Math 104 Midterm #1

blue=question, black=answer.

1: Which Maple command would you use to plot two periods of the sine curve?

```
> plot( sin, 0..4*Pi );
```

2: Sketch the plot created by the following Maple command:

```
>  with(plots):
>  implicitplot( x^2+y*y = 1, x=-1..1, y=-1..1);
```

It's a circle.

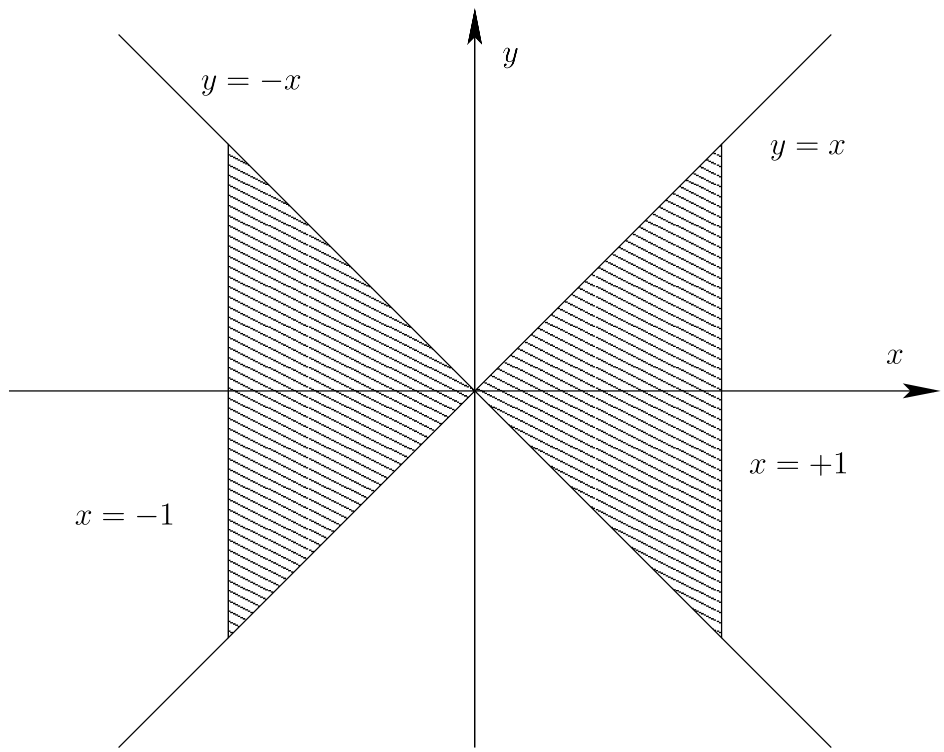
3: Here Maple cannot find an analytic solution. Which Maple command would compute a numeric solution?

```
> int( sin(x) / x, x=0..Pi/2 );
```

$$\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{x}\, \mathrm{d}x$$

```
> evalf( % );
```

4: What is the formula for the shaded area between the graphs in the following picture:



$$\int_{-1}^{0}\Big((-x)-x\Big)\,\mathrm{d}x+\int_{0}^{1}\Big(x-(-x)\Big)\,\mathrm{d}x$$

5: Find the area between $y=-x$ and $y=\sin(x)$, $0\leq x\leq\frac{\pi}{4}$.

$$\int_{0}^{\frac{\pi}{4}}\Big(\sin(x)-(-x)\Big)\mathrm{d}x=\frac{1}{32}\,\pi^{2}-\frac{1}{2}\,\sqrt{2}+1$$

6: A solid lies between the planes perpendicular to the x-axis at $x=0$ and $x=\frac{\pi}{4}$. The cross sections perpendicular to the x-axis are discs of diameter $\sqrt{\tan(x)}$. What is the volume of the solid?

$$\int_{0}^{\frac{\pi}{4}}\pi\left(\frac{1}{2}\sqrt{\tan(x)}\right)^{2}\mathrm{d}x=\frac{\pi}{8}\ln 2$$

7: The region bounded by $y=|1-|x||$ and the line $y=0$ is rotated around the x-axis. Find the volume of the solid of revolution.

Solid of revolution consists of two cones, glued at the base. The base is a disc of radius 1 $\Rightarrow$ Area $= \pi$. Height of each cone is 1 $\Rightarrow$ Volume of 1 cone $= \frac{1}{3} \cdot 1 \cdot \pi = \frac{\pi}{3}$. Hence total volume $= \frac{2\pi}{3}$.

8: Where is the center of mass of a homogeneous triangular plate with vertices $(0,0)$, $(0,1)$ and $(1,0)$? You may remember that the center of mass of a triangle is on a line trough a vertex and the midpoint of the opposing side. In the (x,y)-plane those lines have equations

$$y = x, \quad y = \frac{1}{2} - \frac{1}{2}x, \quad y = -2(x - \frac{1}{2})$$

The common solution is $(x,y) = (\frac{1}{3}, \frac{1}{3})$, this is the center of mass.

9: Find the center of mass of a thin plate covering the region bounded by $y = 1 - x^2$ and the x-axis if the density function is $\delta = \frac{1}{2}$.

$$M = \int_{-1}^{1} \delta \mathrm{d}x (1 - x^2) = 2/3$$

The moment for rotation around the y-axis, m_y, is zero because of symmetry. The coordinates of the center of mass of a thin strip parallel to the y-axis are $(\tilde{x}, \tilde{y}) = (x, \frac{1-x^2}{2})$:

$$m_x = \int_{-1}^{1} \mathrm{d}m \frac{1-x^2}{2} = \int_{-1}^{1} \delta \mathrm{d}x (1-x^2) \frac{1-x^2}{2} = \frac{4}{15}$$

Hence the center of mass has coordinates

$$(x,y) = \left(\frac{m_y}{M}, \frac{m_x}{M} \right) = \left(0, \frac{2}{5} \right)$$

10: Rotate the region bounded by $y = x^2$, $y = 0$ and $x = 1$ around the y-axis. Find the volume of the solid of revolution. Shell method:

$$V = \int_{0}^{1} 2\pi x y \mathrm{d}x = \int_{0}^{1} 2\pi x \cdot x^2 \mathrm{d}x = \frac{\pi}{2}$$

11: Find the length of the curve $x = \frac{1}{3}y^3 + \frac{1}{y}$ from $y = 2$ to $y = 3$. $x(2) = \frac{19}{6}$, $x(3) = \frac{28}{3}$. Curve length L is

$$L = \int_{\frac{19}{6}}^{\frac{28}{3}} \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} \mathrm{d}y = \int_{\frac{19}{6}}^{\frac{28}{3}} \sqrt{y^4 - 1 + y^{-4}} \mathrm{d}y$$

This integral cannot be evaluated in terms of elementary functions.

12: Find the area of the surface of revolution of the curve $y = \frac{1}{3}x^3$, $0 \le x \le 1$ around the x-axis.

$$A = \int_0^1 2\pi y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \mathrm{d}x = \left[\frac{1}{9}\pi \left(x^4 + 1\right)^{3/2}\right]_0^1 = \frac{\pi}{9}(2\sqrt{2} - 1)$$

13: The boundary of the surface of revolution of the previous exercise is a circle. Compute the area of the cone over this circle, with the tip of the cone at the origin. Base is circle of radius $y(1) = \frac{1}{3}$. Use formula for area of cone frustrum:

$$A = \sqrt{1 + \left(\frac{1}{3}\right)^2} \cdot 2\pi \frac{0 + \frac{1}{3}}{2} = \frac{1}{9}\pi\sqrt{10}$$

14: Sort the following numbers: 0, π, $\ln 2$, e, 1, $\sqrt{2}$.

$$0 \; < \; \ln 2 \; < \; 1 \; < \; \sqrt{2} \; < \; e \; < \; \pi$$

15: Find the equation for the tangent at the curve $y = e^x$ at $x = 0$.

$$y = 1 + x$$

16: Solve the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{1}{x}$, $y(1) = 2$.

$$y(x) = \ln(x) + x + 1$$

17: Find a curve whose length between $x = 1$ and $x = 2$ is $\int_1^2 \sqrt{1 + \frac{1}{x^2}} \, \mathrm{d}x$

$$y = -\frac{1}{x} \; 1 \le x \le 2$$