 *Optimal Auctions*

Formula (8.11) can be used more generally to determine the seller's optimal auctions, those that maximize her expected profit. From (8.11), we see that the seller would like to choose an auction in which each $\Pi_i(0)$ is as small as possible, and to choose functions $Q_i(\cdot)$ that maximize

$$\sum_{i=1}^{n} \int_0^1 \left(v_i - \frac{1 - F(v_i)}{f(v_i)} - v_S \right) Q_i(v_i) f(v_i) dv_i \,. \tag{8.17}$$

Assuming the seller cannot coerce the bidders into participating, the smallest that $\Pi_i(0)$ can be in any feasible auction is zero. Setting $\Pi_i(0) = 0$ is no problem; indeed, this is thr case in all the auctions we have considered. The problem is to find functions $Q_i(\cdot)$ that maximize (8.17). These functions cannot be chosen freely — they are constrained by the rules of probability (if there is more than one bidder). For example, it is impossible for $Q_i(v) = 1$ for all $i \leq n$ and all types v in an interval $[a, b]$. Because only one bidder can win, and there is positive probability that all the bidders' types are in $[a, b]$, no bidder can be sure of winning just because his type is in this interval.

For our problem, we do not need to find the appropriate constraint for the $Q_i(\cdot)$ functions.[23] For each bidder i, let $q_i(v_1,\ldots, v_n)$ be the equilibrium probability that bidder i wins when the vector of bidders' types is $(v_1,\ldots, v_n)$. Then, the probability of winning $Q_i(v_i)$ is obtained from $q_i(v_1,\ldots, v_n)$ by "expecting out" the other values:

$$Q_i(v_i) = \int_0^1 \cdots \int_0^1 q_i(v_1,\ldots, v_n) \, [\prod_{j \neq i} f(v_j) dv_j]. \tag{8.18}$$

(There are n–1 integrals here.) Use (8.18) to replace $Q_i(v_i)$ in (8.17), and move the summation inside the integrals:

$$\int_0^1 \cdots \int_0^1 \left\{ \sum_{i=1}^{n} \left(v_i - \frac{1 - F(v_i)}{f(v_i)} - v_S \right) q_i(v_1,\ldots, v_n) \right\} [\prod_{i=1,n} f(v_i) dv_i]. \tag{8.19}$$

(There are n integrals here.)

Now, instead of choosing functions $Q_i(\cdot)$ to maximize (8.17), choose functions $q_i(\cdot)$ to maximize (8.19). They are constrained only to be probabilities: for each $(v_1,\ldots, v_n)$,

$$q_i(v_1,\ldots, v_n) \geq 0 \text{ for } i = 1,\ldots, n, \quad \text{and} \quad \sum_{i=1}^{n} q_i(v_1,\ldots, v_n) \leq 1. \tag{8.20}$$

(The sum is less than one when the probability of a sale is less than one.)

The obvious thing to try is to maximize (8.19) pointwise, *i*.e., to find for each $(v_1,\ldots, v_n)$ a vector $(q_1,\ldots, q_n)$ that maximizes

$$\sum_{i=1}^{n}\left(v_i - \frac{1-F(v_i)}{f(v_i)} - v_S\right)q_i \tag{8.21}$$

subject to the constraints $(q_1,\ldots, q_n) \in [0, 1]^n$ and $\Sigma q_i \leq 1$. Functions $q_i(\cdot)$ can be defined by letting $(q_1(v_1,\ldots, v_n), \ldots, q_n(v_1,\ldots, v_n))$ be a solution to this problem for each $(v_1,\ldots, v_n)$. The difficulty, however, is that these functions may not be the equilibrium probability-of-winning functions for *any* auction. To avoid this difficulty, we make the following monotonicity assumption.

ASSUMPTION M: $v - \dfrac{1-F(v)}{f(v)}$ increases in v on [0, 1].

Assumption M is not terribly restrictive — most standard distributions satisfy it. (We ran into this monotonicity assumption implicitly at (8.15). It was not so important there.)

Maximizing (8.21) subject to $(q_1,\ldots, q_n) \in [0, 1]^n$ and $\Sigma q_i \leq 1$ is now simple. Note that (8.21) is a weighted average of the n terms in big brackets. It is maximized by putting zero weight on all but the largest of those terms. By Assumption M, the largest of those terms are those with the largest v_i's. Hence, $q_i = 0$ if $v_i \neq \max(v_1,\ldots, v_n)$. Even the largest terms in big brackets should get a zero weight if they are negative, but not if they are positive. Assumption M implies that a unique v_0^* is defined by

$$v_0^* - \frac{1-F(v_0^*)}{f(v_0^*)} - v_S = 0. \tag{8.22}$$

(We saw this before as (8.16).) The i^{th} term in big brackets in (8.21) is positive if and only if $v_i > v_0^*$. Hence $(q_1,\ldots, q_n)$ maximizes (8.21) at $(v_1,\ldots, v_n)$ if and only if

$$\begin{aligned}
&\text{(a)} \quad q_1 = \cdots = q_{\text{n}} = 0 \text{ if each } v_i < v_0^*, \\
&\text{(b)} \quad \sum_{i=1}^{n} q_i = 1 \ \text{ if } \max(v_1,\ldots, v_n) > v_0^*, \ \text{ and} \\
&\text{(c)} \quad q_i > 0 \quad \Rightarrow \quad v_i = \max(v_1,\ldots, v_n).
\end{aligned} \tag{8.23}$$

This should look familiar: the probability-of-winning functions of our four auctions satisfy (8.23), if their reserve price and entry fees are set appropriately. Specifically, if (r, c) satisfies $v_0(r, c) = v_0^*$, then the equilibrium probability-of-winning functions for any of the four auctions with this (r, c) are given by,

$$\begin{aligned}
&q_i(v_1,\ldots, v_n) = 0 && \text{if } v_i < v_0^* \ \text{ or } v_i \neq \max(v_1,\ldots, v_n), \text{ and} \\
&q_i(v_1,\ldots, v_n) = 1/k && \text{if } v_i = \max(v_1,\ldots, v_n) \geq v_0^* \ \text{ and } k = |\{j \mid v_j = \max(v_1,\ldots, v_n)\}|
\end{aligned}$$

These functions clearly satisfy (8.23) and hence maximize (8.21). They therefore maximize (8.19). We conclude that for any (r, c) satisfying $v_0(r, c) = v_0^*$, each of the auctions SPA(r, c), EA(r, c), FPA(r, c) and DA(r, c) is an optimal auction.

Remarkably, among all possible auctions we could dream up, the four standard ones are optimal for the seller. However, this conclusion does require Assumption M.[24]

Remark: Reconsider (8.21). If the terms in big brackets were simply $v_i - v_S$, the entire expression (if added to v_S) would give the expected social surplus from the probabilistic trade defined by the probabilities $q_1,\ldots, q_n$. It would be welfare maximizing to choose these probabilities to maximize this expression. Under complete information, this is what the seller would do, as making the social pie as large as possible is how she, a perfectly discriminating monopolist, would maximize her share of it. However, since the buyers' values are private information, the seller cannot perfectly price discriminate. Each type $v_i > 0$ of bidder receives an "information rent", $\Pi_i(v_i) > 0$. In view of the actual (8.21), we see that the seller's optimal $(q_1,\ldots, q_n)$ would be socially optimal if instead of having value v_i, the type v_i bidder had value $\hat{v}_i = v_i - (1 - F(v_i))/f(v_i)$. This $\hat{v}_i$ has been called (by Myerson) the *virtual value* of the bidder.

9. BRIEF LITERATURE GUIDE

This is an incomplete and biased sampling of the auction literature, starting with the classic paper that introduced second price auctions and revenue equivalence:

Vickrey, William, "Counterspeculation, auctions and competitive sealed tenders," *Journal of Finance* **16**, March 1961, 8-37.

Useful surveys of auction theory are the following:

McAfee, R. Preston and John McMillan, "Auctions and bidding," *Journal of Economic Literature* **25**, June 1987, 699-738.

Milgrom, Paul, "Auctions and bidding: a primer," *Journal of Economic Perspectives* **3**, Summer 1989, 3-22.

Riley, John *G*., "Expected revenues from open and sealed bid auctions," *Journal of Economic Perspectives* **3**, Summer 1989, 41-50.

Wilson, Robert, "Strategic analysis of auctions," in *Handbook of Game Theory,* Vol 1 (ed. R. Aumann and S. Hart), Elsevier Sci. Pub., 1992.

Wolfstetter, Elmer, "Auctions: An introduction," forthcoming in *Journal of Economic Surveys*, 1995.

Optimal auctions for risk neutral bidders with independent types are derived in:

Myerson, Roger, "Optimal auction design," *Mathematics of Operations Research* **6**, 1981, 58-73.

Riley, John *G*. and William *F.* Samuelson, "Optimal auctions," *American Economic Review* **71**, June 1981, 381-392.

The relationship between optimal auctions and monopoly pricing is exposited in

Bulow, Jeremy and D. John Roberts, "The simple economics of optimal auctions," *Journal of Political Economy* **97**, 1989, 1060-1090.

Standard and optimal auctions for risk averse bidders are studied in the following:

Maskin, Eric and John *G*. Riley, "Optimal auctions with risk averse buyers," *Econometrica* **52**, 1984, 1473-1518.

Matthews, Steven A., "Selling to risk averse buyers with unobservable tastes," *Journal of Economic Theory* **30**, 1983, 370-400.

Matthews, Steven A., "Risk aversion and the efficiency of first and second price auctions," Northwestern University, mimeo, March 1980.

Matthews, Steven A., "Comparing auctions for risk averse buyers: A buyer's point of view," *Econometrica* **55**, No. 3 (May 1987), 633-646.

A simple common value model of an oil auction appears in the following paper:

Matthews, Steven A., “Information acquisition in discriminatory auctions,” in *Bayesian Models in Economic Theory*, ed. M. Boyer and R. Kihlstrom, Elsevier Science Pub., 1984, 181-207.

The following influential paper presents and analyzes a general symmetric auction environment which includes, as special cases, independent and affiliated types, and private or common values:

Milgrom, Paul and Robert J. Weber, “A theory of auctions and competitive bidding,” *Econometrica* **50**, November 1982, 1089-1122.

The following articles are about collusion in auctions:

Hendricks, Kenneth and Robert H. Porter, “Collusion in auctions,” *Annales d’Economie et de Statistique*, No. 15/16, 1989, 217-230.

Mailath, George and Peter Zemsky, “Collusion in second price auctions with heterogeneous bidders,” *Games and Economic Behavior*, **3**, 1991, 467-486.

McAfee, R. Preston and John McMillan, “Bidding rings,” *American Economic Review*, **82**, 1992, 579-599.

The following articles contain empirical work on auctions:

Ashenfelter, Orley, “How auctions work for wine and art,” *Journal of Economic Perspectives* **3**, Summer 1989, 23-36.

Cassady, Ralph Jr., *Auctions and Auctioneering*, Berkeley: University of California Press, 1967.

Hendricks, K. and H. J. Paarsch, “A survey of recent empirical work concerning auctions,” forthcoming in the *Canadian Journal of Economics*, 1994.

Laffont, J.-J., H. Ossard and *Q*. Vuong, “Econometrics of first-price auctions,” forthcoming in *Econometrica*, 1995.

Robert H. Porter, “The role of information in U.S. offshore oil and gas lease auctions,” *Econometrica*, **63**, January 1995, 1-28.

The following articles survey experimental work on auctions:

Kagel, John H., “Auctions: A survey of experimental research,” in *Handbook of Experimental Economics*, ed. (John H. Kagel and Alvin E. Roth), Princeton University Press, Princeton N.J., 1995.

Smith, Vernon L., “Bidding,” in *The New Palgrave: Allocation, Information, and Markets*, ed. J. Eatwell, M. Milgate, and *P*. Newman, New York: W. W. Norton & Co., 1989.

10. Exercises

1. Consider SPA(r, 0) with $n = 1$ bidder.
   (a) What is an equilibrium bidding strategy?
   (b) What value of r maximizes the seller's expected profit?
   (c) Discuss the efficiency of the auction with this optimal reserve price.

2. Derive, from first principles using Theorem 4.2, the value of r that maximizes the seller's expected profit in the SPA(r, 0) with $n > 1$ bidders.

3. Same as problem 1, but with a FPA.

4. Same as problem 2, but with a FPA.

5. Consider the *all-pay auction*: sealed bids are submitted, the high bidder wins, and *each bidder*, including each loser, pays an amount equal to his own bid. Suppose $r = c = 0$. Find the symmetric equilibrium.

6. Consider the *buying-bids auction*. It is a FPA in the sense that the high bidder wins and pays as the sale price an amount equal to his bid. The reserve price is zero. But the entry fee is *negative* and depends on the magnitude of the submitted bid. Specifically, the seller pays a bidder wishing to submit a bid b the amount

$$E(b) = \int_0^b F(x)^{n-1} dx.$$

   Find the symmetric equilibrium of this auction. Compare this auction to the other auctions we have studied. What nice properties does it have?

7. Consider an auction in our private values setting in which sealed bids are submitted, the high bidder wins, and the price he pays is the third highest bid. Suppose the reserve price and entry fee are both zero.
   (a) Show that truthful bidding is not a dominant strategy.
   (b) Will the equilibrium be to bid higher or lower than one's value?
   (*c*) Find the equilibrium in the uniform case.

8. Assume the values are uniformly distributed on [0, 1]. Let bidder i be risk averse, with utility function (for money) $u_i(m) = m^a$, where $a \in (0, 1]$ is a constant. Thus, if a bidder pays p_w when he wins and 0 when he loses, his expected utility is

$$\Pr[\text{win}](v_i - p_w)^a.$$

   (a) Find the equilibrium of the SPA(0, 0).
   (b) Find the equilibrium of the FPA(0, 0).
   (c) Does revenue equivalence hold here?

9. The same as problem 8, but assume each bidder has a different risk parameter, and these parameters are private information and uniformly distributed. That is, $u_i(m) = m^{a_i}$, only bidder i knows a_i, and the others view a_i as a random variable distributed uniformly on [0, 1]. [A strategy is now a function $b(v_i, a_i)$.]

11. NOTES

1 The model also applies to (procurement) auctions in which there is one buyer and several potential sellers. Just let the bidders' bids and values (defined below) be negative.

2 In other economic models, v_i would be called the "reservation price" of bidder i. We avoid this term in order to not confuse it later with an auction's "reserve price".

3 Oil tract auctions are "common value" auctions. See, e.g., Matthews (1984), Porter (1995), and Milgrom and Weber (1982). (Full references are in Section 9.)

4 Even in an art auction, (A1) and (A2) might fail. Suppose the bidders are art dealers who will face the same uncertain resale price about which each has private information. Then their values are neither private nor independent, just as in an oil auction.

5 The "dot" denotes functions. Thus, $F(\cdot)$ is a function, but $F(v)$, its value at v, is a number.

6 Without (A4), the bidder would have a nonlinear utility function u_i such that instead of maximizing his expected profit as shown in (1.1), he would maximize his expected utility, $\Pr[\text{win}]\, u_i(v_i - p_w) + \Pr[\text{lose}]\, u_i(-p_\ell)$.

7 A procedure close to a second price auction is used for selling rare stamps. For a discussion of the empirical issue, see M. Rothkopf, T. Teisberg, and E. Kahn, "Why Are Vickrey Auctions Rare," *Journal of Political Economy* **98**, February 1990, 94-109.

8 Statisticians call $\tilde{v}_{(1)}$ and $\tilde{v}_{(2)}$, respectively, the n^{th} *order statistic* and the $(n-1)^{\text{th}}$ *order statistic* of the *random sample* $\tilde{v}_1, \ldots, \tilde{v}_n$.

9 Economists use the phrases "truthful bidding" or "truthful reporting," which have unwarranted moral connotations. Bidding other than truthfully is not immoral in any accepted sense. In an auction in which the winner pays his own bid, bidding one's true value is usually more dumb than moral.

10 For $v_i = r$, either action, "No" or $b = r$, is dominant.

11 The argument below holds even if $\tilde{z}$ does not have a density function. If your are familiar with Riemann-Stieltjes integrals, just replace $h(z)dz$ by $dH(z)$ in the integrals below.

12 I toast the reader who reads a proof to its end: .

13 This simplifying assumption would be important, and hence bad for a model of an English auction, if the values were not private and independent. In a common value setting, the learning that a bidder in a real English auction does when he sees others drop out is important. See Milgrom and Weber (1982).

14 The *winning* bidder's bid in the first price auction, $b^*(\tilde{v}_{(1)})$, is equal to the expectation of the highest of his competitors's values, which is necessarily the second highest value. For you probability experts: if one bidder's value is the highest, $\tilde{v}_{(1)}$, then the highest of his competitors' values, $\tilde{y}$, is necessarily the second highest of all the values, $\tilde{v}_{(2)}$. So,

$$E[\tilde{w}^1] = E[b^*(\tilde{v}_{(1)})] = E\{E[\tilde{y} \mid \tilde{y} \le \tilde{v}_{(1)}]\}$$
$$= E\{E[\tilde{v}_{(2)} \mid \tilde{v}_{(2)} \le \tilde{v}_{(1)}]\} = E[\tilde{v}_{(2)}] = [\tilde{w}^2].$$

15 A differentiable function $f\colon \mathfrak{R}^n \to \mathfrak{R}$ is *pseudoconcave* if for all $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathfrak{R}^n$, $(\boldsymbol{y} - \boldsymbol{x}) \cdot \nabla f(\boldsymbol{x}) \le 0$ implies $f(\boldsymbol{y}) \le f(\boldsymbol{x})$. Pseudoconcavity is a sufficient second order condition because, if $f(\cdot)$ is pseudoconcave and $\nabla f(\boldsymbol{x}^*) = \boldsymbol{0}$, then $\boldsymbol{x}^*$ maximizes $f(\cdot)$. The graph of a pseudoconcave $f\colon \mathfrak{R} \to \mathfrak{R}$ is single-peaked (though the peak can be at $\pm\infty$).

16 The FPA rules alone imply that $\hat{Q}(b)$ is weakly increasing in b. A common error is to think that this and $\hat{Q}(b(v)) \ge \hat{Q}(b(z))$ imply $b(v) \ge b(z)$. They don't!

17 Why choose (6.8) to divide? Well, we want to prove that type v bids more than type z. We have just shown that type v has a greater probability of winning. *If* type v also bid less, then $(b(v), \hat{Q}(b(v)))$ would be unambiguously preferable to $(b(z), \hat{Q}(b(z)))$, since it would have a greater win probability and a lower price. It is (6.8) that contradicts this, since it says that type z must prefer $(b(z), \hat{Q}(b(z)))$ to $(b(v), \hat{Q}(b(v)))$.

18 Let $\mathrm{u}(b, Q \mid v_i) = (v_i - b)Q$ denote a type v_i bidder's utility function over pairs (b, Q). The slope (marginal rate of substitution) of type $\mathrm{v_i}$'s indifference curve at (b, Q) is

$$\frac{-u_b(b, Q \mid v_i)}{u_Q(b, Q \mid v_i)} = \frac{Q}{v_i - b}.$$

This decreases in v_i when $Q > 0$. Since $v > z$, we see that $Q/(v - b) < Q/(z - b)$.

19 From $P(v) = G(v)b(v)$ and $P'(v) = vg(v)$, we obtain the same differential equation we derived in (6.4): $G(v)b'(v) + g(v)b(v) = vg(v)$.

[20] Because there is a continuum of prices p, $b_i(\cdot)$ is not well-defined for some strategies $\beta_i(\cdot,\cdot)$. This is of little economic consequence. If the wheel "ticks" the price down in discrete increments, $b_i(\cdot)$ is well-defined.

[21] The Revelation Principle states that the composition of an auction with one of its equilibria is a "revelation auction" for which truthtelling is an equilibrium. Here, the auction/equilibrium combination is $\langle \hat{Q}(\cdot), \hat{P}(\cdot) \rangle$, and the revelation auction is $\langle Q(\cdot), P(\cdot) \rangle$.

[22] OK — this proof cheats. It relies on the unwarranted assumption that $Q(\cdot)$ and $P(\cdot)$ are differentiable. These functions are endogenous, and so we should not assume *anything* about them. In fact, in many auctions they are not differentiable, e.g., they often have kinks at the marginal type v_0. The following is a correct proof of Theorem 8.1.

Choose two values, v and $z = v - \delta$. The revealed preference relations between these two types, based on (8.3) (see (7.5) and (7.6) in Section 7), can be manipulated to:

$$\delta Q(v - \delta) \;\le\; \Pi(v) - \Pi(v - \delta) \;\le\; \delta Q(v). \tag{*}$$

This shows that $\Pi(\cdot)$ is absolutely continuous and hence differentiable almost everywhere and equal to the integral of its derivative. By choosing $\delta > 0$ in (*), we see that $Q(\cdot)$ is nondecreasing. Hence, $Q(\cdot)$ is continuous almost everywhere. At every v where it is continuous, dividing (*) by δ and letting $\delta \to 0$ shows that $\Pi'(v)$ exists and $\Pi'(v) = Q(v)$. This proves $\Pi(\cdot)$ is the integral of $Q(\cdot)$, *i*.e., Theorem 8.1.

[23] For the nature of this constraint and a setting where its use cannot be avoided, see Matthews, Steven A. (1984), "On the implementability of reduced form auctions," *Econometrica*, **52**, 1519-1522, and references cited there.

[24] Myerson (1981) dispenses with Assumption M. See also Bulow and Roberts (1989).